

## EXTREMAL MULTIPLIERS OF THE DRURY–ARVESON SPACE

MICHAEL T. JURY AND ROBERT T. W. MARTIN

(Communicated by Stephan R. Garcia)

**ABSTRACT.** We give a new characterization of the so-called *quasi-extreme* multipliers of the Drury–Arveson space  $H_d^2$  and show that every quasi-extreme multiplier is an extreme point of the unit ball of the multiplier algebra of  $H_d^2$ .

### 1. INTRODUCTION

In [7] and [8] we introduced the notion of a *quasi-extreme* multiplier of the Drury–Arveson space  $H_d^2$  and gave a number of equivalent formulations of this property. (The relevant definitions are recalled in Section 2.) The purpose of this paper is to give one further characterization of quasi-extremity in the general case, from which it will follow that every quasi-extreme multiplier of  $H_d^2$  is in fact an extreme point of the unit ball of the multiplier algebra  $\mathcal{M}(H_d^2)$ . Our main result is the following theorem.

**Theorem 1.1.** *A contractive multiplier  $b$  of  $H_d^2$  is quasi-extreme if and only if the only multiplier  $a$  satisfying*

$$(1.1) \quad M_a^* M_a + M_b^* M_b \leq I$$

*is  $a \equiv 0$ .*

**Corollary 1.2.** *If  $b \in \text{ball}(\mathcal{M}(H_d^2))$  is quasi-extreme, then  $b$  is an extreme point of  $\text{ball}(\mathcal{M}(H_d^2))$ .*

*Proof.* Since the corollary concerning extreme points follows immediately, we prove it here. If  $b$  is not extreme, then there exists a nonzero  $a \in \text{ball}(\mathcal{M}(H_d^2))$  such that both  $b \pm a$  lie in  $\text{ball}(\mathcal{M}(H_d^2))$ , that is, are contractive multipliers of  $H_d^2$ . We then have the operator inequalities

$$M_{b+a}^* M_{b+a} \leq I, \quad M_{b-a}^* M_{b-a} \leq I.$$

Averaging these inequalities gives

$$M_a^* M_a + M_b^* M_b \leq I,$$

so by Theorem 1.1,  $b$  is not quasi-extreme. □

The converse statement, namely whether or not every extreme point is quasi-extreme in our sense, remains open.

The remainder of the paper is devoted to proving Theorem 1.1. The outline of the paper is as follows: in Section 2 we recall the Drury–Arveson space and its

---

Received by the editors March 2, 2017, and, in revised form, March 13, 2017, and December 21, 2017.

2010 *Mathematics Subject Classification.* Primary 47B32.

The second author acknowledges support of NRF CPRR Grant 90551.

multipliers, and review the necessary results concerning the de Branges–Rovnyak type spaces  $\mathcal{H}(b)$  contractively contained in  $H_d^2$ , and in particular the solutions to the Gleason problem in these spaces. We define the *quasi-extreme* multipliers and review some equivalent formulations of this property that will be used later. In Section 3 we introduce what we call the *minimal functional model* of a multiplier, and in Section 4 we use it to prove one direction of Theorem 1.1. Finally in Section 5 we recall the notion of a free lifting of a contractive multiplier and complete the proof of Theorem 1.1.

## 2. THE DRURY–ARVESON SPACE, MULTIPLIERS, AND QUASI-EXTREMITY

The *Drury–Arveson space* is the Hilbert space of holomorphic functions defined on the unit ball  $\mathbb{B}^d \subset \mathbb{C}^d$  with reproducing kernel

$$k_w(z) = k(z, w) = \frac{1}{1 - zw^*}, \quad z, w \in \mathbb{B}^d.$$

(Here we use the notation  $z = (z_1, z_2, \dots, z_d)$  so that  $zw^* = \sum_{j=1}^d z_j \overline{w_j}$ .) General facts about the  $H_d^2$  spaces may be found in the recent survey [12].

A holomorphic function  $b$  on  $\mathbb{B}^d$  will be called a *multiplier* if  $bf \in H_d^2$  whenever  $f \in H_d^2$ . In this case the operator  $M_b : f \rightarrow bf$  is bounded, and we let  $\mathcal{M}(H_d^2)$  denote the Banach algebra of multipliers, equipped with the operator norm. For the reproducing kernel  $k_w$  we have  $M_b^* k_w = b(w)^* k_w$ . It follows that  $\|M_b\| \leq 1$  if and only if the expression

$$k_w^b(z) = k^b(z, w) := \frac{1 - b(z)b(w)^*}{1 - zw^*}$$

defines a positive kernel on  $\mathbb{B}^d$ . When this is the case we let  $\mathcal{H}(b)$  denote the associated reproducing kernel Hilbert space, called the de Branges–Rovnyak space of  $b$ . The space  $\mathcal{H}(b)$  is a space of holomorphic functions on  $\mathbb{B}^d$ , contained in  $H_d^2$ , and the inclusion map  $\mathcal{H}(b) \subset H_d^2$  is contractive for the respective Hilbert space norms. We write  $\|\cdot\|_b$  and  $\langle \cdot, \cdot \rangle_b$  for the norm and inner product in  $\mathcal{H}(b)$ , respectively.

