ON OPTIMAL SCOTT SENTENCES OF FINITELY GENERATED ALGEBRAIC STRUCTURES

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ABSTRACT. Scott showed that for every countable structure $\mathcal{A}$, there is a sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$, called a *Scott sentence* for $\mathcal{A}$, whose countable models are exactly the isomorphic copies of $\mathcal{A}$. Thus, the least quantifier complexity of a Scott sentence of a structure is an invariant that measures the complexity "describing" the structure. Knight et al. have studied the Scott sentences of many structures. In particular, Knight and Saraph showed that a finitely generated structure always has a Σ^0_3 Scott sentence. We give a characterization of the finitely generated structures for which the Σ^0_3 Scott sentence is optimal. One application of this result is to give a construction of a finitely generated group where the Σ^0_3 Scott sentence is optimal.

1. INTRODUCTION

Given a countable structure $\mathcal{M}$, we can describe $\mathcal{M}$, up to isomorphism, by a sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ which allows countable conjunctions and disjunctions. (See Section 1.1 for the formal description of this logic; in this brief introduction, we will write sentences in an informal way.) To measure the complexity of $\mathcal{M}$, we want to write the simplest possible description of $\mathcal{M}$. For example, one can describe the countably infinite-dimensional $\mathbb{Q}$-vector space by the vector space axioms together with the sentence

> for all n, there are $x_1, \ldots, x_n$ such that for all $r_1, \ldots, r_n \in \mathbb{Q}$, if
> $r_1 x_1 + \cdots + r_n x_n = 0$, then some $r_i = 0$.

This sentence has a universal quantifier, followed by an existential quantifier, followed by a universal quantifier. There is a hierarchy of sentences depending on the number of quantifier alternations. The Σ^0_n sentences have n alternations of quantifiers, beginning with existential quantifiers; the Π^0_n sentences have n alternations of quantifiers, beginning with universal quantifiers; and the d-Σ^0_n sentences are the conjunction of a Σ^0_n and a Π^0_n sentence. The hierarchy is ordered as follows, from the simplest formulas on the left, to the most complicated formulas on the right:
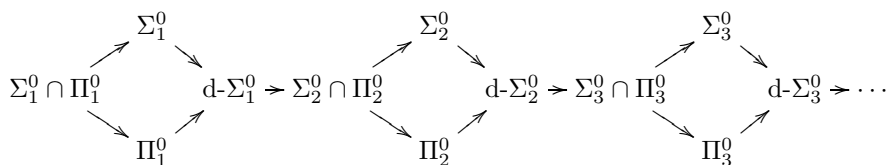
We use this hierarchy to measure the complexity of a sentence. The sentence given above describing the infinite-dimensional $\mathbb{Q}$-vector space is a Π_3^0 sentence, and it turns out that this is the best possible; there is no d-Σ_2^0 description of this vector space. There is a d-Σ_2^0 description of any finite-dimensional $\mathbb{Q}$-vector space, and so these structures are "simpler" than the infinite-dimensional vector space.

In this paper, we consider descriptions of finitely generated structures, and particularly of finitely generated groups. Any finitely generated structure $\mathcal{M}$, with generating tuple $\bar{a}$, has a Σ_3^0 description of the form:

> there is a tuple $\bar{x}$, satisfying the same atomic formulas as $\bar{a}$ (i.e., for all atomic formulas and negations of atomic formulas true of $\bar{a}$, the formula is true of $\bar{x}$) such that every element is generated by $\bar{x}$ (i.e., for all y, there is a term t in the language such that $y = t(\bar{x})$).

However, many finitely generated groups have a simpler description which is d-Σ_2^0. For the group $\mathbb{Z}$, for example, the Π_2^0 axioms of torsion-free abelian groups, together with the following two sentences, which are Π_2^0 and Σ_2^0, respectively, form a d-Σ_2^0 description:

> for all x and y, there are $n, m \in \mathbb{Z}$, not both zero, such that $nx = my$

and

> there is $x \neq 0$ which has no proper divisors.

Indeed, all previously known examples of finitely generated groups had d-Σ_2^0 Scott sentences, including all polycyclic (including nilpotent) groups and many finitely generated solvable groups [5]. The main result of this paper is an example of a computable finitely generated group that has no d-Σ_2^0 Scott sentence. Our group has a Σ_3^0 m-complete index set.

This paper is divided into two main sets of results. The first is a general investigation of conditions for a finitely generated structure to have (or not have) a d-Σ_2^0 Scott sentence. The second is an application of these general results to constructing the group mentioned above. We also include some results on finitely generated fields and rings.

1.1. **Scott sentences.** The infinitary logic $\mathcal{L}_{\omega_1\omega}$ is the logic which allows countably infinite conjunctions and disjunctions but only finite quantification. If the conjunctions and disjunctions of a formula φ are all over computable sets of indices for formulas, then we say that φ is computable.

We use the following recursive definition to define the complexity of classes:

- An $\mathcal{L}_{\omega_1\omega}$ formula is both Σ_0^0 and Π_0^0 if it is quantifier free and does not contain any infinite disjunction or conjunction.
- An $\mathcal{L}_{\omega_1\omega}$ formula is Σ_α^0 if it is a countable disjunction of formulas of the form $\exists\bar{x}\phi$, where each ϕ is Π_β^0 for some $\beta < \alpha$.
- An $\mathcal{L}_{\omega_1\omega}$ formula is Π_α^0 if it is a countable disjunction of formulas of the form $\forall\bar{x}\phi$, where each ϕ is Σ_β^0 for some $\beta < \alpha$.

We say a formula is d-Σ_α^0 if it is a conjunction of a Σ_α^0 formula and a Π_α^0 formula.

Scott [10] showed that if $\mathcal{A}$ is a countable structure in a countable language, then there is a sentence φ of $\mathcal{L}_{\omega_1\omega}$ whose countable models are exactly the isomorphic copies of $\mathcal{A}$. Such a sentence is called a *Scott sentence* for $\mathcal{A}$. We remark that because $\alpha \wedge \neg(\beta \wedge \neg\gamma)$ is equivalent to $(\alpha \wedge \neg\beta) \vee (\alpha \wedge \gamma)$, the complexity classes n-Σ_α^0 of Scott sentences collapse for $n \geq 2$.

