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STOCHASTIC FIXED POINTS AND NONLINEAR PERRON–FROBENIUS THEOREM

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Abstract. We provide conditions for the existence of measurable solutions to the equation $\xi(T\omega) = f(\omega, \xi(\omega))$, where $T : \Omega \to \Omega$ is an automorphism of the probability space Ω and $f(\omega, \cdot)$ is a strictly nonexpansive mapping. We use results of this kind to establish a stochastic nonlinear analogue of the Perron–Frobenius theorem on eigenvalues and eigenvectors of a positive matrix. We consider a random mapping $D(\omega)$ of a random closed cone $K(\omega)$ in a finite-dimensional linear space into the cone $K(T\omega)$. Under the assumptions of monotonicity and homogeneity of $D(\omega)$, we prove the existence of scalar and vector measurable functions $\alpha(\omega) > 0$ and $x(\omega) \in K(\omega)$ satisfying the equation $\alpha(\omega)x(T\omega) = D(\omega)x(\omega)$ almost surely.

1. Introduction

Let $V = \mathbb{R}^n$ be a finite-dimensional real vector space with some norm $\|\cdot\|$. A subset K of V is called a *cone* if it contains with any vectors x and y any nonnegative linear combination $\alpha x + \beta y$ of these vectors. A cone is called *proper* if $K \cap (-K) = \{0\}$.

Let $K \subseteq V$ be a closed proper cone in V with nonempty interior K°; we will call such cones *solid*. The cone K induces the partial ordering $\leq_K$ in the space V defined as follows: $x \leq_K y$ if and only if $y - x \in K$. We shall write $x \prec_K y$ if $x \leq_K y$, $x \neq y$, and $x <_K y$ if $y - x \in K^\circ$.

Let L be another solid cone in V. A mapping $D : K \to L$ is called *monotone* if $D(x) \leq_L D(y)$ for any vectors $x, y \in K$ satisfying $x \leq_K y$. It is called *completely monotone* if each of the relations $x \leq_K y$, $x \prec_K y$ or $x <_K y$ between two vectors $x, y \in K$ implies the corresponding relation $D(x) \leq_L D(y)$, $D(x) \prec_L D(y)$, or $D(x) <_L D(y)$ between the vectors $D(x), D(y) \in L$. A mapping D is termed *strictly monotone* if the relation $x \prec_K y$ implies $D(x) <_L D(y)$.

Denote by V^* the dual to the space V. Elements of V^* are linear functionals $\phi(x) = \langle \phi, x \rangle$ on V. For any cone K, denote by

$$K^* = \{\phi \in V^* \ : \ \phi(x) \geq 0 \text{ for all } x \in K\}$$

the cone *dual* to K. If K is a solid cone, then so is K^*(see [25, Lemma 1.2.4]). Every functional in the interior of K^* is *strictly positive*, i.e., $\phi(x) > 0$ for all $0 \neq x \in K$.

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For any linear functional ϕ in the interior of K^*, put

$$\Sigma_\phi^K = \{x \in K : \ \phi(x) = 1\}.$$

The set Σ_ϕ^K is nonempty, compact, and convex (*ibid*).

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $T : \Omega \to \Omega$ be its automorphism, i.e., a one-to-one mapping such that T and T^{-1} are measurable and preserve the measure P. Let $\cdots \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ be a filtration on Ω such that each σ-algebra $\mathcal{F}_t$ is completed by $\mathcal{F}$-measurable sets of measure 0. Assume this filtration is invariant with respect to T, i.e. $\mathcal{F}_{t+1} = T^{-1}\mathcal{F}_t$ for each t. Suppose that for every $\omega \in \Omega$, a solid cone $K(\omega) \subseteq V$ depending $\mathcal{F}_0$-measurably[1] on ω is given. Put $K_t(\omega) = K(T^t\omega)$, $t = 0, \pm 1, \pm 2, \ldots$. Let $D(\omega, x)$ be a mapping of the cone $K_0(\omega)$ into the cone $K_1(\omega)$. Define

$$D_t(\omega, x) = D(T^{t-1}\omega, x), \ t = 0, \pm 1, \pm 2, \ldots.$$

For shortness, we will write $D(\omega)$ in place of $D(\omega, x)$ and $D_t(\omega)$ in place of $D_t(\omega, x)$. Put

$$C(t, \omega) = D_t(\omega) D_{t-1}(\omega) \cdots D_1(\omega), \ t = 1, 2, \ldots,$$

where the product means the composition of maps, and $C(0, \omega) = Id$ (the identity map). We have

$$C(t, T^s\omega) C(s, \omega) = C(t + s, \omega), \ t, s \geq 0,$$

i.e., the mapping $C(t, \omega)$ is a *cocycle* over the dynamical system $(\Omega, \mathcal{F}, P, T)$ (see Arnold [1]). The mapping $C(t, \omega)$ transforms elements of the cone $K_0(\omega)$ into elements of the cone $K_t(\omega)$.