Properties of the spaces  $\mathcal{H}(b)$  when  $d > 1$  were studied in [7], inspired among other things by the results of Sarason in the one-variable case [10], [11]. In one variable, the  $\mathcal{H}(b)$  spaces are invariant under the backward shift; in several variables we instead (following Ball, Bolotnikov, and Fang [2]) consider solutions to the *Gleason problem*: given a function  $f \in \mathcal{H}(b)$ , we seek functions  $f_1, \dots, f_d \in \mathcal{H}(b)$  such that

$$(2.1) \quad f(z) - f(0) = \sum_{j=1}^d z_j f_j(z).$$

From [2] we know that this problem always has a solution; in fact there exist (not necessarily unique) bounded operators  $X_1, \dots, X_d$  acting on  $\mathcal{H}(b)$  such that the functions  $f_j := X_j f$  solve (2.1) for any  $f \in \mathcal{H}(b)$ . Moreover, these  $X_j$  can be chosen to be *contractive* in the following sense: for every  $f \in \mathcal{H}(b)$ ,

$$(2.2) \quad \sum_{j=1}^d \|X_j f\|_b^2 \leq \|f\|_b^2 - |f(0)|^2.$$

These contractive solutions were studied further in [7], where we proved the following (see also [8] for the vector-valued case).

**Proposition 2.1.** *A set of bounded operators  $(X_1, \dots, X_d)$  is a contractive solution to the Gleason problem in  $\mathcal{H}(b)$  if and only if the  $X_j$  act on reproducing kernels by the formula*

$$(2.3) \quad X_j k_w^b = w_j^* k_w^b - b(w)^* b_j$$

for some choice of functions  $b_1, \dots, b_d \in \mathcal{H}(b)$  which satisfy

- (i)  $\sum_{j=1}^d z_j b_j(z) = b(z) - b(0),$
- (ii)  $\sum_{j=1}^d \|b_j\|_b^2 \leq 1 - |b(0)|^2.$

The set of all contractive solutions  $X$  is in one-to-one correspondence with the set of all tuples  $b_1, \dots, b_d$  satisfying these conditions [8, Theorem 4.10]. We will call such sets of  $b_j$  *admissible*, or say that such a set is a *contractive Gleason solution* for  $b$ .

It turns out that for some contractive multipliers  $b$ , the operators  $X_j$  of the proposition are unique, that is, there is only one admissible tuple. When this happens we will call the multiplier  $b$  *quasi-extreme*. (The original definition of quasi-extreme in [7] is different, involving the so-called noncommutative Aleksandrov–Clark state for  $b$ , but this definition will be easier to work with for the present purposes.) In [7] and [8] we gave a number of equivalent formulations of quasi-extremity; we recall only a few of them here.

**Proposition 2.2.** *Let  $b$  be a contractive multiplier of  $H_d^2$  and  $d > 1$ . The following are equivalent:*

- i)  $b$  is quasi-extreme.
- ii) There is a unique contractive solution  $(X_1, \dots, X_d)$  to the Gleason problem in  $\mathcal{H}(b)$ .
- iii) Every contractive Gleason solution for  $H(b)$  is extremal, i.e., given any contractive solution,  $(X_1, \dots, X_d)$ , the equality  $\sum_{j=1}^d \|X_j f\|_b^2 = \|f\|_b^2 - |f(0)|^2$  holds for every  $f \in \mathcal{H}(b)$ .
- iv) There is a unique admissible tuple  $(b_1, \dots, b_d)$  satisfying the conditions of Proposition 2.1.
- v) All admissible tuples  $(b_1, \dots, b_d)$  are extremal, i.e.,

$$\sum_{j=1}^d \|b_j\|_b^2 = 1 - |b(0)|^2,$$

for any admissible tuple.

- vi)  $\mathcal{H}(b)$  does not contain the function  $b$ .

In [8] these equivalences were extended to the case of operator-valued  $b$ .

What will be most useful in what follows is item (v); in particular,  $b$  is *not* quasi-extreme if and only if there exists an admissible tuple  $(b_1, \dots, b_d)$  which obeys the strict inequality

$$\sum_{j=1}^d \|b_j\|_b^2 < 1 - |b(0)|^2.$$

### 3. THE MINIMAL FUNCTIONAL MODEL

We begin by recalling the relevant facts about transfer function realizations [3] and the *generalized functional models* of [1, 2].

Let  $\mathcal{X}, \mathcal{U}, \mathcal{Y}$  be Hilbert spaces and let  $\mathcal{X}^d$  denote the direct sum of  $d$  copies of  $\mathcal{X}$ . By a  $d$ -colligation we mean an operator  $\mathbf{U} : \mathcal{X} \oplus \mathcal{U} \rightarrow \mathcal{X}^d \oplus \mathcal{Y}$  expressed in the block matrix form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}.$$

The colligation is called *contractive*, *isometric*, *unitary*, etc. if  $\mathbf{U}$  is an operator of that type. For points  $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ , it will be convenient to identify  $z$  with the row contraction:

$$z : \mathcal{X}^d \rightarrow \mathcal{X}; \quad z \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} := z_1 x_1 + \dots + z_d x_d.$$