We can measure the complexity of a countable structure by looking for a Scott sentence of minimal complexity, as measured by the quantifier complexity hierarchy of computable formulas described above. [9] showed that if $\mathcal{A}$ has a Π_α^0 Scott sentence and a Σ_α^0 Scott sentence, then it must have a d-Σ_β^0 Scott sentence for some $\beta < \alpha$. So for a given structure, the optimal Scott sentence is Σ_α^0, Π_α^0, or d-Σ_α^0 for some α.

We refer the interested readers to Chapter 6 of [2] for a more complete description of $\mathcal{L}_{\omega_1\omega}$ formulas and Scott sentences.

1.2. **Index set complexity.** Given a structure $\mathcal{A}$ and a Scott sentence φ for $\mathcal{A}$, we want to determine whether φ is an optimal Scott sentence for $\mathcal{A}$ or whether there is a simpler Scott sentence which we have not yet found. We can use index set calculations to resolve this problem.

Definition 1.1. Let $\mathcal{A}$ be a structure. The index set $I(\mathcal{A})$ is the set of all indices e such that the eth Turing machine Φ_e gives the atomic diagram of a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$. We can also relativize this to any set X: $I^X(\mathcal{A})$ is the set of all indices e such that the eth Turing machine Φ_e^X with oracle X gives the atomic diagram of a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$.

There is a connection between index sets and Scott sentences; as shown below.

Proposition 1.2. *If a countable structure $\mathcal{A}$ has an X-computable Σ_α^0 (respectively, Π_α^0 or d-Σ_α^0) Scott sentence, then the index set $I^X(\mathcal{A})$ is in $\Sigma_\alpha^0(X)$ (respectively, $\Pi_\alpha^0(X)$ or d-$\Sigma_\alpha^0(X)$).*

So if, for example, we have a computable Σ_3^0 Scott sentence for a structure $\mathcal{A}$, we will try to show that the index set $I(\mathcal{A})$ is Σ_3^0 m-complete. If we can do this, then we know that our Scott sentence is optimal. In general, any $\mathcal{L}_{\omega_1\omega}$ sentence is X-computable for some X.

1.3. **Summary of prior results.** There are many results using the strategy above to find the complexities of optimal Scott sentences of structures. For example, Knight et al. [3], [4] determined the complexities of optimal Scott sentences for finitely generated free abelian groups, reduced abelian groups, free groups, and many other structures.

However, this strategy does not work when the complexity of the optimal Scott sentence is strictly higher than the complexity of the index set. Indeed, Knight and McCoy gave the first such example in [6], showing there is a subgroup G of $\mathbb{Q}$ such that $I(G)$ is d-Σ_2^0, but it has no computable d-Σ_2^0 Scott sentence.

It was observed in [7] that any computable finitely generated group, and indeed any computable finitely generated structure, has a computable Σ_3^0 Scott sentence. In [5], it was shown that many classes of "nice" groups in the sense of geometric group theory, including polycyclic groups (which includes nilpotent groups and abelian groups) and certain solvable groups; all have computable d-Σ_2^0 Scott sentences. However, none of these examples achieves the Σ_3^0 bound that was given in [7].

1.4. **New results.** In this paper, we give an example of a finitely generated group which has no d-Σ_2^0 Scott sentence. As mentioned above, we do this by showing that the index set is Σ_3^0 m-complete.

Theorem 1.3. *There is a finitely generated computable group G whose index set is Σ_3^0 m-complete.*

The proof is in two parts. First, in Section 2, we develop some general results on when a finitely generated structure of any kind has a d-Σ_2^0 Scott sentence. These results are of interest beyond the application to groups.

Definition 1.4. Let $\mathcal{A}$ be a finitely generated structure. Then $\mathcal{A}$ is *self-reflective* if it contains a proper Σ_1^0-elementary substructure isomorphic to itself. ($\mathcal{B}$ is a Σ_1^0-elementary substructure of $\mathcal{A}$, and we write $\mathcal{B} \preceq_1 \mathcal{A}$, if, for each existential formula $\varphi(\bar{x})$ and $\bar{b} \in \mathcal{B}$, $\mathcal{A} \models \varphi(\bar{b})$ if and only if $\mathcal{B} \models \varphi(\bar{b})$.)

We prove, using an index-set calculation, the equivalence of (1) and (2) in the following characterization of finitely generated structures with no d-Σ_2^0 Scott sentence.

Theorem 1.5. *Let $\mathcal{M}$ be a finitely generated structure. The following are equivalent:*

(1) *$\mathcal{M}$ has a d-Σ_2^0 Scott sentence,*
(2) *$\mathcal{M}$ is not self-reflective,*
(3) *for all (or some) generating tuples of $\mathcal{M}$, the orbit is defined by a Π_1^0 formula.*

The equivalence of (3) to (1) has been proved by Alvir, Knight, and McCoy [1].

Second, in Section 4, we apply this characterization to finitely generated groups. Using small cancellation theory and HNN extensions, we produce a computable group G which is self-reflective. Thus—using Theorem 1.5—this group has no d-Σ_2^0 Scott sentence. Using the group ring construction, we generalize this in Section 5 to produce a ring which is self-reflective.

We also apply our results to finitely generated fields in Section 3. A simple argument shows that no finitely generated field is self-reflective. Thus we have the following.

Theorem 1.6. *Every finitely generated field has a d-Σ_2^0 Scott sentence.*

1.5. **Open questions.** Here we leave several open questions. First, a special class of finitely generated groups are the finitely presented groups. Is there a (computable) finitely presented group with no d-Σ_2^0 Scott sentence?

Question 1.7. Does every finitely presented group with solvable word problem have a d-Σ_2^0 Scott sentence?

Second, one can consider structures other than fields and groups. A natural class to consider is rings. Using the group ring construction, we get a self-reflective ring. However, if we insist that the ring be commutative, then such a construction no longer works.