It can be shown that the interior of the dual cone $K^*(\omega)$ depends $\mathcal{F}_0$-measurably on ω, and so there exists an $\mathcal{F}_0$-measurable linear functional $\phi(\omega)$ such that $\phi(\omega)$ belongs to the interior of $K^*(\omega)$ for each ω (this follows from [5, Theorems III.22 and III.30]). We fix the functional $\phi(\omega)$ and define $\hat{K}(\omega) = \Sigma_{\phi(\omega)}^{K(\omega)}$. The set $\hat{K}(\omega)$ is nonempty, compact and convex.

Let us extend the mapping $D(\omega, x)$ to all $x \in V$ by setting $\bar{D}(\omega, x) = D(\omega, x)$ if $x \in K(\omega)$ and $\bar{D}(\omega, x) = \infty$ for $x \notin K(\omega)$, where "∞" stands for a one-point compactification of V. We will impose the following conditions:

(**D1**) $D(\omega, x)$ is continuous in $x \in K(\omega)$ and $\bar{D}(\omega, x)$ is $\mathcal{F}_1 \times \mathcal{V}$-measurable in $(\omega, x) \in \Omega \times V$, where $\mathcal{V}$ is the Borel σ-algebra on V.

(**D2**) $D(\omega, x)$ is positively homogeneous (of degree one) in $x \in K(\omega)$:

$$D(\omega, \lambda x) = \lambda D(\omega, x) \ \text{ for any } \lambda > 0, \ x \in K(\omega).$$

(**D3**) $D(\omega, x)$ is a completely monotone mapping from $K_0(\omega)$ into $K_1(\omega)$.

Furthermore, we will assume that the cocycle $C(t, \omega)$ satisfies the following condition:

(**C**) For almost all $\omega \in \Omega$ there is a natural number l_ω such that the mapping $C(l_\omega, \omega)$ is strictly monotone.

[1] We say that the set-valued mapping $\omega \mapsto K(\omega) \subseteq V$ is $\mathcal{F}_0$-measurable (or the set $K(\omega)$ depends $\mathcal{F}_0$-measurably on ω) if its graph $\{(\omega, x) : x \in K(\omega)\}$ belongs to $\mathcal{F}_0 \times \mathcal{V}$, where $\mathcal{V}$ is the Borel σ-algebra on V.

The main result of this paper is as follows.

Theorem 1.

(a) *There exist an $\mathcal{F}_0$-measurable vector function $x(\omega)$ and an $\mathcal{F}_1$-measurable scalar function $\alpha(\omega)$ such that*

$$\alpha(\omega) > 0,\ x(\omega) \in K^\circ(\omega),\ \langle \phi(\omega), x(\omega)\rangle = 1$$

for all ω and

$$\alpha(\omega)x(T\omega) = D(\omega)x(\omega) \text{ (a.s.)}. \tag{1.1}$$

(b) *The pair of functions $(\alpha(\omega), x(\omega))$, where $\alpha(\omega) \geq 0$, $x(\omega) \in K(\omega)$, and $\langle \phi(\omega), x(\omega)\rangle = 1$, satisfying* (1.1) *is determined uniquely up to the equivalence with respect to the measure P.*

(c) *If $t \to \infty$, then*

$$\left\| \frac{C(t, T^{-t}\omega)a}{\langle \phi(\omega), C(t, T^{-t}\omega)a\rangle} - x(\omega)\right\| \to 0 \quad \text{(a.s.)}, \tag{1.2}$$

where convergence is uniform in $0 \neq a \in K_{-t}(\omega)$.

This result may be regarded as a stochastic nonlinear generalization of the Perron–Frobenius theorem: $x(\cdot)$ and $\alpha(\cdot)$ play the roles of an "eigenvector" and an "eigenvalue" of the random mapping $D(\omega)$ with respect to the dynamical system $T : \Omega \to \Omega$. The original versions of this classical theorem were discovered at the beginning of the 20th century by Perron [37, 38], who investigated eigenvalues and eigenvectors of matrices with strictly positive entries, and by Frobenius [15–17], who extended Perron's results to irreducible nonnegative matrices. Extensions of the Perron–Frobenius results to nonlinear mappings were obtained by H. Nikaidô [33], M. Morishima [30], T. Fujimoto [18], Y. Oshime [34–36], and others. Those extensions were motivated by applications in mathematical economics, in particular, to the so-called nonlinear Leontief model [39]. For reviews of nonlinear versions of the Perron–Frobenius theory, we refer the reader to the monographs by Nussbaum [31, 32] and the papers by Kohlberg [23] and Gaubert and Gunawardena [19].