Observe that  $\|z\|^2 = \|zz^*\|_{\mathcal{L}(\mathcal{X})} = \sum_{j=1}^d |z_j|^2$ , so  $\|z\| = |z|_{\mathbb{C}^d} < 1$  if and only if  $z \in \mathbb{B}^d$ . If  $\mathbf{U}$  is a contractive colligation, the *transfer function* for  $\mathbf{U}$  is

$$S(z) = D + C(I - zA)^{-1}zB.$$

The transfer function  $S(z)$  is a holomorphic function in  $\mathbb{B}^d$  taking values in the space of bounded operators from  $\mathcal{U}$  to  $\mathcal{Y}$ . Moreover,  $S(z)$  acts as a contractive multiplier from  $H_d^2 \otimes \mathcal{U}$  to  $H_d^2 \otimes \mathcal{Y}$ . (For our purposes we will only need to consider finite-dimensional  $\mathcal{U}$  and  $\mathcal{Y}$ .) Conversely, it is a theorem of Ball, Trent, and Vinnikov [3] that  $S$  is a contractive multiplier of  $H_d^2 \otimes \mathcal{U}$  into  $H_d^2 \otimes \mathcal{Y}$  if and only if it possesses a transfer function realization. In [2], it was shown that such a transfer function could always be chosen to be of a special form, called a *generalized functional model* realization. In particular (in the case  $\mathcal{U} = \mathcal{Y} = \mathbb{C}$ ) this is a realization such that the state space  $\mathcal{X}$  is equal to  $\mathcal{H}(b)$ , and:

- $(A_1, \dots, A_d)$  is a contractive solution to the Gleason problem in  $\mathcal{H}(b)$ ,
- $(B_1, \dots, B_d)$  is a contractive Gleason solution for  $b$ ,
- $Cf = f(0)$ , and
- $D\lambda = b(0)\lambda$ .

The corresponding colligation is contractive and its transfer function is  $b(z)$ .

Since  $Cf = f(0) = \langle f, k_0^b \rangle$ , we will write  $k_0^{b*}$  for  $C$ , and using the notation of the previous section we will write generalized functional model colligations in the form

$$\mathbf{U} = \begin{bmatrix} X_1 & b_1 \\ \vdots & \vdots \\ X_d & b_d \\ k_0^{b*} & b(0) \end{bmatrix}.$$

**3.1. The minimal functional model.** We now construct what we will call the *minimal functional model* of  $b$ . In the next section we will use it to prove the first half of Theorem 1.1 and deduce some of its further properties.

Fix a (nonconstant) contractive multiplier  $b$  and consider the de Branges–Rovnyak kernel

$$(3.1) \quad \frac{1 - b(z)b(w)^*}{1 - zw^*} = \langle k_w^b, k_z^b \rangle.$$

We define two Hilbert spaces  $\mathcal{D}$  and  $\mathcal{R}$  as follows:

$$(3.2) \quad \mathcal{D} = \overline{\text{span}} \left\{ \begin{bmatrix} k_w^b \\ b(w)^* \end{bmatrix} : w \in \mathbb{B}^d \right\} \subset \mathcal{H}(b) \oplus \mathbb{C},$$

$$(3.3) \quad \mathcal{R} = \overline{\text{span}} \left\{ \begin{bmatrix} w_1^* k_w^b \\ \vdots \\ w_d^* k_w^b \\ 1 \end{bmatrix} : w \in \mathbb{B}^d \right\} \subset \mathcal{H}(b)^{(d)} \oplus \mathbb{C}.$$

The factorization (3.1) shows that the map defined for each  $w \in \mathbb{B}^d$  by

$$(3.4) \quad \mathbf{V} : \begin{bmatrix} k_w^b \\ b(w)^* \end{bmatrix} \rightarrow \begin{bmatrix} w_1^* k_w^b \\ \vdots \\ w_d^* k_w^b \\ 1 \end{bmatrix}$$

extends to a linear isometry (that is, a unitary) from  $\mathcal{D}$  onto  $\mathcal{R}$  (this is the well-known “lurking isometry” argument). We then extend it to a map (still denoted  $\mathbf{V}$ ) from all of  $\mathcal{H}(b) \oplus \mathbb{C}$  into  $\mathcal{H}(b)^{(d)} \oplus \mathbb{C}$  by declaring  $\mathbf{V}$  to be 0 on  $\mathcal{D}^\perp \subset \mathcal{H}(b) \oplus \mathbb{C}$ . The extended  $\mathbf{V}$  is thus a partial isometry with initial space  $\mathcal{D}$  and final space  $\mathcal{R}$ .

Let us pause to make some remarks on the spaces  $\mathcal{D}$  and  $\mathcal{R}$ . First we observe that the space  $\mathcal{R}^\perp$  consists of all the vectors

$$(3.5) \quad \begin{bmatrix} g_1 \\ \vdots \\ g_d \\ \lambda \end{bmatrix} \in \mathcal{H}(b)^{(d)} \oplus \mathbb{C} \quad \text{such that} \quad w_1 g_1(w) + \cdots + w_d g_d(w) + \lambda \equiv 0.$$

When  $d = 1$ , this forces both  $g$  and  $\lambda$  to be 0, so we always have  $\mathcal{R} = \mathcal{H}(b) \oplus \mathbb{C}$  in that case. In contrast, it can be shown that  $\mathcal{R}$  is always a proper subspace when  $d > 1$ , though we do not prove this here since it will not be required.

What is the space  $\mathcal{D}^\perp$ ? If the vector  $\begin{bmatrix} f \\ \lambda \end{bmatrix} \in \mathcal{H}(b) \oplus \mathbb{C}$  is orthogonal to  $\mathcal{D}$ , then

$$(3.6) \quad \langle f, k_w^b \rangle + \lambda b(w) = f(w) + \lambda b(w) = 0$$

for all  $w \in \mathbb{B}^d$ . If  $\lambda = 0$ , this forces  $f = 0$ . If  $\lambda \neq 0$ , then  $f = -\lambda b$ , which means  $b \in \mathcal{H}(b)$ . Conversely, if  $b \in \mathcal{H}(b)$ , then  $\begin{bmatrix} -b \\ 1 \end{bmatrix}$  belongs to  $\mathcal{D}^\perp$ . There are thus two cases: if  $b \notin \mathcal{H}(b)$  (that is, if  $b$  is quasi-extreme), then  $\mathcal{D}^\perp = \{0\}$ ; otherwise, if  $b$  is not quasi-extreme, then  $\mathcal{D}^\perp$  is one dimensional, spanned by  $\begin{bmatrix} -b \\ 1 \end{bmatrix}$ .