Question 1.8. Does every commutative ring have a d-Σ_2^0 Scott sentence?

One can also place further restrictions on the ring. A natural restriction is that there be no zero-divisors.

Question 1.9. Does every integral domain have a d-Σ_2^0 Scott sentence?

We expect the answer to be yes, as integral domains have a good dimension theory.

2. General theory

Our goal in this section is to prove Theorem 1.5. The proof is in two parts. First we will show that if $\mathcal{A}$ is not self-reflective, then it has a d-Σ_2^0 Scott sentence. Second, we will show that if $\mathcal{A}$ is self-reflective, then its index set is as complicated as possible.

Theorem 2.1. *Let $\mathcal{A}$ be a finitely generated structure. If $\mathcal{A}$ is not self-reflective, then $\mathcal{A}$ has a d-Σ_2^0 Scott sentence.*

Proof. Let $\bar{g}$ be a generating tuple for $\mathcal{A}$. Let p be the atomic type of $\bar{g}$. For any tuple $\bar{g}'$ satisfying p, the substructure generated by $\bar{g}$ is isomorphic to $\mathcal{A}$. Since $\mathcal{A}$ is not self-reflective, if $\bar{g}'$ does not generate $\mathcal{A}$, then there is a tuple $\bar{a}$ and a quantifier-free formula $\psi(\bar{x}, \bar{y})$ with $\mathcal{A} \models \psi(\bar{g}', \bar{a})$ such that there is no $\bar{b} \in \mathcal{A}$ such that $\mathcal{A} \models \psi(\bar{g}, \bar{b})$. Let S be the set of formulas $\psi(\bar{x}, \bar{y})$ such that for some tuple $\bar{g}'$ satisfying the atomic type p but not generating $\mathcal{A}$, and some $\bar{a}$, $\mathcal{A} \models \psi(\bar{g}', \bar{a})$, there is no $\bar{b} \in \mathcal{A}$ such that $\mathcal{A} \models \psi(\bar{g}, \bar{b})$.

Using the set S, we can now define the Scott sentence for $\mathcal{A}$. The Scott sentence for $\mathcal{A}$ is the conjunction of the Σ_2^0 sentence which says:

> *there exists a tuple $\bar{x}$ satisfying p and such that for all $\bar{z}$ and $\psi \in S$, $\bar{x}\bar{z}$ does not satisfy ψ,*

and the Π_2^0 sentence which says:

> *for all tuples $\bar{x}$ which satisfy p, either for all y, $y \in \langle \bar{x} \rangle$, or there is a formula $\psi \in S$ and a tuple $\bar{z}$ such that $\bar{x}, \bar{z}$ satisfies ψ.*

This latter sentence is of the form $(\forall \bar{x})\,[\theta \to (\alpha \vee \beta)]$, where θ is Π_1^0, α is Π_2^0, and β is Σ_1^0.

It is easy to see that $\mathcal{A}$ models this sentence. Now suppose that $\mathcal{M}$ is any structure which satisfies this sentence. Since $\mathcal{M}$ satisfies the Σ_2^0 part of the sentence, there is a tuple $\bar{h} \in \mathcal{M}$ which satisfies the atomic type p and such that for all $\bar{c} \in \mathcal{M}$ and $\psi \in S$, $\mathcal{M} \nvDash \psi(\bar{h}, \bar{c})$. We claim that $\bar{h}$ generates $\mathcal{M}$; since $\bar{h}$ satisfies the atomic type p, this would imply that $\mathcal{M}$ is isomorphic to $\mathcal{A}$. Indeed, by the Π_2^0 part of the sentence, either $\bar{h}$ generates $\mathcal{M}$ or there is a formula $\psi \in S$ and a tuple $\bar{c}$ such that $\mathcal{M} \models \psi(\bar{h}, \bar{c})$. The latter cannot happen, and so $\bar{h}$ generates $\mathcal{M}$. $\qquad \square$

We will now show that if $\mathcal{A}$ is self-reflective, then (relativizing everything to $\mathcal{A}$) its index set is Σ_3^0 m-complete.

Theorem 2.2. *Let $\mathcal{A}$ be X-computable and self-reflective. Then $I^X(\mathcal{A})$ is $\Sigma_3^0(X)$ m-complete (relative to X).*

Proof. We will assume that $\mathcal{A}$ is computable; the general result can be obtained by relativizing. Fix a Σ_3^0 set S. We may assume that S is of the form

$$n \in S \iff (\exists e)\, W_{f(e,n)} \text{ is infinite}$$

for some computable function f. We will define, uniformly in n, a computable structure $\mathcal{B}_n$ such that if $n \in S$, then $\mathcal{B}_n \cong \mathcal{A}$, and if $n \notin S$, then $\mathcal{B}_n$ is not finitely generated. We may assume that at each stage s, there is at most one e for which an element is enumerated into $W_{f(e,n)}$.

For convenience, we will suppress n, writing $\mathcal{B}$ for $\mathcal{B}_n$ and $f(e)$ for $f(e,n)$. We will build $\mathcal{B}$ with domain ω as a union of finite substructures (in a finite sublanguage) $\mathcal{B}[s]$, viewing the language as a relation language as is usual for this kind of construction.

Since $\mathcal{A}$ sits properly inside itself as a Σ_1^0-elementary substructure, we can create an infinite chain

$$\mathcal{A}_0 \prec_1 \mathcal{A}_1 \prec_1 \mathcal{A}_2 \prec_1 \cdots \prec_1 \mathcal{A}^*,$$

where each $\mathcal{A}_i$ is (effectively) isomorphic to $\mathcal{A}$ and $\mathcal{A}_i$ is a c.e. (but not necessarily computable) subset of $\mathcal{A}_{i+1}$. The structure $\mathcal{A}^*$ is the union of all of the $\mathcal{A}_i$'s and is not finitely generated (and hence not isomorphic to $\mathcal{A}$).