The first result on stochastic generalizations of the Perron–Frobenius theorem for linear maps $D(\omega)$ (nonnegative random matrices) was obtained in [10]. The result was extended and applied to mathematical models in statistical physics and evolutionary biology by Arnold et al. [2]. Recently, important progress has been made in a series of papers by Mierczyński and Shen [27–29], who established stochastic versions of the Krein–Rutman theorem (generalizing the Perron–Frobenius theorem to infinite-dimensional Banach spaces). Under some general conditions, it was shown that a positive linear random dynamical system in an ordered Banach space admits a family of generalized principal Floquet subspaces, a generalized principal Lyapunov exponent, and a generalized exponential separation. These results extend to the stochastic case the classical Krein–Rutman theorem for strongly positive and compact operators in strongly ordered Banach spaces. In the paper [26] by Lian and Wang, a stochastic Krein–Rutman theory for general *k-cones* was developed, extending the classical results both from the deterministic to the stochastic case and from $k = 1$ to $k > 1$.

The first stochastic nonlinear analogue of the Perron–Frobenius theorem was obtained in the paper by Evstigneev and Pirogov [12]. In that paper, $D(\omega)$ was a mapping of the set $\mathbb{R}^n_+$ of nonnegative n-dimensional vectors into itself. Now we generalize this result to more general random cones $K(\omega) \subseteq V$.

Problems related to stochastic (linear and nonlinear) Perron–Frobenius theorems arise in various areas of pure and applied mathematics, in particular, in statistical physics, ergodic theory, mathematical biology, and mathematical finance; see, e.g., Arnold et al. [2], Kifer [21, 22], and Dempster et al. [7]. Extensions of this theory to set-valued mappings $D(\omega, x)$ (von Neumann–Gale dynamical systems [3], [13]) have important applications in mathematical economics [14] and finance [8].

Several comments about the assumptions imposed are in order. Let K and L be solid cones in V. Consider a *concave* mapping $D : K \to L$, i.e., a mapping satisfying

$$D(\theta x + (1-\theta)y) \geq_L \theta D(x) + (1-\theta)D(y) \tag{1.3}$$

for all $x, y \in K$ and $\theta \in [0, 1]$. Clearly, if D is homogeneous, then D is concave if and only if it is *superadditive*:

$$D(x + y) \geq_L D(x) + D(y). \tag{1.4}$$

For a superadditive mapping $D : K \to L$, the relation $x \prec_K y$ between two vectors $x, y \in K$ implies the corresponding relation $D(x) \prec_L D(y)$ between the vectors $D(x), D(y) \in L$ if and only if

(**M1**) $D(h) \succ_L 0$ for all $h \succ_K 0$.

The relation $x <_K y$ implies the corresponding relation $D(x) <_L D(y)$ if and only if

(**M2**) $D(h) >_L 0$ for all $h >_K 0$.

The mapping $D(x)$ is strictly monotone if and only if

(**M3**) $D(h) >_L 0$ for all $h \succ_K 0$.

We can also see from (1.4) that any superadditive mapping is monotone. By using this, we obtain that if D is concave and homogeneous, then (**M2**) is equivalent to

(**M4**) $D(h_*) >_L 0$ for some $h_* \geq_K 0$.

Clearly, (**M4**) follows from (**M2**). Conversely, (**M4**) implies (**M2**) because for any $h >_K 0$ we have $h \geq_K \lambda h_*$, where $\lambda > 0$ which yields $D(h) \geq_L \lambda D(h_*) >_L 0$. Thus, for a concave homogeneous mapping, its complete monotonicity is equivalent to the validity of (**M1**) and (**M2**) (or (**M1**) and (**M4**)), and its strict monotonicity is equivalent to (**M3**).

The remainder of the paper is organized as follows. In Section 2, some general facts regarding the Hilbert–Birkhoff metric are given. In Section 3, a stochastic version of the fixed point principle, which plays a key role in the proof of Theorem 1, is formulated. Sections 4 and 5 provide proofs of the main results.

2. Hilbert–Birkhoff metric

Given a solid cone K and a strictly positive linear functional ϕ, the Hilbert–Birkhoff ([20], [4]) metric on the set $Y := \Sigma_\phi^K \cap K^\circ$ is defined as follows. For any $x, y \in Y$ put

$$M(x/y) = \inf\{\beta > 0 : x \leq_K \beta y\}, \ m(x/y) = \sup\{\alpha > 0 : \alpha y \leq_K x\}$$

and

$$d(x, y) = \log\left[\frac{M(x/y)}{m(x/y)}\right]. \tag{2.1}$$

It can be shown (see [25, Propositions 2.1.1 and 2.5.4]) that the function $d(x, y)$ is a complete separable metric on Y and that the topology generated by it on Y

coincides with the Euclidean topology on Y. Furthermore, there exists a constant $M > 0$ such that

$$\|x - y\| \leq M(e^{d(x,y)} - 1), \tag{2.2}$$

for all $x, y \in Y$ (see [25, formula (2.21)]).