**Proposition 3.1.** *The partial isometry  $\mathbf{V}$  just defined is a generalized functional model realization for  $b$ , with state space  $\mathcal{X} = \mathcal{H}(b)$  and input and output spaces  $\mathcal{U} = \mathcal{Y} = \mathbb{C}$ .*

*Proof.* Write  $\mathbf{V}$  as a colligation:

$$\mathbf{V} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(b) \\ \mathbb{C} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}(b)^{(d)} \\ \mathbb{C} \end{bmatrix}.$$

The bottom row  $\begin{bmatrix} C & D \end{bmatrix}$  is a linear functional on  $\mathcal{H}(b) \oplus \mathbb{C}$  uniquely determined by the fact that for each  $w \in \mathbb{B}^d$  it sends  $\begin{bmatrix} k_w^b \\ b(w)^* \end{bmatrix}$  to 1 and sends  $\mathcal{D}^\perp$  to 0. It

is straightforward to check that this forces  $C = k_0^*$  and  $D = b(0)$ . Next, since  $\mathbf{V}^*\mathbf{V} = 1$  on  $\mathcal{D}$ , we obtain for all  $w$

$$(3.7) \quad \begin{bmatrix} A_1^* & \cdots & A_d^* & C^* \\ B_1^* & \cdots & B_d^* & D^* \end{bmatrix} : \begin{bmatrix} w_1^* k_w^b \\ \vdots \\ w_d^* k_w^b \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} k_w^b \\ b(w)^* \end{bmatrix}.$$

Since we have  $C = k_0^{b*}$ , from the first row we obtain

$$(3.8) \quad \left( \sum_{j=1}^d w_j^* A_j^* \right) k_w^b + k_0^b = k_w^b.$$

Taking the inner product of an arbitrary  $f \in \mathcal{H}(b)$  against this equality, and rearranging, we obtain

$$(3.9) \quad f(w) - f(0) = \sum_{j=1}^d w_j (A_j f)(w),$$

which shows that the  $A_j$  solve the Gleason problem in  $\mathcal{H}(b)$ . The fact that  $\mathbf{V}$  is a partial isometry shows that this is a contractive solution as defined above. Similarly, from the second row, using the fact that  $D = b(0)$ , we have

$$(3.10) \quad \sum_{j=1}^d w_j B_j(w) = b(w) - b(0),$$

so that the  $B_j$  form a solution to the Gleason problem for  $b$ . And again since  $\mathbf{V}$  is a partial isometry, we obtain  $\sum_{j=1}^d \|B_j\|_{\mathcal{H}(b)}^2 + |b(0)|^2 \leq 1$ . Thus  $\mathbf{V}$  is a functional model realization of  $b$ .  $\square$

More or less by definition, any generalized functional model must agree with  $\mathbf{V}$  on  $\mathcal{D}$  and map  $\mathcal{D}^\perp$  contractively into  $\mathcal{R}^\perp$ . Since  $\mathbf{V}$  is 0 on  $\mathcal{D}^\perp$  we will call  $\mathbf{V}$  the *minimal functional model* of  $b$ .

#### 4. THE $a$ -FUNCTION

In this section we prove the first half of Theorem 1.1.

**Proposition 4.1.** *If  $b$  is not quasi-extreme, then there exists a nonzero multiplier  $a$  such that*

$$M_a^* M_a + M_b^* M_b \leq I.$$

In the one-variable case if  $b$  is not extreme, then there is an outer function  $a$  defined by the property that

$$(4.1) \quad |a(\zeta)|^2 + |b(\zeta)|^2 = 1 \quad \text{a.e. on } \mathbb{T};$$

we can assume that  $a(0) > 0$ . In the above  $\mathbb{T}$  denotes the unit circle. For this  $a$  we immediately have  $M_a^* M_a + M_b^* M_b = I$ . It is known in general that an equality of this sort cannot hold when  $d > 1$  except in trivial cases (where the functions are constant); see [5]. In any case, when  $d > 1$  we do not have any direct recourse to the theory of outer functions, so different methods are required.

Nonetheless, the proof of (4.1) is in a sense constructive:  $a$  will be given in terms of a transfer function realization.

It is remarkable that the algebraic construction given here, if carried out in one variable, produces exactly the outer function in (4.1). This follows from our transfer function realization and Sarason's computation of the Taylor coefficients of  $a$  [10]; we prove this at the end of the section.