At each stage s, the domain of $\mathcal{B}[s]$ will be the union of finitely many unary relations $R_0[s] \subseteq \cdots \subseteq R_{k_s}[s]$. We will also have computable partial embeddings $j[s] \colon \mathcal{B}[s] \to \mathcal{A}^*$ such that $j[s](R_k[s]) \subseteq \mathcal{A}_k$.

We will build R_0 isomorphic to $\mathcal{A}_0$, R_1 isomorphic to $\mathcal{A}_1$, and so on, via j. While $W_{f(e)}$ does not have any elements enumerated into it, we will keep building R_e to copy $\mathcal{A}_e$. However, when an element is enumerated into $W_{f(e)}$, we will collapse each R_j, $j > e$, into R_e. If e is least such that $W_{f(e)}$ is infinite, then $\mathcal{B}$ will consist just of the domain R_e, as each R_j, $j > e$, will be collapsed infinitely many times, and $\mathcal{B}$ will be isomorphic to $\mathcal{A}$. On the other hand, if each $W_{f(e)}$ is finite, then $\mathcal{B}$ will be isomorphic to $\mathcal{A}^*$, and hence $\mathcal{B}$ will not be isomorphic to $\mathcal{A}$.

Construction. Begin at stage 0 with $\mathcal{B}[0]$ empty and $k_0 = 0$, with $R_0[0]$ empty.

Action at stage $s + 1 = 3t + 1$. Set $k = k_s$. We will have $k_{s+1} = k$. For each $n = 0, \ldots, k$, let a_n be the first element of $\mathcal{A}_n$ not in $j[s](R_n[s])$. Define $\mathcal{B}[s+1] \supseteq \mathcal{B}[s]$ so that $j[s+1] \colon \mathcal{B}[s+1] \to \mathcal{A}^*$ is a partial embedding, extending $j[s]$, whose range also contains $a_0, \ldots, a_k$. Given $x \in \mathcal{B}[s+1]$, set $R_n[s+1]$ to be $R_n[s]$ plus the elements x such that $j(x)$ is among the first s elements of $\mathcal{A}_n$.

Action at stage $s + 1 = 3t + 2$. Set $k_{s+1} = k_s + 1$ and $j[s+1] = j[s]$. Let $R_{k_{s+1}}$ be empty. For each $n = 0, \ldots, k_s$, let $R_n[s+1] = R_n[s]$.

Action at stage $s + 1 = 3t + 3$. If for some $e < k_s$, an element entered $W_{f(e)}$ at stage t, do the following. Otherwise, do nothing. Let $k_{s+1} = e$. Let $\bar{u}$ be the elements of $R_e[s]$ and let $\bar{v}$ be the other elements of $\mathcal{B}[s]$ which are not in $R_e[s]$. Let $\psi(\bar{x}, \bar{y})$ be the conjunction of the atomic diagram of $\mathcal{B}[s]$ so that $\mathcal{B}[s] \models \psi(\bar{u}, \bar{v})$. Then $\mathcal{A}_{k_s} \models \psi(j[s](\bar{u}), j[s](\bar{v}))$. Since $j[s](\bar{u}) \in \mathcal{A}_e \prec_1 \mathcal{A}_{k_s}$, there is a tuple $\bar{a} \in \mathcal{A}_e$ such that $\mathcal{A}_e \models \psi(j[s](\bar{u}), \bar{a})$. Then define $R_e[s+1] = R_e[s] \cup \{\bar{v}\}$ and define $j[s+1] \supseteq j[s] \upharpoonright_{R_e[s]}$ to map $\bar{v}$ to $\bar{a}$. For $n < e$, define $R_n[s+1] = R_n[s]$.

Note that at every stage s, $j[s](R_n) \subseteq \mathcal{A}_n$.

End construction.

Let $k = \liminf_s k_s$. If $n \in S$, then k is the least e such that $W_{f(e)}$ is infinite. Otherwise, if $n \notin S$, then $k = \infty$.

Claim 2.3. Fix $n \leq k$. Let s be a stage such that $k_s \geq n$ and after which no element is ever enumerated into $W_{f(e)}$ for any $e < n$. Then:

    (1) for all $t_2 > t_1 \geq s$, $R_n[t_1] \subseteq R_n[t_2]$ and $j[t_1] \upharpoonright R_n[t_1] \subseteq j[t_2]$.
    (2) $R_n = \bigcup_{t \geq s} R_n[t]$ is a substructure (in the relational language) of $\mathcal{B}$.
    (3) $j_n = \bigcup_{t \geq s} j[t] \upharpoonright R_n[t]$ is an isomorphism between R_n and $\mathcal{A}_n$.

Given $m \leq n \leq k$, $R_m \subseteq R_n$.

Proof. (1) is easy to see from the construction. (2) is also clear. For (3) it remains to see that j_n is surjective onto $\mathcal{A}_n$. If $a \in \mathcal{A}_n$ is the least element which is not in the image of j, then there is some stage $t \geq s$ at which each lesser element of $\mathcal{A}_n$

is already in the image of $j[t]$, and a is among the first t elements of $\mathcal{A}_n$. For each lesser element a' of $\mathcal{A}_n$, $a' = j[3t+1](b')$ for some b', and $b' \in R_n[3t+1]$; hence $j[t'](b') = a'$ at each later stage $t' \geq 3t+1$. Then at some stage, say $3t+4$, we put a into the image of j, say with $j(b) = a$, and we have $b \in R_n[3t+4]$ so that $j[t'](b) = a$ at each later stage $t' \geq 3t+4$. This is a contradiction; thus j contains all of $\mathcal{A}_n$ in its image. $\square$

Claim 2.4. $\mathcal{B} = \bigcup_{n \leq k} R_n$.

Proof. If an element enters $W_{f(e)}$ at stage t and no element ever enters $W_{f(e')}$, for $e' < e$, after stage t, then $\mathcal{B}[3t+3] = R_e[3t+3] \subseteq R_e$. If $k < \infty$, then there are infinitely many stages $3t+3$ at which $\mathcal{B}[3t+3] = R_k[3t+3]$, and so $\mathcal{B} = R_k$. If $k = \infty$, then there is a sequence $(e_1, t_1), (e_2, t_2), (e_3, t_3), \ldots$, with $e_1 < e_2 < e_3 < \cdots$ and $t_1 < t_2 < t_3 < \cdots$, at which $\mathcal{B}[3t_i+3] \subseteq R_{e_i}[3t_i+3] \subseteq R_{e_i}$. Then $\mathcal{B} = \bigcup_{n \leq k} R_n$. $\square$

Claim 2.5. If $m \in S$, then $\mathcal{B}_m \cong \mathcal{A}$.

