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ON SYMMETRIC HOMEOMORPHISMS ON THE REAL LINE

HU YUN, WU LI, AND SHEN YULIANG

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Abstract. We introduce a complex Banach manifold structure on the space of normalized symmetric homeomorphisms on the real line.

1. Introduction

We begin with some basic notation. Let $\mathbb{U} = \{z = x + iy : y > 0\}$ and $\mathbb{L} = \{z = x + iy : y < 0\}$ denote the upper and lower half plane in the complex plane $\mathbb{C}$, respectively. $\mathbb{R} = \partial\mathbb{U} = \partial\mathbb{L}$ is the real line, and $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the extended real line in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\Delta = \{z : |z| < 1\}$ denote the unit disk. $\Delta^* = \hat{\mathbb{C}} - \overline{\Delta}$ is the exterior of Δ, and $S^1 = \partial\Delta = \partial\Delta^*$ is the unit circle.

Let $\mathrm{Hom}^+(\mathbb{R})$ denote the set of all increasing homeomorphisms of $\mathbb{R}$ onto itself. A homeomorphism $h \in \mathrm{Hom}^+(\mathbb{R})$ is said to be quasisymmetric if there exists some $M > 0$ such that

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t))} \leq M \tag{1.1}$$

for all $x \in \mathbb{R}$ and $t > 0$. Beurling-Ahlfors [BA] proved that $h \in \mathrm{Hom}^+(\mathbb{R})$ is quasisymmetric if and only if there exists some quasiconformal homeomorphism of $\mathbb{U}$ onto itself which has boundary values h.

The universal Teichmüller space T is a universal parameter space for all Riemann surfaces and can be defined as the space of all normalized quasisymmetric homeomorphisms on the real line, namely, $T = QS(\mathbb{R})/\,\mathrm{Aff}(\mathbb{R})$. Here, $QS(\mathbb{R})$ denotes the group of all quasisymmetric homeomorphisms of the real line, and $\mathrm{Aff}(\mathbb{R})$ the subgroup of all real affine mappings $z \mapsto az + b$, $a > 0, b \in \mathbb{R}$. It is known that the universal Teichmüller space T is an infinite-dimensional complex Banach manifold, and $QS(\mathbb{R})$ has a smooth Banach manifold structure such that $QS(\mathbb{R})$ is diffemorphic to $T \times \mathrm{Aff}(\mathbb{R})$ (see [Ga], [GL], [Le], [Na]).

A quasisymmetric homeomorphism h is said to be symmetric if

$$\lim_{t\to 0+} \frac{h(x+t) - h(x)}{h(x) - h(x-t))} = 1 \tag{1.2}$$

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uniformly for all $x \in \mathbb{R}$. Let $S(\mathbb{R})$ denote the set of all symmetric homeomorphisms of the real line. This class was first studied in [Ca] when Carleson discussed the absolute continuity of a quasisymmetric homeomorphism. It was investigated in depth later by Gardiner-Sullivan [GS] during their study of little Teichmüller spaces and asymptotic Teichmüller spaces (see also [EGL1, EGL2], [EMS]). In particular, it was proved that a quasisymmetric homeomorphism h is symmetric if and only if h can be extended as an asymptotically conformal mapping f to the upper half plane, and that the Beurling-Ahlfors extension of h is asymptotically conformal when h is symmetric (see [Ca], [GS], [Ma]). Here by an asymptotically conformal mapping f of the upper half plane onto itself we mean that its complex dilatation $\mu = \overline{\partial} f/\partial f$ satisfies the condition $\mu(x+iy) \to 0$ uniformly for all $x \in \mathbb{R}$ when $y \to 0+$. Consequently, $S(\mathbb{R})$ is a subgroup of $QS(\mathbb{R})$. We denote $T_* = S(\mathbb{R})/\operatorname{Aff}(\mathbb{R})$ and call it the symmetric Teichmüller space.

In this paper, we will endow the symmetric Teichmüller space T_* with a complex Banach manifold structure under which T_* can be biholomorphically embedded as a bounded domain in a certain Banach space. We will also point out an essential difference, a fact which appears to have gone unnoticed, between the symmetric Teichmüller space T_* and the little Teichmüller space T_0 that will be defined in the last section.