*Proof of Proposition 4.1.* Consider the minimal functional model  $\mathbf{V}$  constructed in the previous section; we maintain the notation used there. Consider the vector

$$(4.2) \quad \mathbf{u} := \begin{bmatrix} -b \\ 1 \end{bmatrix}$$

which spans the space  $\mathcal{D}^\perp$ . Let  $a_0 := \|\mathbf{u}\|^{-1} = (1 + \|b\|_{\mathcal{H}(b)}^2)^{-1/2}$ . Since  $\mathbf{V}$  is a partial isometry with initial space  $\mathcal{D}$ , we have  $\mathbf{V}^*\mathbf{V} + |a_0|^2 \mathbf{u}\mathbf{u}^* = I$ , so we can extend  $\mathbf{V}$  vertically by  $a_0\mathbf{u}^*$  to obtain an isometric colligation acting between  $\mathcal{H}(b) \oplus \mathbb{C}$  and  $\mathcal{H}(b)^{(d)} \oplus \mathbb{C}^2$ ; explicitly,

$$\tilde{\mathbf{V}} := \begin{bmatrix} \mathbf{V} \\ a_0\mathbf{u}^* \end{bmatrix} = \begin{bmatrix} X_1 & b_1 \\ \vdots & \vdots \\ X_d & b_d \\ k_0^{b*} & b(0) \\ -a_0b^* & a_0 \end{bmatrix}.$$

By the realization theory for contractive multipliers, the transfer function of this  $\tilde{\mathbf{V}}$  is a  $\mathbb{C}^2$ -valued contractive multiplier given by

$$(4.3) \quad \begin{bmatrix} s_1(z) \\ s_2(z) \end{bmatrix} = \begin{bmatrix} b(0) \\ a_0 \end{bmatrix} + \begin{bmatrix} k_0^{b*} \\ -a_0b^* \end{bmatrix} \left( I - \sum_{j=1}^d z_j X_j \right)^{-1} \begin{bmatrix} \sum_{j=1}^d z_j b_j \\ \end{bmatrix}.$$

Now  $s_1(z)$  is equal to  $b(z)$ , since it is just the transfer function associated to the minimal functional model  $\mathbf{V}$ , which was constructed as a realization of  $b$  to begin with. Thus, taking  $a(z) = s_2(z)$  proves the theorem (note that  $a$  is nonzero since  $a(0) = a_0 \neq 0$ ).  $\square$

**4.1. Further properties of the minimal functional model.** Let us collect some further properties of the minimal functional model and its associated Gleason solution  $X$ ; these results should be compared with the results of Sarason ([10]) for the backward shift in  $\mathcal{H}(b)$  in the one-variable case.

**Proposition 4.2.** *Suppose  $b$  is not quasi-extreme. There exists a unique contractive solution  $X = (X_1, \dots, X_d)$  to the Gleason problem in  $\mathcal{H}(b)$  with the property that*

$$(4.4) \quad X_j b = b_j,$$

where the  $b_j$  are those associated to  $X_j$  in Proposition 2.1.

*Proof.* The  $X_j$  that we seek are precisely those given by the minimal functional model  $\mathbf{V}$ . By its definition,  $\mathbf{V}$  annihilates the vector  $\mathbf{u} = \begin{bmatrix} -b \\ 1 \end{bmatrix}$ , which means for each  $j$ ,

$$(4.5) \quad -X_j b + b_j = 0,$$

as desired. Since any other functional model must be nonzero on  $\mathcal{D}^\perp$ , it would not annihilate  $\mathbf{u}$ , and the uniqueness follows.  $\square$

**Proposition 4.3.** *Let  $b$  be a nonquasi-extreme multiplier. If  $X_j$  is the minimal solution to the Gleason problem in  $\mathcal{H}(b)$  (coming from the minimal functional model  $\mathbf{V}$ ), then*

$$(4.6) \quad I - \sum_{j=1}^d X_j^* X_j = k_0^b k_0^{b*} + |a_0|^2 b b^*.$$

*Proof.* This is immediate from the fact that the extended  $\tilde{\mathbf{V}}$  is an isometry (one need only write out the identity  $\tilde{\mathbf{V}}^* \tilde{\mathbf{V}} = I$  as a block matrix).  $\square$

**4.2. The one-variable case.** We analyze the foregoing construction in the one-variable case. Here the Drury–Arveson space becomes the classical Hardy space  $H^2(\mathbb{D})$  and its multiplier algebra is the space of bounded analytic functions  $H^\infty(\mathbb{D})$ , equipped with the supremum norm. In this case it is known [11] that  $b \in \text{ball}(H^\infty)$  is quasi-extreme if and only if it is an extreme point of  $\text{ball}(H^\infty)$ , which is equivalent to the condition

$$(4.7) \quad \int_{\mathbb{T}} \log(1 - |b|^2) \, dm = -\infty.$$

(See [6, p. 138]). Conversely, if  $b$  is not (quasi-)extreme, this integral is finite, and hence there exists (as noted at the beginning of this section) an outer function  $a \in \text{ball}(H^\infty)$  satisfying

$$(4.8) \quad |a(\zeta)|^2 + |b(\zeta)|^2 = 1$$

for almost every  $|\zeta| = 1$ ; this  $a$  is unique if we impose the normalization  $a(0) > 0$ .

In this setting, there is, of course, ever only one solution to the Gleason problem in  $\mathcal{H}(b)$ , namely the usual backward shift operator on holomorphic functions

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$

Following Sarason [10] we denote the restriction

$$X = S^*|_b.$$

All of the above discussions of transfer function realizations apply here, so  $b$  is realized by the colligation

$$\mathbf{V} = \begin{bmatrix} X & Xb \\ k_0^{b*} & b(0) \end{bmatrix}.$$

Now let  $a$  be the outer function of (4.8) with  $a(0) > 0$ . We expand  $a$  as a power series

$$a(z) = \sum_{n=0}^{\infty} \hat{a}(n) z^n.$$

Sarason [10, Lemma 6] proves the following formula for the Taylor coefficients  $\hat{a}(n)$ .