Proof. We have $k < \infty$. Then $\mathcal{B}_m = \bigcup_{n \leq k} R_n = R_k$, and R_k is isomorphic to $\mathcal{A}$ via j_k. $\square$

Claim 2.6. If $m \notin S$, then $\mathcal{B}_m$ is not finitely generated.

Proof. Fix a tuple $\bar{g} \in \mathcal{B}_m$. Then $\bar{g} \in R_n$ for some n. Pick $a \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n$. Since $a \notin \mathcal{A}_n$, $a \notin j(R_n)$. Thus there is $h \in R_{n+1} \setminus R_n$ with $j(h) = a$. Thus R_n is a proper substructure of $\mathcal{B}$. Since $\bar{g} \in R_n$, $\bar{g}$ cannot generate $\mathcal{B}$. $\square$

This completes the proof of the theorem. $\square$

Proof of $(1) \Rightarrow (2)$ *in Theorem* 1.5. Let $\mathcal{A}$ be a finitely generated self-reflective structure which has a d-Σ_2^0 Scott sentence. Let $X \geq_T \mathcal{A}$ be such that this Scott sentence is X-computable. Then by Theorem 2.2, the index set $I^X(\mathcal{A})$ is $\Sigma_3^0(X)$ m-complete relative to X, contradicting that $I^X(\mathcal{A})$ is in d-$\Sigma_2^0(X)$. $\square$

3. Finitely generated fields

It is not hard to show that no finitely generated field is self-reflective, and hence every finitely generated field has a d-Σ_2^0 Scott sentence.

Proof of Theorem 1.6. Let F be a finitely generated field of characteristic p which is possibly zero. We claim that F is not self-reflective, and hence by Theorem 2.1, F has a d-Σ_2^0 Scott sentence.

Let $\mathbb{F}_p$ be the prime field of characteristic p. Write $F = \mathbb{F}_p(a_1, \ldots, a_m, b_1, \ldots, b_n)$, with $a_1, \ldots, a_m$ a transcendence basis for F over $\mathbb{F}_p$, and let $\varphi \colon F \to E \subsetneq F$ be an isomorphism between F and a proper subfield E of F. We claim that E is not a Σ_1^0-elementary substructure of F.

Let $a_1', \ldots, a_m'$ be the images of $a_1, \ldots, a_m$ under φ, and let $b_1', \ldots, b_n'$ be the images of $b_1, \ldots, b_n$ under φ. Since F and $E = \mathbb{F}_p(\bar{a}', \bar{b}')$ are isomorphic, $a_1', \ldots, a_m'$ are a transcendence base for E, and so $\bar{a}, \bar{b}$ are algebraic over $\mathbb{F}_p(\bar{a}', \bar{b}')$. Thus the atomic type $\mathrm{tp}_{\mathrm{at}}(\bar{a}, \bar{b}/\mathbb{F}_p(\bar{a}', \bar{b}'))$ is isolated by a formula $\psi(\bar{a}', \bar{b}', \bar{x}, \bar{y})$. We claim that there is no tuple $\bar{c}, \bar{d} \in E$ with $E \models \psi(\bar{a}', \bar{b}', \bar{c}, \bar{d})$. Suppose to the contrary that there was such a tuple $\bar{c}, \bar{d}$; then $E(c, d)$ would be isomorphic to F over E. But since $\bar{c}, \bar{d} \in E$, $E(c, d) = E$, and so E is isomorphic to F over E. This cannot happen, as E is a proper subfield of F. This is a contradiction; thus E is not a Σ_1^0-elementary substructure of F, proving the theorem. $\square$

4. Finitely generated groups

In this section, we first introduce some group theory background on HNN extensions (Section 4.1) and small cancellation theory (Section 4.2). Then we will use this machinery to construct a self-reflective group in Section 4.3. We refer the interested reader to [8, §IV, V] for more details on the group theoretic tools we are using here.

4.1. HNN extensions.

Definition 4.1. For a group G with presentation $G = \langle S \mid R \rangle$ and an isomorphism $\alpha : H \to K$ between two subgroups $H, K \subseteq G$, we define the *HNN extension* of G by α to be

$$G *_\alpha = \langle S, t \mid R, \{tht^{-1} = \alpha(h)\}_{h \in H} \rangle.$$

The key lemma about HNN extensions we will need is the following, which says that every trivial word in the HNN extension is either already trivial in G or "reducible" by a conjugation of t or t^{-1}.

Lemma 4.2 (Britton's Lemma). *With the notation above, let*

$$w = g_0 t^{e_1} g_1 t^{e_2} \cdots t^{e_n} g_n \in G *_\alpha$$

with $g_i \in G$ and $e_i = \pm 1$. Suppose $w = 1$; then one of the following is true:

(1) *$n = 0$ and $g_0 = 1$ in G,*
(2) *there is k such that $e_k = 1$, $e_{k+1} = -1$, and $g_k \in H$, or*
(3) *there is k such that $e_k = -1$, $e_{k+1} = 1$, and $g_k \in K$.*

One can show using Britton's Lemma that the natural homomorphism from G to $G *_\alpha$ is an embedding, so we can think of G as a subgroup of $G *_\alpha$.

4.2. Small cancellation.

Definition 4.3. We say a presentation $\langle S \mid R \rangle$ is *symmetrized* if every relation is cyclically reduced and the relation set R is closed under inverse and cyclic permutation.

Let $\langle S \mid R \rangle$ be a symmetrized presentation. We say a word $u \in F(S)$ is a *piece* if there are two $r_1 \neq r_2 \in R$ such that u is an initial subword of both r_1 and r_2. We also say the presentation satisfies the $C'(\lambda)$ *small cancellation hypothesis* if for every relation r and every piece u with $r = uv$, we have $|u| < \lambda |r|$.