Remark. An important example of K is the cone $\mathbb{R}^n_+$ consisting of all nonnegative vectors in $V = \mathbb{R}^n$. Suppose $\phi(x) = \sum_{i=1}^n x_i$ for $x = (x_1, \dots, x_n)$. Then we have

$$\Sigma^K_\phi = \{x \geq 0 : \sum_{i=1}^n x_i = 1\},\ Y = \{x > 0 : \sum_{i=1}^n x_i = 1\},$$

$$M(x/y) = \max_i (x_i/y_i),\ m(x/y) = \min_j (x_j/y_j),$$

$$d(x, y) = \log \left[\max_i (x_i/y_i) \cdot \max_j (y_j/x_j)\right].$$

Here, the inequalities $x \geq 0$ and $x > 0$ are understood coordinate-wise.

The Hilbert–Birkhoff metric is a particularly useful tool in the study of monotone homogeneous maps on cones. A mapping $f : X \to Y$ from a metric space (X, d_X) into a metric space (Y, d_Y) is called *nonexpansive* if $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$. It is called *strictly nonexpansive* if the inequality in the above formula is strict for all $x \neq y$ in X. The usefulness of the Hilbert–Birkhoff metric lies in the fact that linear, and some nonlinear, mappings of cones are nonexpansive with respect to this metric.

Let K and L be solid cones in V and let $\phi_1 \in (K^*)^\circ$, $\phi_2 \in (L^*)^\circ$. Put $Y_1 = \Sigma^K_{\phi_1} \cap K^\circ$ and $Y_2 = \Sigma^L_{\phi_2} \cap L^\circ$, and suppose $d_i(x, y)$ is Hilbert–Birkhoff metric on Y_i, $i = 1, 2$.

Theorem 2. *If $f : K \to L$ is a monotone and homogeneous (of degree 1) mapping such that $f(x) \succ_L 0$ for all $x \succ_K 0$, then the mapping $g : Y_1 \to Y_2$ given by $g(x) = f(x)/\langle \phi_2, f(x)\rangle$ is nonexpansive with respect to the metric $d_1(x, y)$ on Y_1 and the metric $d_2(x, y)$ on Y_2. Moreover, if f is strictly monotone and homogeneous, then g is strictly nonexpansive.*

Proof. Let $x, y \in Y_1$ and write $\alpha = m(x/y)$, $\beta = M(x/y)$. Since K is a closed cone, we have $\alpha y \leq_K x \leq_K \beta y$, and so $\alpha f(y) \leq_L f(x) \leq_L \beta f(y)$ because f is monotone and homogeneous. Thus,

$$\alpha \frac{\langle \phi_2, f(y)\rangle}{\langle \phi_2, f(x)\rangle} g(y) \leq_L g(x) \leq_L \beta \frac{\langle \phi_2, f(y)\rangle}{\langle \phi_2, f(x)\rangle} g(y),$$

which implies

$$d_2(g(x), g(y)) \leq \log(\beta/\alpha) = d_1(x, y).$$

Let f be strictly monotone. If $x, y \in Y_1$ and $x \neq y$, we have $x \neq \lambda y$ for all $\lambda > 0$ (otherwise if $x = \lambda y$ for some $\lambda > 0$, then $1 = \phi_1(y) = \phi_1(x) = \lambda \phi_1(y)$, which yields $\lambda = 1$ and $x = y$). Then $\alpha y \prec_K x \prec_K \beta y$, and so $\alpha f(y) <_L f(x) <_L \beta f(y)$. Hence there exist $\mu > \alpha$ and $\tau < \beta$ such that $\mu f(y) \leq_L f(x) \leq_L \tau f(y)$. Therefore,

$$d_2(g(x), g(y)) \leq \log(\frac{\tau}{\mu}) < \log(\frac{\beta}{\alpha}) = d_1(x, y).$$

The proof is complete. $\square$

3. Stochastic fixed-point principle

In the proof of Theorem 1, we will use a stochastic generalization of the following well-known result regarding strictly nonexpansive mappings (see, e.g., [9], [23]). Let f be a strictly nonexpansive mapping from a compact space X into itself. Then f has a unique fixed point $\overline{x}$, and $f^k(x) \to \overline{x}$ as $k \to \infty$ for each $x \in X$. (We denote by $f^k(x)$ the kth iterate of the mapping f.) A stochastic version of the above contraction principle was obtained in the paper by Evstigneev and Pirogov [11]. Here we establish a more general version of this result. Let us formulate it.