2. Preliminaries and statement of the main result

In this section, we recall some basic definitions and results on the universal Teichmüller space and state the main result of the paper. For primary references, see Gardiner-Lakic [GL], Lehto [Le] and Nag [Na].

2.1. **Universal Teichmüller space.** Let $M(\mathbb{U})$ denote the open unit ball of the Banach space $L^\infty(\mathbb{U})$ of essentially bounded measurable functions on the upper half plane $\mathbb{U}$. For $\mu \in M(\mathbb{U})$, let f^μ be the unique quasiconformal mapping of $\mathbb{U}$ onto itself which has complex dilatation μ and keeps the points 0, 1, and ∞ fixed. We say two elements μ and ν in $M(\mathbb{U})$ are equivalent, denoted by $\mu \sim \nu$, if $f^\mu = f^\nu$ on the real line $\mathbb{R}$. We let $[\mu]$ denote the equivalence class of μ. Then the correspondence $[\mu] \mapsto f^\mu|_{\mathbb{R}}$ establishes a one-to-one map from $M(\mathbb{U})/_\sim$ onto the universal Teichmüller space T. $T = M(\mathbb{U})/_\sim$ is known as the Bers model of the universal Teichmüller space. We let Φ denote the natural projection from $M(\mathbb{U})$ onto T so that $\Phi(\mu)$ is the equivalence class $[\mu]$. $[0]$ is called the base point of T. The Teichmüller distance between two points $[\mu_1]$ and $[\mu_2]$ in T is defined as

$$\tau([\mu_1],[\mu_2]) \doteq \inf\left\{\frac{1}{2}\log\frac{1+\|\frac{\nu_1-\nu_2}{1-\overline{\nu}_1\nu_2}\|_\infty}{1-\|\frac{\nu_1-\nu_2}{1-\overline{\nu}_1\nu_2}\|_\infty} : [\nu_1]=[\mu_1], [\nu_2]=[\mu_2]\right\}. \tag{2.1}$$

Earle-Eells [EE] proved that the universal Teichmüller space T is contractible by means of a Beurling-Ahlfors extension. In fact, they showed that the Beurling-Ahlfors extension induces a continuous section $s : T \to M(\mathbb{U})$ of the natural projection Φ by sending a point $[\mu]$ to the Beltrami coefficient of the Beurling-Ahlfors extension of $f^\mu|_{\mathbb{R}}$.

Let Ω be an arbitrary simply connected domain in the extended complex plane $\hat{\mathbb{C}}$ which is conformally equivalent to the upper half plane. Recall that the hyperbolic metric λ_Ω in Ω can be defined by

$$\lambda_\Omega(f(z))|f'(z)| = \frac{1}{2y}, \quad z = x + iy \in \mathbb{U}, \tag{2.2}$$

where $f : \mathbb{U} \to \Omega$ is any conformal mapping. Let $B(\Omega)$ denote the Banach space of functions ϕ holomorphic in Ω with norm

$$\|\phi\|_{B(\Omega)} = \sup_{z\in\Omega} |\phi(z)|\lambda_\Omega^{-2}(z). \tag{2.3}$$

It is easy to see that a conformal mapping $g : \Omega_1 \to \Omega_2$ induces an isometric isomorphism $\phi \mapsto (\phi \circ g)(g')^2$ from $B(\Omega_2)$ onto $B(\Omega_1)$.

It is known that the universal Teichmüller space T is an infinite-dimensional complex Banach manifold. To make this precise, we consider the map $\mathcal{S} : M(\mathbb{U}) \to B(\mathbb{L})$ defined as $\mathcal{S}(\mu) = S(f_\mu|_{\mathbb{L}})$, where f_μ is the unique quasiconformal mapping of the complex plane $\mathbb{C}$ which has complex dilatation μ in $\mathbb{U}$, is conformal in $\mathbb{L}$, and keeps the points 0, 1, and ∞ fixed, while $S(f)$ denotes the Schwarzian derivative of a locally univalent function f of a domain in the extended plane $\hat{\mathbb{C}}$, defined as $(f''/f')' - 1/2(f''/f')^2$. It is known that $\mathcal{S}$ is a holomorphic split submersion and descends down to a one-to-one map $\mathcal{B} : T \to B(\mathbb{L})$, which is known as the Bers embedding. Via the Bers embedding, T carries a natural complex structure so that the natural projection $\Phi : M \to T$ is a holomorphic split submersion and $\mathcal{B}$ is a biholomorphism from T onto its image.