Furthermore, we shall say a nonsymmetrized presentation satisfies the small cancellation hypothesis if it does once we replace the relation set with its symmetrized closure. We shall also refer to a group as a *small cancellation group* when it is clear which presentation we are using.

The key lemma we will need for small cancellation groups is the following, which says that every presentation of the trivial word must contain a long common subword with a relator.

Lemma 4.4 (Greendlinger's Lemma). *Let $G = \langle S \mid R \rangle$ be a $C'(\lambda)$ small cancellation group with $0 \leq \lambda \leq \frac{1}{6}$. Let w be a nontrivial freely reduced word representing the trivial element of G. Then there is a cyclic permutation r of a relation in R or its inverse with $r = uv$ such that u is a subword of w, and $|u| > (1 - 3\lambda)|r|$.*
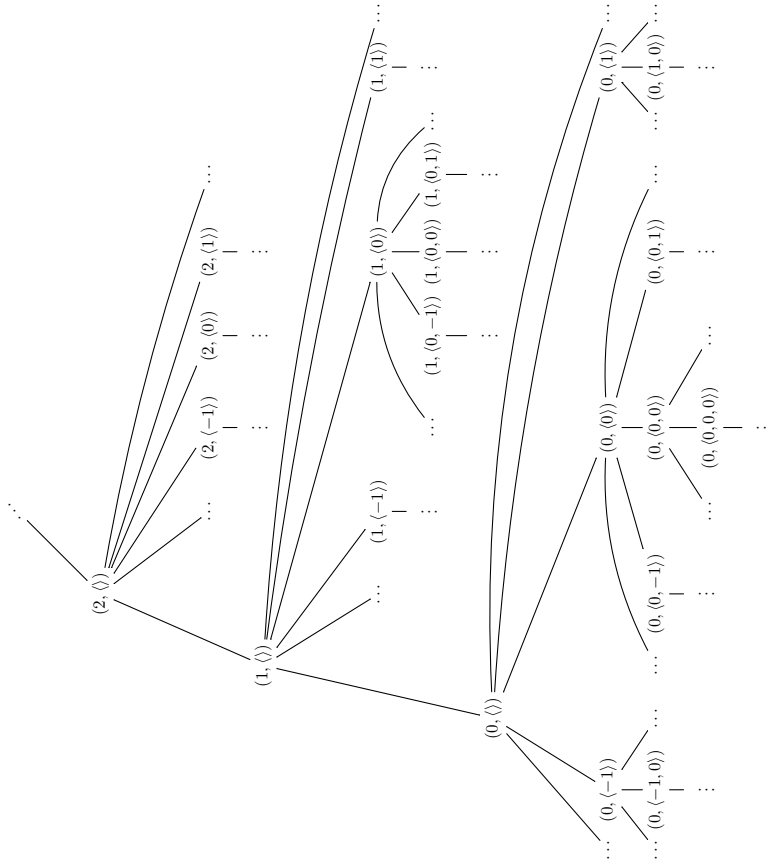
FIGURE 1. The tree T

We say that a word w is *Dehn-minimal* if it does not contain any subword v that is also a subword of a relator $r = vu$ such that $|v| > |r|/2$. Greendlinger's Lemma implies that, given a $C'(1/6)$ presentation of a group, we can solve the word problem using the following observation: a Dehn-minimal word is equivalent to the identity if and only if it is the trivial word. Given a word w, we replace w by equivalent words of shorter length until we have replaced w by a Dehn-minimal word w'. Then w is equivalent to the identity if and only if w' is the trivial word. This is Dehn's algorithm.

4.3. **A self-reflective group.** Let T be the tree (directed acyclic graph) with vertex set $V(T) = \{(n, \tau) \mid n \in \omega \text{ and } \tau \in \mathbb{Z}^{<\omega}\}$. The parent $(n, \tau)^-$ of (n, τ) is (n, τ^-) if $\tau \neq \langle\rangle$, and $(n + 1, \langle\rangle)$ otherwise. See Figure 1.

Let $u(x, y) = xyx^2y \cdots x^{100}y$ be a word in $F(x, y)$. Let K be the group on generators $V(T) \cup \{a\} \cup B$ (where $B = \{b_i \mid i \in \mathbb{Z}\}$) with relations:

- $u((n, \tau), a) = (n, \tau)^-$ for every $(n, \tau) \in T$.
- $u((n, \tau), b_i) = (n, \tau^\frown\langle i\rangle)$ for every $(n, \tau) \in T$ and $i \in \mathbb{Z}$.

Note that K is generated by $(0, \langle\rangle)$, a, and B: we can generate any vertex $(n, \langle\rangle)$ by $(1, \langle\rangle) = u((0, \langle\rangle), a)$, $(2, \langle\rangle) = u((1, \langle\rangle), a)$, and so on, and then we can generate, for

example, $(2, \langle 5, 3 \rangle)$, as $(2, \langle 5, 3 \rangle) = u(u((2, \langle \rangle), b_5), b_3)$. Also note that K is a $C'(\frac{1}{10})$ small cancellation group. Noting that any reduced word in B is Dehn-minimal, we see that B freely generates a free subgroup of K.

Claim 4.5. Let v be a word in $V(T), a$, and B such that v is Dehn-minimal. Then v is in the subgroup $F(B)$ of K generated by B if and only if v is a word in B.