As before, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $T : \Omega \to \Omega$ be its automorphism, and let $\cdots \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ be a filtration such that each $\mathcal{F}_t$ contains all sets in $\mathcal{F}$ of measure 0. Let $(V, \mathcal{V})$ be a standard[2] measurable space and let $X(\omega) \subseteq V$ be a nonempty set depending $\mathcal{F}_0$-measurably on $\omega \in \Omega$. Let $f(\omega, x)$ be a mapping assigning to every $\omega \in \Omega$ and every $x \in X(\omega)$ an element $f(\omega, x) \in X(T\omega)$. Our main goal in this section is to provide conditions under which the equation

$$\xi(T\omega) = f(\omega, \xi(\omega)) \text{ (a.s.)} \tag{3.1}$$

has a solution in the class of measurable mappings $\xi : \Omega \to V$ such that $\xi(\omega) \in X(\omega)$ for all ω. We also will be interested in the uniqueness of this solution and the properties of its stability. Equations of the type (3.1) arise in connection with various questions of the theory of random dynamical systems (Arnold [1]). Our study of such equations is motivated by their applications in the stochastic Perron–Frobenius theory.

Let us extend $f(\omega, x)$ to the whole space V by setting $\bar{f}(\omega, x) = f(\omega, x)$ if $x \in X(\omega)$ and $\bar{f}(\omega, x) = \infty$ if $x \notin X(\omega)$, where the symbol "∞" denotes a point added to V.

Assume that the following conditions hold:

(**A1**) The mapping $\bar{f}(\omega, x)$ ($\omega \in \Omega$, $x \in V$) is $\mathcal{F}_1 \times \mathcal{V}$-measurable.

For each ω, let $Y(\omega)$ be a nonempty subset of $X(\omega)$ equipped with a separable metric $\rho(\omega, x, y)$, $x, y \in Y(\omega)$. Let us introduce the following assumptions:

(**A2**)

(a) The set-valued mapping $\omega \mapsto Y(\omega)$ is $\mathcal{F}_0$-measurable.

(b) The function

$$\bar{\rho}(\omega, x, y) := \begin{cases} \rho(\omega, x, y), & \text{if } x, y \in Y(\omega), \\ +\infty, & \text{otherwise} \end{cases}$$

is $\mathcal{F}_0 \times \mathcal{V} \times \mathcal{V}$-measurable.

(c) For each ω, the Borel measurable structure on $Y(\omega)$ induced by the metric $\rho(\omega, x, y)$ coincides with the measurable structure induced on $Y(\omega)$ by the σ-algebra $\mathcal{V}$.

(d) For each $\omega \in \Omega$ and $x \in Y(\omega)$, we have $f(\omega, x) \in Y(T\omega)$, and the mapping $f(\omega, \cdot) : Y(\omega) \to Y(T\omega)$ is continuous with respect to the metric $\rho(\omega, x, y)$ on $Y(\omega)$ and the metric $\rho(T\omega, x, y)$ on $Y(T\omega)$.

Note that $Y(\omega) \in \mathcal{V}$ for each ω by virtue of (a).

For every $k = 0, \pm 1, \pm 2, \ldots$ define

$$X_k(\omega) = X(T^k\omega),\ Y_k(\omega) = Y(T^k\omega),\ \rho_k(\omega, x, y) = \rho(T^k\omega, x, y),$$

[2] A measurable space is called standard if it is isomorphic to a Borel subset of a complete separable metric space.

$$f_k(\omega, x) = f(T^{k-1}\omega, x) \; [x \in X_{k-1}(\omega)]. \tag{3.2}$$

For each $m = 0, 1, 2, \dots$ put

$$f^{(m)}(\omega, x) = f_0(\omega) f_{-1}(\omega) \cdots f_{-m}(\omega)(x) \; [x \in X_{-m-1}(\omega)], \tag{3.3}$$

$$X^{(m)}(\omega) = f^{(m)}(\omega, X_{-m-1}(\omega)).$$

The product $f^{(m)}(\omega, x) = f_0(\omega) f_{-1}(\omega) \cdots f_{-m}(\omega)$ means the composition of the mappings. Note that for each $m = 0, 1, \dots$ the map $f_{-m}(\omega, x)$ acts from $\Omega \times X_{-m-1}(\omega)$ into $X_{-m}(\omega)$, and so $f^{(m)}(\omega, x)$ acts from $\Omega \times X_{-m-1}(\omega)$ into $X_0(\omega)$. The extended mappings $\bar{f}_{-m}(\omega, x)$ and $\bar{f}^{(m)}(\omega, x)$ are $\mathcal{F}_{-m} \times \mathcal{V}$-measurable and $\mathcal{F}_0 \times \mathcal{V}$-measurable, respectively. The functions $\bar{\rho}_{-m}(\omega, x, y)$ are measurable with respect to $\mathcal{F}_{-m} \times \mathcal{V} \times \mathcal{V} \subseteq \mathcal{F}_0 \times \mathcal{V} \times \mathcal{V}$.