2.2. **Statement of main result.** Let $B_*(\mathbb{L})$ be the subspace of $B(\mathbb{L})$ which consists of those functions ϕ such that $y^2\phi(x+iy) \to 0$ uniformly for $x \in \mathbb{R}$ as $y \to 0-$. It is easy to see that $B_*(\mathbb{L})$ is closed in $B(\mathbb{L})$. To see this, let a sequence (ϕ_n) in $B_*(\mathbb{L})$ and $\phi \in B(\mathbb{L})$ be given such that $\|\phi_n - \phi\|_{B(\mathbb{L})} \to 0$ as $n \to \infty$. Then for each $\epsilon > 0$, we may choose some n_0 such that $\|\phi_{n_0} - \phi\|_{B(\mathbb{L})} < \epsilon$. Since $\phi_{n_0} \in B_*(\mathbb{L})$, there exists some $\delta > 0$ such that $y^2|\phi_{n_0}(x + iy)| < \epsilon$ whenever $x \in \mathbb{R}$ and $-\delta < y < 0$. Thus,

$$y^2|\phi(x + iy)| \le y^2|\phi_{n_0}(x + iy)| + y^2|\phi_{n_0}(x + iy) - \phi(x + iy)| < 2\epsilon$$

whenever $x \in \mathbb{R}$ and $-\delta < y < 0$, which implies that $\phi \in B_*(\mathbb{L})$. Consequently, $B_*(\mathbb{L})$ is closed in $B(\mathbb{L})$, which implies that $B_*(\mathbb{L})$ itself is a complex Banach space.

Now let $L_*(\mathbb{U})$ be the subspace of $L^\infty(\mathbb{U})$ which consists of those functions μ such that $\mu(x+iy) \to 0$ uniformly for $x \in \mathbb{R}$ as $y \to 0+$. By the same reasoning as above, we find out that $L_*(\mathbb{U})$ is closed in $L^\infty(\mathbb{U})$. Set $M_*(\mathbb{U}) = M(\mathbb{U}) \cap L_*(\mathbb{U})$. Then the correspondence $[\mu] \mapsto f^\mu|_{\mathbb{R}}$ establishes a one-to-one map from $M_*(\mathbb{U})/_\sim$ onto the symmetric Teichmüller space T_*, and the Beurling-Ahlfors section $s : T \to M(\mathbb{U})$ maps T_* into $M_*(\mathbb{U})$ (see [Ca], [GS], [Ma]). By the continuity of $s : T \to M(\mathbb{U})$, we conclude that T_* is contractible.

In this paper, we will endow the symmetric Teichmüller space T_* with a complex Banach manifold structure. The main result is the following.

Theorem 2.1. *$\mathcal{S}$ maps $M_*(\mathbb{U})$ into $B_*(\mathbb{L})$ and is a holomorphic split submersion from $M_*(\mathbb{U})$ onto its image. Consequently, T_* has a unique complex structure such that $\mathcal{B} : T_* \to B_*(\mathbb{L})$ is a biholomorphic map from T_* onto a domain in $B_*(\mathbb{L})$. Under this complex structure, the natural projection Φ from $M_*(\mathbb{U})$ onto T_* is a holomorphic split submersion.*

3. An integral representation of the Schwarzian derivative $\mathcal{S}(\mu)$

In this section, we will prove that $\mathcal{S}$ maps $M_*(\mathbb{U})$ into $B_*(\mathbb{L})$. We first establish an integral representation of the Schwarzian derivative $\mathcal{S}(\mu)$, which was inspired by the computation from Astala-Zinsmeister [AZ].