**Proposition 4.4.** *We have  $|a(0)|^2 = \frac{1}{1 + \|b\|_b^2}$ , and for  $n \geq 1$ ,*

$$\langle X^n b, b \rangle_{\mathcal{H}(b)} = \frac{-\hat{a}(n)}{a(0)}.$$



It is then straightforward to verify, by comparing Taylor coefficients, that this coincides with the  $a$  constructed via the transfer function realization above, since we had  $|a(0)|^2 = |a_0|^2 = \frac{1}{1+\|b\|_b^2}$  by definition, and

$$(4.9) \quad a(z) = a_0 - a_0 b^* (I - zX)^{-1} (zXb) = a_0 - a_0 \sum_{n=1}^{\infty} z^n \langle X^n b, b \rangle.$$

We do not know if the  $a$  function is outer when  $d > 1$  (here *outer* means that the operator of multiplication by  $a$  in  $H_d^2$  has dense range).

## 5. CONCLUSION OF THE PROOF OF THEOREM 1.1

In this section we prove the second half of Theorem 1.1.

**Proposition 5.1.** *If  $b$  is a multiplier of  $H_d^2$  and there exists a nonzero multiplier  $a$  such that*

$$M_a^* M_a + M_b^* M_b \leq I,$$

*then  $b$  is not quasi-extreme.*

The proof requires an elementary-seeming lemma, which nonetheless appears easiest to prove using the notion of a *free lifting* of a multiplier. We review the relevant results, prove the lemma, and finally prove Proposition 5.1.

We recall quickly the construction of the *free or noncommutative Toeplitz algebra* of Popescu. This is a canonical example of a *free semigroup algebra* as described by Davidson and Pitts [4], which contains proofs of all the claims made here. Fix an alphabet of  $d$  letters  $\{1, \dots, d\}$  and let  $\mathbb{F}_d^+$  denote the set of all words  $w$  in these  $d$  letters, including the empty word  $\emptyset$ . The set  $\mathbb{F}_d^+$  is a semigroup under concatenation: if  $w = i_1 \cdots i_n$  and  $v = j_1 \cdots j_m$ , we define

$$wv = i_1 \cdots i_n j_1 \cdots j_m.$$

Let  $F_d^2$  denote the Hilbert space (called the *Fock space*) with orthonormal basis  $\{\xi_w\}_{w \in \mathbb{F}_d^+}$ . This space comes equipped with a system of isometric operators  $L_1, \dots, L_d$  which act on basis vectors  $\xi_w$  by *left creation*:

$$L_i \xi_w = \xi_{iw}.$$

The operators  $L_1, \dots, L_d$  obey the relations

$$L_i^* L_j = \delta_{ji} I;$$

in other words, they are isometric with orthogonal ranges. The *free semigroup algebra*  $\mathcal{L}_d$  is the WOT-closed algebra of bounded operators on  $F_d^2$  generated by  $L_1, \dots, L_d$ . Each operator  $F \in \mathcal{L}_d$  has a Fourier-like expansion

$$(5.1) \quad F \sim \sum_{w \in \mathbb{F}_d^+} f_w L^w,$$

where, for a word  $w = i_1 \cdots i_n$ , by  $L^w$  we mean the product  $L_{i_1} L_{i_2} \cdots L_{i_n}$ . The coefficients  $f_w$  are determined by the relation

$$f_w = \langle F \xi_{\emptyset}, \xi_w \rangle_{F_d^2},$$

and the Cesaro means of the series converge WOT to  $F$ . To each  $F \in \mathcal{L}_d$  we can associate a  $d$ -variable holomorphic function  $\lambda(F)$  as follows: to each word  $w = i_1 \cdots i_n$  let  $z^w$  denote the product

$$z^w = z_{i_1} z_{i_2} \cdots z_{i_n}.$$

(Observe that  $z^w = z^v$  precisely when  $w$  is obtained by permuting the letters of  $v$ .) Then for  $F \in \mathcal{L}_d$  we define  $\lambda(F)$  by the series

$$\lambda(F)(z) = \sum_{w \in \mathbb{F}_d^+} f_w z^w.$$

The series converges uniformly on compact subsets of  $\mathbb{B}^d$  and is always a multiplier of  $H_d^2$ . In fact, Davidson and Pitts prove that the map  $\lambda$  is completely contractive from  $\mathcal{L}_d$  to  $\mathcal{M}(H_d^2)$ . Conversely, if  $f \in \mathcal{M}(H_d^2)$  and  $\|f\| \leq 1$ , then there exists (by commutant lifting) an  $F \in \mathcal{L}_d$  (not necessarily unique) such that  $\|F\| \leq 1$  and  $\lambda(F) = f$ . We call such an  $F$  a *free lifting* of  $f$ . Free liftings also always exist for matrix-valued multipliers, so in particular if, say,

$$\begin{pmatrix} f \\ g \end{pmatrix}$$

is a contractive  $2 \times 1$  multiplier, then there exist  $F, G \in \mathcal{L}_d$  such that  $\lambda(F) = f$ ,  $\lambda(G) = g$ , and

$$\begin{pmatrix} F \\ G \end{pmatrix}$$

is contractive.

We will need the following lemma, which we prove using free liftings.