Proof. The "if" direction is obvious. For the "only if" direction, assume we have a word v, in $V(T)$, a, and B, which is equal to a reduced word v' in B. If v' was the trivial word, then since v is Dehn-minimal, v would also be the trivial word. So we may assume that v' is not the trivial word. Also, we may assume without loss of generality that v and v' have no common prefix, so $v^{-1}v'$ is a reduced word. Then, by applying Greendlinger's Lemma to $v^{-1}v'$, we get a subword u of $v^{-1}v'$ which is also a subword of a relator r, with $|u| > (\frac{7}{10})|r|$. Noting that none of the relators of K have two consecutive b_i's, we see that the subword u of $v^{-1}v'$ given by Greendlinger's Lemma has to be contained in v^{-1}, except possibly the last letter of u. If u' is the part of u which is contained in v^{-1}, we have $|u'| \geq |u| - 1 > \frac{1}{2}|r|$ as $|r| > 100$. This contradicts the Dehn-minimality of v. $\square$

Now let G be the HNN extension $\langle K, t \mid tb_it^{-1} = \alpha(b_i) = b_{i+1} \rangle$ of K by the partial isomorphism $\alpha(b_i) = b_{i+1}$. G is then finitely generated by $(0, \langle \rangle), a, b_0$, and t.

Lemma 4.6. *G is self-reflective.*

Proof. Let $H \subseteq G$ be the subgroup generated by $(1, \langle \rangle), a, b_0$, and t. We claim that H is a proper Σ_1^0-elementary subgroup of G which is isomorphic to G.

Claim 4.7. H is isomorphic to G.

Proof. Define the homomorphism $\iota : G \rightarrow H \subseteq G$ given by sending (n, τ) to $(n + 1, \tau)$ and fixing a, b_i, and t. We must check that this does indeed define a homomorphism:

- $u(\iota(n, \tau), \iota(a)) = u((n+1, \tau), a) = (n+1, \tau)^- = \iota((n, \tau)^-)$ for every $(n, \tau) \in T$.
- $u(\iota(n, \tau), \iota(b_i)) = u((n + 1, \tau), b_i) = (n + 1, \tau\hat{\ }\langle i \rangle) = \iota((n, \tau\hat{\ }\langle i \rangle))$.
- $\iota(t)\iota(b_i)\iota(t)^{-1} = tb_it^{-1} = b_{i+1} = \iota(b_{i+1})$.

Since ι maps relators of G to relators of G, it defines a homomorphism.

Now we will check that ι is an embedding. Suppose $\iota(v) = 1$ for some word v in $V(T), a, B$, and t. Without loss, we may assume $v \neq 1$ is a word of minimum length among the words representing the same group element. By abusing notation, we will use $\iota(v)$ to denote the word obtained by replacing each (n, σ) in v by $(n+1, \sigma)$; this is a word that represents the group element $\iota(v)$.

Now since $\iota(v) = 1$, by Britton's Lemma, either $\iota(v)$ does not contain t, t^{-1} or it contains a subword tut^{-1} or $t^{-1}ut$ with $u \in F(B)$. We claim that we must be in the first case, where $\iota(v)$ (and hence v) does not contain t or t^{-1}. In the second case, if $\iota(v)$ does contain a subword tut^{-1} or $t^{-1}ut$ with $u \in F(B)$, we can write $u = \iota(u')$, where $tu't^{-1}$ or $t^{-1}u't$ appears as a subword of v as ι leaves t unchanged. If we can show that u is Dehn-minimal and so, by Lemma 4.5, u is actually a word in B, then, as ι leaves B unchanged, u' would also be a word in B. By conjugating each b_i in u' by t (or t^{-1}) to get b_{i+1} (or b_{i-1}), we get a shorter word representing the same element, contradicting the minimality of v. We will now argue that u is

Dehn-minimal. If u was not Dehn-minimal, this would be witnessed by a subword w of a relator r, with $|w| > \frac{1}{2}|v|$. Then looking at all of the relators of K, we see that $w = \iota(w')$ and $r = \iota(r')$, where w' is a subword of a relator r' of K, and also a subword of u'. Thus v is not of minimal length, a contradiction. So u is Dehn-minimal, and thus $\iota(v)$ does not contain t, t^{-1}.

Since $\iota(v) = 1$ and contains only $V(T)$, a, and B, by Greendlinger's Lemma, $\iota(v)$ is not Dehn-minimal. However, since any relator r that holds on $\iota(V(T))$, a, and B is the image, under ι, of a relator that holds on $V(T)$, a, and B, this shows that v is also not Dehn-minimal, a contradiction. □

Claim 4.8. H is a proper subgroup of G.

Proof. We will show that $(0, \langle \rangle) \notin H$. Suppose $(0, \langle \rangle) \in H$. Choose a shortest spelling v of $(0, \langle \rangle)$ in $\iota(V(T)), a, B$, and t. By applying Britton's Lemma to $(0, \langle \rangle)^{-1}v$ and using the same argument as above, we see that v does not contain t. Thus, we may apply Greendlinger's Lemma on $(0, \langle \rangle)^{-1}v$ to get a subword that is also a subword of some relator r with length more than half of the length of r. However, this subword cannot contain $(0, \langle \rangle)^{-1}$, as the only relation containing $(0, \langle \rangle)^{-1}$ but not $(0, \sigma)$ for any $\sigma \neq \langle \rangle$ is $u((0, \langle \rangle), a) = (1, \langle \rangle)$, but any long subword of it will contain more than one instance of $(0, \langle \rangle)$. Thus, the subword must be strictly in v and contradict the minimality of v. □

Claim 4.9. H is a Σ^0_1-elementary subgroup of G.

Proof. We only need to show that for every tuple $\overline{g} \in G$ and every quantifier-free formula $\psi(\overline{x}, \overline{y})$ such that $G \models \psi((1, \langle \rangle), a, b_0, t, \overline{g})$, there is a tuple $\overline{h} \in H$ such that $H \models \psi((1, \langle \rangle), a, b_0, t, \overline{h})$. It suffices to show the (stronger) statement that for every finite subset $1 \notin S \subset G$, there is a (retraction) $\kappa : G \to G$ such that $\kappa|_H = \text{id}_H$, $\kappa(G) = H$, and $1 \notin \kappa(S)$. Fixing a shortest spelling in $V(T), a, B$, and t for each element in S, we define κ by fixing the generators of H and sending $(0, \langle \rangle)$ to $(1, \langle n \rangle)$ for n sufficiently large relative to the (length and subscripts of the) spelling of elements of S.