Proposition 3.1. *For each $\mu \in M(\mathbb{U})$, the Schwarzian derivative $\mathcal{S}(\mu)$ has the following expression:*

$$\mathcal{S}(\mu)(z) = -\frac{6}{\pi} f_\mu'(z) \iint_{\mathbb{U}} \frac{\overline{\partial} f_\mu(\zeta)}{(\zeta - z)^2 (f_\mu(\zeta) - f_\mu(z))^2} d\xi d\eta,\ z \in \mathbb{L}. \tag{3.1}$$

Proof. We borrow some discussion by Astala-Zinsmeister [AZ]. Consider a quasiconformal mapping g in the complex plane, which is conformal in Δ^* and has the expression

$$g(w) = w + \sum_{n=0}^{\infty} b_n w^{-n},\ w \in \Delta^*. \tag{3.2}$$

A direct computation yields that

$$\lim_{w\to\infty} w^4 S(g)(w) = -6b_1.$$

On the other hand, we have the following identity by the Green formula:

$$\begin{aligned}
\iint_{\Delta} \overline{\partial} g(\zeta) d\xi d\eta &= \lim_{w\to\infty} \iint_{\Delta} \frac{w^2}{(\zeta - w)^2} \overline{\partial} g(\zeta) d\xi d\eta \\
&= \lim_{w\to\infty} \frac{1}{2i} \int_{\partial\Delta} \frac{w^2}{(\zeta - w)^2} g(\zeta) d\zeta \\
&= \lim_{w\to\infty} \frac{1}{2i} \int_{\partial\Delta^*} \frac{w^2}{(1 - \bar{\zeta} w)^2} g(\zeta) d\bar{\zeta} \\
&= -\lim_{w\to\infty} \iint_{\Delta^*} \frac{w^2}{(1 - \bar{\zeta} w)^2} g'(\zeta) d\xi d\eta \\
&= -\iint_{\Delta^*} \frac{g'(\zeta)}{\bar{\zeta}^2} d\xi d\eta \\
&= \pi b_1.
\end{aligned}$$

Consequently, we have

$$\frac{6}{\pi} \iint_{\Delta} \overline{\partial} g(\zeta) d\xi d\eta = -\lim_{w\to\infty} w^4 S(g)(w). \tag{3.3}$$

Now let $\mu \in M(\mathbb{U})$ be given. For simplicity, we set $f = f_\mu$. For any $z \in \mathbb{L}$, consider $\gamma(w) = \frac{zw - \bar{z}}{w-1}$, $\Gamma(w) = \frac{(z-\bar{z}) f'(z)}{w - f(z)}$. Then $g \doteq \Gamma \circ f \circ \gamma$ is a quasiconformal mapping in the complex plane, which is conformal in Δ^* and satisfies the normalized condition (3.2). So (3.3) holds. Noting that $S(g) = (S(f) \circ \gamma)(\gamma')^2$, we obtain

$$\lim_{w\to\infty} w^4 S(g)(w) = \lim_{w\to\infty} w^4 S(f)(\gamma(w))(\gamma'(w))^2 = S(f)(z)(z - \bar{z})^2. \tag{3.4}$$

On the other hand, since $\overline{\partial} g = (\Gamma' \circ f \circ \gamma)(\overline{\partial} f \circ \gamma)\overline{\gamma'}$, we have

$$\iint_{\Delta} \overline{\partial} g(\zeta) d\xi d\eta = \iint_{\Delta} (\Gamma' \circ f \circ \gamma)(\overline{\partial} f \circ \gamma)\overline{\gamma'} d\xi d\eta$$

$$= \iint_{\mathbb{U}} (\Gamma' \circ f)\overline{\partial} f (\gamma^{-1})' d\xi d\eta$$

$$= f'(z)(z-\bar{z})^2 \iint_{\mathbb{U}} \frac{\overline{\partial} f(\zeta)}{(\zeta - z)^2 (f(\zeta) - f(z))^2} d\xi d\eta. \tag{3.5}$$

Now (3.1) follows from (3.3)-(3.5) directly. □

Corollary 3.2. *For each $\mu \in M(\mathbb{U})$, it holds that*

$$|\mathcal{S}(\mu)(z)|^2 \leq \frac{144}{(1 - \|\mu\|_\infty^2)\pi y^2} \iint_{\mathbb{U}} \frac{|\mu(\zeta)|^2}{|\zeta - z|^4} d\xi d\eta,\ z = x + iy \in \mathbb{L}. \tag{3.6}$$