(**A3**) There is a sequence of $\mathcal{F}_0$-measurable sets $\Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega$ such that $P(\Omega_m) \to 1$, and for each $m = 0, 1, \dots$ and $\omega \in \Omega_m$ the following conditions are satisfied:

(a) the set $X^{(m)}(\omega)$ is contained in $Y(\omega)$ and is compact with respect to the metric $\rho(\omega, x, y)$;

(b) for all $x, y \in Y_{-m-1}(\omega)$ with $x \neq y$, we have

$$\rho(\omega, f^{(m)}(\omega, x), f^{(m)}(\omega, y)) < \rho_{-m-1}(\omega, x, y). \tag{3.4}$$

Since the sequence of the sets Ω_m is nondecreasing, there exists an $\mathcal{F}_0$-measurable function $m(\omega)$ with nonnegative integer values such that for each $\omega \in \bar{\Omega} := \Omega_1 \cup \Omega_2 \cup \cdots$ (and hence for almost all ω), we have $\omega \in \Omega_m$, $m \geq m(\omega)$. We can define $m(\omega) = \min\{i : \omega \in \Omega_i\}$ if $\omega \in \bar{\Omega}$, and $m(\omega) = 0$ otherwise.

Theorem 3.

(i) *There exists an* $\mathcal{F}_0$*-measurable mapping* $\xi : \Omega \to V$ *such that* $\xi(\omega) \in Y(\omega)$*, equation* (3.1) *holds, and*

$$\lim_{m(\omega) \leq m \to \infty} \sup_{x \in X_{-m-1}(\omega)} \rho(\omega, \xi(\omega), f_0(\omega) \cdots f_{-m}(\omega)(x)) = 0 \;\; \textit{(a.s.)}. \tag{3.5}$$

(ii) *If* $\eta : \Omega \to V$ *is any (not necessarily measurable) mapping for which* $\eta(\omega) \in X(\omega)$ *and equation* (3.1) *holds, then* $\eta = \xi$ *with probability one.*

According to (3.5), the sequence $f_0 \cdots f_{-m}(x)$ converges to $\xi(\omega)$ in the metric $\rho(\omega, x, y)$ uniformly in $x \in X_{-m-1}(\omega)$ with probability one. Note that the distance $\rho(\omega, \cdot, \cdot)$ between $f_0 \cdots f_{-m}(x)$ and $\xi(\omega)$ involved in (3.5) is defined only if $f_0 \cdots f_{-m}(x) \in Y(\omega)$. By virtue of condition (a) in (**A3**), this inclusion holds for almost all ω, and all $m \geq m(\omega)$ and $x \in X_{-m-1}(\omega)$; therefore the limit in (3.5) is taken over $m \geq m(\omega)$.

4. Proof of the stochastic fixed point principle

Proof of Theorem 3.

1*st step.* Observe that $X^{(0)}(\omega) \supseteq X^{(1)}(\omega) \supseteq X^{(2)}(\omega) \supseteq \cdots$ and $X^{(m)}(\omega) \neq \emptyset$ for each m and ω. Consider the sets Ω_m $(m = 0, 1, \dots)$ described in (**A3**) and their union $\bar{\Omega}$. According to (**A3**), $P(\bar{\Omega}) = 1$ and each $\omega \in \bar{\Omega}$ belongs to all Ω_m, $m \geq m(\omega)$. For $\omega \in \bar{\Omega}$, all the sets $X^{(m)}(\omega)$, $m \geq m(\omega)$, are contained in $Y(\omega)$ and compact, and so the set $X^{\infty}(\omega) := \bigcap_{m=0}^{\infty} X^{(m)}(\omega) \subseteq Y(\omega)$ is nonempty and compact as an intersection of a nested sequence of nonempty compacta $X^{(m)}(\omega)$, $m \geq m(\omega)$.