**Lemma 5.2.** *If  $b$  is a multiplier and there exists a nonzero multiplier  $a$  satisfying  $M_a^* M_a + M_b^* M_b \leq I$ , then an  $a$  can be chosen satisfying this inequality and such that  $a(0) \neq 0$ .*

*Proof.* By the above remarks there exist free liftings  $A, B$  of  $a$  and  $b$  to the free semigroup algebra  $\mathcal{L}_d$  such that the column  $\begin{pmatrix} B \\ A \end{pmatrix}$  is contractive. The element  $A$  has Fourier expansion

$$A \sim \sum a_w L^w$$

with  $a_\emptyset = 0$  (since  $a(0) = 0$ ). Choose a word  $v$  of minimal length such that  $c_v \neq 0$ . It follows that

$$\tilde{A} = L_v^* A = \sum_w c_w L_v^* L_w = \sum_u \tilde{c}_u L_u$$

is a contractive free multiplier, and  $\tilde{A}(0) := \tilde{c}_\emptyset = c_v \neq 0$ , and we then have that

$$\begin{pmatrix} B \\ \tilde{A} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & L_v^* \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}$$

is contractive. Since the Davidson–Pitts symmetrization map  $\lambda$  is completely contractive, on putting  $\tilde{a} = \lambda(\tilde{A})$  we have  $\tilde{a}(0) \neq 0$  and

$$\begin{pmatrix} b \\ \tilde{a} \end{pmatrix}$$

is a contractive  $2 \times 1$  multiplier, which proves the lemma.  $\square$

*Remark.* This is really the same proof that works in the disk (without the need for the free lifting step). In the disk we just get that  $\tilde{a}$  satisfies  $a(z) = z^n \tilde{a}(z)$  for some  $n$ , and hence

$$M_a^* M_a = M_{\tilde{a}}^* M_{\tilde{a}}$$

(since  $M_z$  is an isometry). More generally, we could let  $a = \theta F$  be the inner-outer factorization of  $a$ ; since  $M_\theta$  is isometric we would have

$$M_a^* M_a = M_F^* M_\theta^* M_\theta M_F = M_F^* M_F.$$

*Proof of Proposition 5.1.* Suppose that  $b$  is a contractive multiplier and there exists a nonzero multiplier  $a$  so that

$$M_a^* M_a + M_b^* M_b \leq I.$$

By the lemma we may assume that  $a(0) \neq 0$ . We will construct an admissible tuple  $b_1, \dots, b_d$  such that

$$\sum_{j=1}^d \|b_j\|_b^2 \leq 1 - |b(0)|^2 - |a(0)|^2 < 1 - |b(0)|^2;$$

by the remark following Proposition 2.2 this proves that  $b$  is not quasi-extreme.

Let

$$c = \begin{pmatrix} b & 0 \\ a & 0 \end{pmatrix}.$$

Then  $c$  is a  $2 \times 2$  contractive multiplier and

$$(5.2) \quad c(0)^* c(0) = \begin{pmatrix} |b(0)|^2 + |a(0)|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

We form the de Branges–Rovnyak space  $\mathcal{H}(c)$  of the function  $c$ , which has the reproducing kernel

$$\begin{aligned} k^c(z, w) &= \frac{I - c(z)c(w)^*}{1 - zw^*} \\ &= \begin{bmatrix} k^b(z, w) & \frac{-b(z)a(w)^*}{1 - zw^*} \\ \frac{-a(z)b(w)^*}{1 - zw^*} & k^a(z, w) \end{bmatrix}. \end{aligned}$$

Now we apply the vector-valued generalization of a basic result from the theory of reproducing kernel Hilbert spaces: let  $H(k)$  be a  $\mathcal{H}$ -valued RKHS of functions on a set  $X$ . An  $\mathcal{H}$ -valued function  $F$  on  $X$  belongs to  $H(k)$  if and only if there is a  $t \geq 0$  such that

$$F(x)F(y)^* \leq t^2 k(x, y)$$

as positive  $\mathcal{L}(\mathcal{H})$ -valued kernel functions on  $X$ . Moreover, the least such  $t$  that works is  $t = \|F\|_{H(k)}$  [9, Theorem 10.17].

Note that in the above we view  $F(x) : \mathbb{C} \rightarrow \mathcal{H}$  as a linear map for any fixed  $x \in X$ . It follows that  $F(y)^* h = \langle F(y), h \rangle_{\mathcal{H}}$  for any  $h \in \mathcal{H}$ . For example, if (as in the case of  $\mathcal{H}(c)$ )  $\mathcal{H} = \mathbb{C}^2$ , then in the standard basis  $F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$  and

$$F(x)F(y)^* = \begin{bmatrix} F_1(x)\overline{F_1(y)} & F_1(x)\overline{F_2(y)} \\ F_2(x)\overline{F_1(y)} & F_2(x)\overline{F_2(y)} \end{bmatrix}.$$

So now let  $C : \mathbb{C}^2 \rightarrow K(c) \otimes \mathbb{C}^d$  be a contractive Gleason solution for  $c$ , e.g., the one appearing in a generalized functional model realization for  $c$  (which exists by