Proof. We recall a classical result of Bergman-Schiffer [BS] which says that for any regular domain Ω bounded by analytic curves,

$$\frac{1}{\pi} \iint_{\hat{\mathbb{C}} - \Omega} \frac{d\xi d\eta}{|\zeta - w|^4} \leq \lambda_\Omega^2(w),\ w \in \Omega. \tag{3.7}$$

In fact, Harmelin [Ha] pointed out that (3.7) holds for any simply connected hyperbolic domain Ω in the Riemann sphere. Since we are not able to find a complete proof of (3.7) in the literature, here we give a direct estimate of the integral in (3.7) for convenience. Though our estimate is not as precise as (3.7), it is sufficient for our purpose. For each $w \in \Omega$, we denote by $d(w, \partial\Omega)$ the distance between w and $\partial\Omega$ so that $|\zeta - w| > d(w, \partial\Omega)$ when $\zeta \in \hat{\mathbb{C}} \setminus \Omega$. Thus,

$$\frac{1}{\pi} \iint_{\hat{\mathbb{C}} - \Omega} \frac{d\xi d\eta}{|\zeta - w|^4} \leq \frac{1}{\pi} \iint_{|\zeta - w| > d(w, \partial\Omega)} \frac{d\xi d\eta}{|\zeta - w|^4} = \frac{1}{d^2(w, \partial\Omega)}. \tag{3.8}$$

On the other hand, when Ω does not contain ∞, it is well known that (see [Le])

$$\lambda_\Omega(w) \geq \frac{1}{4d(w, \partial\Omega)},\ w \in \Omega. \tag{3.9}$$

By (3.8) and (3.9) we have the desired estimate

$$\frac{1}{\pi} \iint_{\hat{\mathbb{C}} - \Omega} \frac{d\xi d\eta}{|\zeta - w|^4} \leq 16\lambda_\Omega^2(w),\ w \in \Omega, \tag{3.10}$$

when Ω does not contain ∞.

Now set $f = f_\mu$ as above, and $D = f(\mathbb{U})$, $D^* = f(\mathbb{L})$, $k = \|\mu\|_\infty$. By Hölder's inequality we conclude from (3.1) that

$$|\mathcal{S}(\mu)(z)|^2 \leq \frac{36}{\pi^2} |f'(z)|^2 \iint_{\mathbb{U}} \frac{|\mu(\zeta)|^2}{|\zeta - z|^4} d\xi d\eta \iint_{\mathbb{U}} \frac{|\partial f(\zeta)|^2}{|f(\zeta) - f(z)|^4} d\xi d\eta. \tag{3.11}$$

Since D^* does not contain ∞, we obtain from (3.10) that

$$\iint_{\mathbb{U}} \frac{|\partial f(\zeta)|^2}{|f(\zeta) - f(z)|^4} d\xi d\eta \leq \frac{1}{1 - k^2} \iint_{\mathbb{U}} \frac{|\partial f(\zeta)|^2 - |\overline{\partial} f(\zeta)|^2}{|f(\zeta) - f(z)|^4} d\xi d\eta$$

$$= \frac{1}{1 - k^2} \iint_{D} \frac{1}{|\zeta - f(z)|^4} d\xi d\eta$$

$$\leq \frac{16\pi}{1 - k^2} \lambda_{D^*}^2(f(z)).$$

Combining this with (3.11) yields that

$$|\mathcal{S}(\mu)(z)|^2 \leq \frac{576}{(1-k^2)\pi} \lambda^2_{D^*}(f(z))|f'(z)|^2 \iint_{\mathbb{U}} \frac{|\mu(\zeta)|^2}{|\zeta - z|^4} d\xi d\eta$$
$$= \frac{144}{(1-k^2)\pi y^2} \iint_{\mathbb{U}} \frac{|\mu(\zeta)|^2}{|\zeta - z|^4} d\xi d\eta,$$

by (2.2). □

Lemma 3.3. *For each $\mu \in M_*(\mathbb{U})$, it holds that*

$$\lim_{y\to 0-} \iint_{\mathbb{U}} \frac{y^2}{|\zeta - (x+iy)|^4} |\mu(\zeta)|^2 d\xi d\eta = 0$$

uniformly for $x \in \mathbb{R}$.