2nd step. Define $\Omega^* = \bigcap_{k=-\infty}^{+\infty}(T^k\bar{\Omega})$. The set Ω^* is invariant and $P(\Omega^*) = 1$. Let us show that

$$X^\infty(T\omega) = f(\omega, X^\infty(\omega)),\ \omega \in \Omega^*. \tag{4.1}$$

Equality (4.1) is equivalent to

$$X^\infty(\omega) = f(T^{-1}\omega, X^\infty(T^{-1}\omega)),\ \omega \in \Omega^*, \tag{4.2}$$

because $\omega \in \Omega^*$ if and only if $T^{-1}\omega \in \Omega^*$. To prove (4.2) let us observe that

$$f(T^{-1}\omega, \bigcap_{m=0}^{\infty} X^{(m)}(T^{-1}\omega)) = \bigcap_{m=0}^{\infty} f(T^{-1}\omega, X^{(m)}(T^{-1}\omega)),\ \omega \in \Omega^*. \tag{4.3}$$

The inclusion "$\subseteq$" in (4.3) holds always. The opposite inclusion follows from the continuity of $f(T^{-1}\omega, \cdot)$ on $Y(T^{-1}\omega)$ and the fact that $X^{(m)}(T^{-1}\omega)$ are nested and compact in $Y(T^{-1}\omega)$ for all m large enough. By using (4.3), we obtain

$$\begin{aligned} f(T^{-1}\omega, X^\infty(T^{-1}\omega)) &= f(T^{-1}\omega, \bigcap_{m=0}^{\infty} X^{(m)}(T^{-1}\omega)) \\ = \bigcap_{m=0}^{\infty} f(T^{-1}\omega, X^{(m)}(T^{-1}\omega)) &= \bigcap_{m=0}^{\infty} f_0(\omega, X^{(m)}(T^{-1}\omega)) \\ &= \bigcap_{m=0}^{\infty} X^{(m+1)}(\omega) = X^\infty(\omega),\ \omega \in \Omega^*. \end{aligned}$$

The fourth equality in this chain of relations holds because

$$\begin{aligned} X^{(m)}(T^{-1}\omega) &= f_0(T^{-1}\omega) f_{-1}(T^{-1}\omega) \cdots f_{-m}(T^{-1}\omega)(X_{-m-1}(T^{-1}\omega)) \\ &= f_{-1}(\omega) f_{-2}(\omega) \cdots f_{-m-1}(\omega)(X_{-m-2}(\omega)), \end{aligned}$$

and so

$$f_0(\omega)(X^{(m)}(T^{-1}\omega)) = f_0(\omega) f_{-1}(\omega) \cdots f_{-m-1}(\omega)(X_{-m-2}(\omega)) = X^{(m+1)}(\omega).$$

3rd step. For $\omega \in \Omega^*$, denote the diameter in the metric $\rho(\omega, x, y)$ of the compact set $X^\infty(\omega) \subseteq Y(\omega)$ by $\rho(\omega)$ and put $\rho(\omega) = +\infty$ if $\omega \in \Omega \setminus \Omega^*$. For $m = 0, 1, 2, \ldots$, put $\Omega_m^* := \Omega^* \cap \Omega_m$, and for $\omega \in \Omega$ define

$$\rho^{(m)}(\omega) = \begin{cases} \operatorname{diam} X^{(m)}(\omega), & \text{if } \omega \in \Omega_m^*, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.4}$$

Recall that, for $\omega \in \Omega_m$ and hence for $\omega \in \Omega_m^*$, the set $X^{(m)}(\omega)$ is contained in $Y(\omega)$ and is compact, so that its diameter $\operatorname{diam} X^{(m)}(\omega)$ in the metric $\rho(\omega, x, y)$ is well defined and finite. We claim that $\rho^{(m)}(\omega)$ is an $\mathcal{F}_0$-measurable function of $\omega \in \Omega$. To prove this assertion we observe that for $\omega \in \Omega_m^*$, we have $\operatorname{diam} X^{(m)}(\omega) = \operatorname{diam} f^{(m)}(\omega, X_{-m-1}(\omega))$, where

$$f^{(m)}(\omega, x) = f_0(\omega) f_{-1}(\omega) \cdots f_{-m}(\omega)(x), x \in X_{-m-1}(\omega).$$

Consequently, for each real a, the set Ω_m^a of $\omega \in \Omega_m^*$ satisfying $\operatorname{diam} X^{(m)}(\omega) > a$ is the projection on Ω_m^* of the set

$$\{(\omega, x, y) \in \Omega_m^* \times X_{-m-1}(\omega) \times X_{-m-1}(\omega) :\ \rho(\omega, f^{(m)}(\omega, x),\ f^{(m)}(\omega, y)) > a\ \}, \tag{4.5}$$

which is an $\mathcal{F}_0 \times \mathcal{V} \times \mathcal{V}$-measurable subset in $\Omega_m^* \times V \times V$ by virtue of assumptions (**A1**) and (**A2**). Since V (and hence $V \times V$) is standard and $(\Omega, \mathcal{F}_0, P)$ is a complete probability space, Ω_m^a is $\mathcal{F}_0$-measurable (see, e.g., [6, Theorem III.33]). This implies that $\rho^{(m)}(\omega)$ is $\mathcal{F}_0$-measurable because $\rho^{(m)}(\omega) = +\infty$ outside Ω_m^*. Finally, $\rho(\omega)$ is $\mathcal{F}_0$-measurable because

$$\rho(\omega) = \lim_{m\to\infty} \rho^{(m)}(\omega) \text{ for } \omega \in \Omega^*, \tag{4.6}$$

which follows the fact that $X^{(m)}(\omega)$ are nested and compact in $Y(\omega)$ for all $\omega \in \Omega^*$ and $m \geq m(\omega)$.