Suppose there is $s \in S$ with $\kappa(s) = 1$. Write s in the shortest spelling fixed above. We spell $\kappa(s)$ by replacing every $(0, \tau)$ in the shortest spelling of s by $(1, n\char`^\tau)$. By Britton's Lemma, either there is no t in $\kappa(s)$ or there is a subword tvt^{-1} or $t^{-1}vt$ with $v \in F(B)$. In the second case, by minimality of s and Claim 4.5, we see that v only contains letters b_i, and thus we can reduce the length of s by replacing each b_i by b_{i+1} (or b_{i-1}) to get a shorter spelling of s, a contradiction. Thus, s does not have any t in it.

Now, applying Greendlinger's Lemma to $\kappa(s)$, we get a subword of $\kappa(s)$ that can be replaced by a shorter string. We will argue that a corresponding replacement can also be carried out for s, possibly with a different relator, contradicting the minimality of s. We divide into three cases, depending on which relator is used. First, note that the replacement cannot be given by any relator involving b_m for $|m| \geq n$ since $n \gg 1$ implies s does not contain the letter b_m in it; thus the following three cases exhaust the possibilities.

Case 1. The relator is $u((1, \langle i \rangle\char`^\sigma), a) = (1, \langle i \rangle\char`^\sigma)^-$ for $|i| \geq n$.

Since $n \gg 1$, each instance of $(1, \langle i \rangle\char`^\sigma)$ in $\kappa(s)$ comes from an instance of $(0, \sigma)$ in $s$, and each instance of $(1, \langle i \rangle\char`^\sigma)^-$ comes from an instance of $(0, \sigma)^-$ in s.

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(It is important here that $(0, \langle \rangle)^- = (1, \langle \rangle) = (1, \langle i \rangle)^-$.) Thus we can perform a replacement in s using the relator $u((0, \sigma), a) = (0, \sigma)^-$.

Case 2. The relator is $w((1, \langle i \rangle\,\hat{}\,\sigma), b_k) = (1, \langle i \rangle\,\hat{}\,\sigma\,\hat{}\,\langle k \rangle)$ for $|i| \geq n$.

Since $i \gg 1$, each instance of $(1, \langle i \rangle\,\hat{}\,\sigma)$ in $\kappa(s)$ comes from an instance of $(0, \sigma)$ in s, and each instance of $(1, \langle i \rangle\,\hat{}\,\sigma\,\hat{}\,\langle k \rangle)$ in $\kappa(s)$ comes form an instance of $(0, \sigma\,\hat{}\,\langle k \rangle)$ in s. Thus we can perform a replacement in s using $w((0, \sigma), b_k) = (0, \sigma\,\hat{}\,\langle k \rangle)$.

Case 3. The relator does not involve any letters $(1, \langle i \rangle\,\hat{}\,\sigma)$ with $|i| \geq n$.

In this case, we can apply exactly the same relator to s. $\qquad\square$

Thus we have shown that G contains a copy H of itself as a Σ_1^0-elementary subgroup, and hence is self-reflective. $\qquad\square$

Proposition 4.10. *G is computable.*

Proof. We use the following algorithm to solve the word problem in G: for any string in $V(T), a, B$, and t, we search and replace the following three types of subwords:
  (1) tvt^{-1} with v containing only b_i's. Replace such subwords by deleting t and t^{-1} and replacing each b_i by b_{i-1}.
  (2) $t^{-1}vt$ with v containing only b_i's. Replace such subwords by deleting t^{-1} and t and replacing each b_i by b_{i+1}.
  (3) Subword v such that v is also a subword of a relator r and $|v| > \frac{1}{2}|r|$. Replace such subwords by the rest of the relator r after deleting v.

Since any word can only mention finitely many letters, there are only finitely many possible relators for case (3). Thus, even though we have infinitely many relators, the search in (3) is still finite. Since these replacements shorten the length of the word, for any input word, sequences of such replacements terminate. If the resulting word is trivial, we output "The input word is equal to the identity", otherwise we output "The input word does not equal the identity".

To verify that this algorithm is valid, consider a word that represents the identity on which the algorithm terminates with a nontrivial word v. Since we cannot perform any more replacements of the third kind, v is Dehn-minimal. Thus, by Claim 4.5 and Britton's Lemma, we should be able to do a replacement of either the first or the second kind, a contradiction. $\qquad\square$

5. Finitely generated rings

In this section, we use the group ring construction to produce a ring that is self-reflective. Notice that the group ring $R[G]$ is computable if both G and R are computable.

Theorem 5.1. *Let G be the self-reflective group defined in Section* 4. *Then the group ring $R[G] = \{f : G \to R \mid |\operatorname{supp}(f)| < \infty\}$ over any finitely generated ring R is also self-reflective.*

Proof. Note that any endomorphism α of G induces an endomorphism α^* of $R[G]$ by precomposition and fixing R. Furthermore, if the endomorphism on G is injective, then the induced endomorphism on $R[G]$ is also injective.

Let ι be as defined in Lemma 4.6. Then ι^*, the induced endomorphism of $R[G]$, is also injective and not surjective. Say $B = \iota^*(R[G])$. Note that B is just $R[H]$, where $H = \iota(G)$.

Now, as in Lemma 4.6, it suffices to show that for every finite subset $0 \notin T \subset R[G]$, there is a retraction $\beta : R[G] \to R[G]$ with $\beta(R[G]) = B$, $\beta|_B = \mathrm{id}\,|_B$, and $0 \notin \beta(T)$. Let U be the set of group elements that appear in some members of T, and let $S = \{u_1 u_2^{-1} \mid u_1 \neq u_2 \in U\}$. Since $1 \notin S$, the proof of Lemma 4.6 gives a retraction $\kappa : G \to G$ such that $1 \notin \kappa(S)$. Now the induced endomorphism $\beta = \kappa^*$ is also a retraction. Furthermore, if $\kappa^*(t) = 0$ for some $t \in T$, since $1 \notin \kappa(S)$, κ^* is injective on the support of t, thus we must have $t = 0$, a contradiction. Thus $R[G]$ is also self-reflective. $\qquad\square$

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