Proof. For each $t > 0$, set $\mathbb{U}^1_t = \{\zeta = \xi + i\eta : 0 < \eta < t\}$, $\mathbb{U}^2_t = \{\zeta = \xi + i\eta : \eta > t\}$. Let $\epsilon > 0$ be arbitrarily given. Since $\mu \in M_*(\mathbb{U})$, there exists $t > 0$ such that $|\mu(\zeta)| < \epsilon$ for $\zeta \in \mathbb{U}^1_t$. Set $k = \|\mu\|_\infty$ as above. Then for $z = x + iy \in \mathbb{L}$, we have

$$\iint_{\mathbb{U}} \frac{|z-\bar{z}|^2}{|\zeta - z|^4} |\mu(\zeta)|^2 d\xi d\eta \leq \epsilon^2 \iint_{\mathbb{U}^1_t} \frac{|z-\bar{z}|^2}{|\zeta - z|^4} d\xi d\eta + k^2 \iint_{\mathbb{U}^2_t} \frac{|z-\bar{z}|^2}{|\zeta - z|^4} d\xi d\eta.$$

Set $\gamma(w) = \frac{zw-\bar{z}}{w-1}$ as above, and $\omega(\zeta) = \gamma^{-1}(\zeta) = \frac{\zeta-\bar{z}}{\zeta-z}$. Then ω maps the upper half plane $\mathbb{U}$ onto the unit disk Δ, and $\omega'(\zeta) = \frac{\bar{z}-z}{(\zeta-z)^2}$. Thus,

$$\iint_{\mathbb{U}} \frac{|z-\bar{z}|^2}{|\zeta - z|^4} |\mu(\zeta)|^2 d\xi d\eta \leq \epsilon^2 \iint_{\omega(\mathbb{U}^1_t)} dudv + k^2 \iint_{\omega(\mathbb{U}^2_t)} dudv \leq \pi\epsilon^2 + k^2 \iint_{\omega(\mathbb{U}^2_t)} dudv.$$

Noting that

$$\frac{\gamma(w) - \overline{\gamma(w)}}{2i} = \frac{1}{2i}\left(\frac{zw-\bar{z}}{w-1} - \frac{\bar{z}\bar{w}-z}{\bar{w}-1}\right) = \frac{z-\bar{z}}{2i}\frac{|w|^2-1}{|w-1|^2},$$

we find out that, under the mapping ω, the line $\{\zeta = \xi + i\eta : \eta = t\}$ is mapped onto the circle $\{w : \frac{|w-1|^2}{1-|w|^2} = -\frac{y}{t}\}$, or equivalently, $\{w : |w - \frac{t}{t-y}| = -\frac{y}{t-y}\}$. Consequently, $\omega(\mathbb{U}^2_t)$ is the disk $\{w : |w - \frac{t}{t-y}| < -\frac{y}{t-y}\}$, which implies that

$$\iint_{\mathbb{U}} \frac{|z-\bar{z}|^2}{|\zeta - z|^4} |\mu(\zeta)|^2 d\xi d\eta \leq \pi\epsilon^2 + \pi k^2 \frac{y^2}{(t-y)^2},$$

and so

$$\lim_{y\to 0-} \iint_{\mathbb{U}} \frac{|z-\bar{z}|^2}{|\zeta - z|^4} |\mu(\zeta)|^2 d\xi d\eta \leq \pi\epsilon^2$$

uniformly for $x \in \mathbb{R}$. By the arbitrariness of ϵ, we obtain (3.8) as desired. □

Corollary 3.4. *For each $\mu \in M_*(\mathbb{U})$, it holds that $\mathcal{S}(\mu) \in B_*(\mathbb{L})$.*

Proof. This follows from Corollary 3.2 and Lemma 3.3. □

Remark. Corollary 3.4 implies that $\mathcal{B}(T_*) \subset \mathcal{B}(T) \cap B_*(\mathbb{L})$. It is not clear whether the converse is true.