[2]). That is,  $C = \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}$  obeys

$$zC(z) = z_1 c_1(z) + \dots + z_d c_d(z) = c(z) - c(0)$$

and contractivity means

$$C^*C \leq I - c(0)^*c(0).$$

So each  $c_j(z) \in \mathbb{C}^{2 \times 2}$  and we write

$$c_j(z) = \begin{bmatrix} b_j(z) & * \\ a_j(z) & * \end{bmatrix},$$

and observe that the  $B = \begin{bmatrix} b_1 \\ \vdots \\ b_d \end{bmatrix}$  and the  $A = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}$  are Gleason solutions for  $b, a$  in the sense that

$$b(z) - b(0) = \sum_{j=1}^d z_j b_j(z),$$

and similarly for  $a$ . Note that

$$c_j(z)e_1 = \begin{bmatrix} b_j(z) \\ a_j(z) \end{bmatrix}.$$

We need to check that  $B$  actually belongs to  $\mathcal{H}(b) \otimes \mathbb{C}^d$  and is a contractive Gleason solution for  $b$ : Let  $\{e_1, e_2\}$  denote the standard orthonormal basis of  $\mathbb{C}^2$  and let  $t_j := \|c_j e_1\|_{\mathcal{H}(c)}$ . Then by the vector-valued RKHS proposition discussed above and the form of the reproducing kernel for  $\mathcal{H}(c)$ ,

$$\begin{aligned} (c_j(z)e_1)(c_j(w)e_1)^* &= \begin{bmatrix} b_j(z) \\ a_j(z) \end{bmatrix} \begin{bmatrix} b_j(w)^* & a_j(w)^* \end{bmatrix} \\ &= \begin{bmatrix} b_j(z)b_j(w)^* & b_j(z)a_j(w)^* \\ a_j(z)b_j(w)^* & a_j(z)a_j(w)^* \end{bmatrix} \\ &\leq t_j^2 \begin{bmatrix} k^b(z, w) & \frac{-b(z)a(w)^*}{1-zw^*} \\ \frac{-a(z)b(w)^*}{1-zw^*} & k^a(z, w) \end{bmatrix} \end{aligned}$$

as positive kernel functions. In particular, the  $(1, 1)$  entry of the above equation must be a positive kernel function so that

$$b_j(z)b_j(w)^* \leq t_j^2 k^b(z, w).$$

Again, by the scalar version of the RKHS result this implies that  $b_j \in \mathcal{H}(b)$  and that

$$\|b_j\|_{\mathcal{H}(b)} \leq t_j = \|c_j e_1\|_{\mathcal{H}(c)}.$$

This yields the inequalities

$$\begin{aligned}
 \sum_{k=1}^d \|b_j\|_{\mathcal{H}(b)}^2 &\leq \sum_{k=1}^d t_j^2 \\
 &= \sum_{k=1}^d \|c_j e_1\|_{\mathcal{H}(c)}^2 \\
 &= \sum_{k=1}^d \langle c_j^* c_j e_1, e_1 \rangle_{\mathbb{C}^2} \\
 &= \langle C^* C e_1, e_1 \rangle_{\mathbb{C}^2} \\
 &\leq \langle (I - c(0)^* c(0)) e_1, e_1 \rangle_{\mathbb{C}^2} \\
 &= 1 - |b(0)|^2 - |a(0)|^2, \\
 &< 1 - |b(0)|^2,
 \end{aligned}$$

and the proof is complete.  $\square$

## REFERENCES

- [1] Joseph A. Ball and Vladimir Bolotnikov, *Canonical de Branges-Rovnyak model transfer-function realization for multivariable Schur-class functions*, Hilbert spaces of analytic functions, CRM Proc. Lecture Notes, vol. 51, Amer. Math. Soc., Providence, RI, 2010, pp. 1–39. MR2648864
- [2] Joseph A. Ball, Vladimir Bolotnikov, and Quanlei Fang, *Transfer-function realization for multipliers of the Arveson space*, J. Math. Anal. Appl. **333** (2007), no. 1, 68–92. MR2323477
- [3] Joseph A. Ball, Tavan T. Trent, and Victor Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces*, Operator theory and analysis (Amsterdam, 1997), Oper. Theory Adv. Appl., vol. 122, Birkhäuser, Basel, 2001, pp. 89–138. MR1846055
- [4] Kenneth R. Davidson and David R. Pitts, *Nevanlinna-Pick interpolation for non-commutative analytic Toeplitz algebras*, Integral Equations Operator Theory **31** (1998), no. 3, 321–337. MR1627901
- [5] Kunyu Guo, Junyun Hu, and Xianmin Xu, *Toeplitz algebras, subnormal tuples and rigidity on reproducing  $\mathbf{C}[z_1, \dots, z_d]$ -modules*, J. Funct. Anal. **210** (2004), no. 1, 214–247. MR2052120
- [6] Kenneth Hoffman, *Banach spaces of analytic functions*, Reprint of the 1962 original, Dover Publications, Inc., New York, 1988. MR1102893
- [7] Michael T. Jury, *Clark theory in the Drury-Arveson space*, J. Funct. Anal. **266** (2014), no. 6, 3855–3893. MR3165246
- [8] Michael T. Jury and Robert T.W. Martin, *Aleksandrov-Clark theory for the Drury-Arveson space*, preprint.
- [9] Vern I. Paulsen and Mrinal Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics, vol. 152, Cambridge University Press, Cambridge, 2016. MR3526117
- [10] Donald Sarason, *Shift-invariant spaces from the Brangesian point of view*, The Bieberbach conjecture (West Lafayette, Ind., 1985), Math. Surveys Monogr., vol. 21, Amer. Math. Soc., Providence, RI, 1986, pp. 153–166. MR875239
- [11] Donald Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 10, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1994. MR1289670
- [12] Orr Shalit, *Operator theory and function theory in Drury-Arveson space and its quotients*, <http://adsabs.harvard.edu/abs/2013arXiv1308.1081S>.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611  
*Email address:* `mjury@ufl.edu`

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF CAPE TOWN,  
PRIVATE BAG X1, RONDEBOSCH 7701, SOUTH AFRICA  
*Email address:* `rtwmartin@gmail.com`