4. Proof of Theorem 2.1

Corollary 3.4 says that $\mathcal{S}$ maps $M_*(\mathbb{U})$ into $B_*(\mathbb{L})$. This also implies that $\mathcal{S} : M_*(\mathbb{U}) \to B_*(\mathbb{L})$ is holomorphic since $\mathcal{S} : M(\mathbb{U}) \to B(\mathbb{L})$ is holomorphic. It remains to show that $\mathcal{S} : M_*(\mathbb{U}) \to B_*(\mathbb{L})$ is a split submersion onto its image, or equivalently, $\mathcal{S} : M_*(\mathbb{U}) \to B_*(\mathbb{L})$ has local holomorphic sections. We write the standard proof here (see [Ah], [EN], [SW]).

Let $\phi = S(\mu)$, $\mu \in M_*(\mathbb{U})$ be given. Without loss of generality, we may assume that $\mu = s([\mu])$, that is, f^{μ} is the Beurling-Ahlfors extension of $f^{\mu}|_{\mathbb{R}}$. Set $f = f_{\mu}$, $D = f(\mathbb{U})$, $D^* = f(\mathbb{L})$ as before, and $r = f \circ \overline{f^{-1}}$. Ahlfors [Ah] showed that $r : D \to D^*$ is a quasiconformal reflection and there exists a constant C_1 such that

$$\frac{1}{C_1} \leq |r(z) - z|^2 \lambda_{D^*}^2(r(z)) |\overline{\partial} r(z)| \leq C_1,\ z \in D. \tag{4.1}$$

Consider $B_{\epsilon}(\phi) = \{\psi \in B_*(\mathbb{L}) : \|\psi - \phi\|_{B(\mathbb{L})} < \epsilon\}$ for $\epsilon > 0$. Then for each $\psi \in B_{\epsilon}(\phi)$ there exists a unique locally univalent function f_{ψ} in $\mathbb{L}$ which fixes the points $0, 1, \infty$ such that $S(f_{\psi}) = \psi$. Set $g_{\psi} = f_{\psi} \circ f^{-1}$. Then $S(g_{\psi}) \circ f (f')^2 = \psi - \phi$, and $S(g_{\psi}) \in B(D^*)$ with $\|S(g_{\psi})\|_{B(D^*)} = \|\psi - \phi\|_{B(\mathbb{L})}$. More specifically,

$$\lambda_{D^*}^{-2}(f(z)) |S(g_{\psi})(f(z))| = |z - \bar{z}|^2 |\psi(z) - \phi(z)|,\ z \in \mathbb{L}.$$

When ϵ is small, Ahlfors [Ah] proved that g_{ψ} is univalent and can be extended to a quasiconformal mapping in the whole plane whose complex dilatation μ_{ψ} has the form

$$\mu_{\psi}(z) = \frac{S(g_{\psi})(r(z))(r(z) - z)^2 \overline{\partial} r(z)}{2 + S(g_{\psi})(r(z))(r(z) - z)^2 \partial r(z)},\ z \in D. \tag{4.2}$$

By (4.1) we have for some constant C_2 that

$$|\mu_{\psi}(z)| \leq C_2 |S(g_{\psi})(r(z))| \lambda_D^{-2}(r(z)),\ z \in D. \tag{4.3}$$

Consequently, $f_{\psi} = g_{\psi} \circ f$ is univalent in $\mathbb{L}$ and has a quasiconformal extension to the whole plane whose complex dilatation ν_{ψ} is

$$\nu_{\psi} = \frac{\mu + (\mu_{\psi} \circ f)\tau}{1 + \overline{\mu}(\mu_{\psi} \circ f)\tau},\ \tau = \frac{\overline{\partial f}}{\partial f}. \tag{4.4}$$

It is well known that ν_{ψ} depends holomorphically on ψ (see [Ah], [EN]). Now, it follows from (4.3) that

$$\begin{aligned} |\mu_{\psi}(f(z))| &\leq C_2 |S(g_{\psi})(r(f(z)))| \lambda_{D^*}^{-2}(r(f(z))) \\ &= C_2 |S(g_{\psi})(f(\bar{z}))| \lambda_{D^*}^{-2}(f(\bar{z})) \\ &= C_2 |\psi(\bar{z}) - \phi(\bar{z})| |z - \bar{z}|^2, \end{aligned}$$

which implies that $\mu_{\psi} \circ f \in M_*(\mathbb{U})$, and we conclude by (4.4) that $\nu_{\psi} \in M_*(\mathbb{U})$. Since $\mathcal{S}(\nu_{\psi}) = \psi$, we conclude that $\nu : B_{\epsilon}(\phi) \to M_*(\mathbb{U})$ is a local holomorphic section to $\mathcal{S} : M_*(\mathbb{U}) \to B_*(\mathbb{L})$. This completes the proof of Theorem 2.1. □

5. Concluding remarks

5.1. We recall the little Teichmüller space, which is closely related to the symmetric Teichmüller space T_*. Let $L_0(\mathbb{U})$ be the closed subspace of $L^{\infty}(\mathbb{U})$ which consists of those functions μ such that

$$\inf\{\|\mu|_{\mathbb{U} \setminus K}\|_{\infty} : K \subset \mathbb{U} \text{ compact}\} = 0.$$

Set $M_0(\mathbb{U}) = M(\mathbb{U}) \cap L_0(\mathbb{U})$. $T_0 = M_0(\mathbb{U})/_\sim$ is one of the models of the little Teichmüller space. It is obvious that $L_0(\mathbb{U}) \subset L_*(\mathbb{U})$, and so $T_0 \subset T_*$. Now we point out that T_0 is a nontrivial subset of T_*. To make this precise, we consider the symmetric homeomorphisms on the unit circle S^1.

Let h be an orientation-preserving self-homeomorphism of the unit circle S^1 with $h(1) = 1$. Then h determines two increasing self-homeomorphisms of the real line $\mathbb{R}$. One is $\hat{h}$ determined by

$$e^{2\pi i\hat{h}(x)} = h(e^{2\pi ix}),\ \hat{h}(0) = 0.$$

The other is $\tilde{h}$ determined by

$$\frac{\tilde{h}(x) - i}{\tilde{h}(x) + i} = h(\frac{x - i}{x + i}).$$

By definition, h is said to be quasisymmetric if $\hat{h}$ is quasisymmetric, while h is said to be symmetric if $\hat{h}$ is symmetric. It is well known that h is quasisymmetric if and only if h can be extended a quasiconformal homeomorphism f to the unit disk Δ, or, equivalently, $\tilde{h}$ is quasisymmetric. It is also known that h is symmetric if and only if h can be extended as an asymptotically conformal mapping f to Δ in the sense that its complex dilatation $\mu = \overline{\partial} f/\partial f$ satisfies the condition $\mu(z) \to 0$ when $|z| \to 1-$, or, equivalently, $\tilde{h}$ represents a point in the little Teichmüller space T_0 (see [GL], [GS], [Ma]). Consequently, $\hat{h}$ is a symmetric homeomorphism and thus represents a point in the symmetric Teichmüller space T_* whenever h is a symmetric homeomorphism on the unit circle. However, $\hat{h}$ cannot represent a point in the little Teichmüller space T_0 except for h being the identity map. In fact, $\hat{h}$ can even not represent a so-called Strebel point in the universal Teichmüller space T from extremal quasiconformal mapping theory (see [GL]).

5.2. **A problem.** It is known that there exists the invariant Kobayashi metric on any complex Banach manifold (see [Ko]). It is also known that the Kobayashi metric coincides with the Teichmüller metric on the universal Teichmüller space T and also on the little Teichmüller space T_0 (see [Ga], [GL], [EGL1, EGL2]). We do not know whether or not the Kobayashi metric coincides with the Teichmüller metric on the symmetric Teichmüller space T_*, which is a complex Banach manifold by our Theorem 2.1.

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Department of Mathematics, Soochow University, Suzhou 215006, Peoples' Republic of China

Email address: huyun_80@163.com

Department of Mathematics, Soochow University, Suzhou 215006, Peoples' Republic of China

Email address: wuli187@126.com

Department of Mathematics, Soochow University, Suzhou 215006, Peoples' Republic of China

Email address: ylshen@suda.edu.cn