4*th step.* Let us show that $\rho(\omega) = 0$ (a.s.). Observe that equality (4.1) implies

$$X^\infty(\omega) = f(T^{-1}\omega, X^\infty(T^{-1}\omega)) = f(T^{-1}\omega)(X^\infty(T^{-1}\omega))$$

$$= f(T^{-1}\omega)f(T^{-2}\omega)(X^\infty(T^{-2}\omega)) = \cdots = f(T^{-1}\omega)\cdots f(T^{-m-1}\omega)(X^\infty(T^{-m-1}\omega))$$

$$= f_0(\omega)\cdots f_{-m}(\omega)(X^\infty(T^{-m-1}\omega)) = f^{(m)}(\omega, X^\infty(T^{-m-1}\omega)),\ \omega \in \Omega^*. \tag{4.7}$$

By virtue of (4.7) and condition (b) in (**A3**), for $\omega \in \Omega_m^*$, we have

$$\rho(\omega) = \rho(\omega, X^\infty(\omega)) \leq \rho(T^{-m-1}\omega, X^\infty(T^{-m-1}\omega)) = \rho(T^{-m-1}\omega) \tag{4.8}$$

and

$$\text{if } \rho(\omega) > 0, \text{ then } \rho(\omega) < \rho(T^{-m-1}\omega). \tag{4.9}$$

(Here we also use the fact that $X^\infty(\omega)$ is compact.) Since $P(\Omega_m^*) = P(\Omega^* \cap \Omega_m) \to 1$, inequality (4.8) yields

$$\lim_{m\to\infty} P\{\rho(\omega) \leq \rho(T^{-m}\omega)\} \to 1. \tag{4.10}$$

We claim that (4.10) implies

$$\rho(\omega) = \rho(T^{-m}\omega) \text{ a.s. for all } m. \tag{4.11}$$

To deduce (4.11) from (4.10) we may assume that $\rho(\omega)$ is bounded by some constant C (we can always replace $\rho(\omega)$ by $\arctan \rho(\omega)$). By setting $\Delta_m := \{\omega : \rho(\omega) \leq \rho(T^{-m}\omega)\}$, we write

$$E|\rho(\omega) - \rho(T^{-m}\omega)| \leq E(\rho(T^{-m}\omega) - \rho(\omega))\chi_{\Delta_m} + CP(\Omega \setminus \Delta_m),$$

where χ_{Δ_m} is the indicator function of Δ_m. Further, since $E\rho(T^{-m}\omega) = E\rho(\omega)$, we have

$$\begin{aligned} E(\rho(T^{-m}\omega) - \rho(\omega))\chi_{\Delta_m} &= E(\rho(T^{-m}\omega) - \rho(\omega))\chi_{\Delta_m} - E(\rho(T^{-m}\omega) - \rho(\omega)) \\ &= -E(\rho(T^{-m}\omega) - \rho(\omega))\chi_{\Omega\setminus\Delta_m} \leq CP(\Omega \setminus \Delta_m). \end{aligned}$$

Consequently, $E|\rho(\omega) - \rho(T^{-m}\omega)| \leq 2CP(\Omega \setminus \Delta_m) \to 0$, which implies (4.11).

Suppose $\rho(\omega) > 0$ with strictly positive probability. Then there exists a number m and a set $\Gamma \in \mathcal{F}_0$ contained in Ω_m^* such that $P(\Gamma) > 0$ and $\rho(\omega) > 0$ on Γ. By virtue of (4.9), we have $\rho(\omega) < \rho(T^{-m-1}\omega)$ for $\omega \in \Gamma$. On the other hand, we proved that $\rho(\omega) = \rho(T^{-m-1}\omega)$ for almost all ω. A contradiction.

5*th step.* Since the $\mathcal{F}_0$-measurable function $\rho(\omega)$ is zero a.s., there is a set $\tilde{\Omega} \in \mathcal{F}_0$ of full measure such that

$$\tilde{\Omega} \subseteq \Omega^* \text{ and } \rho(\omega) = 0 \text{ for each } \omega \in \tilde{\Omega}. \tag{4.12}$$

This means that for $\omega \in \tilde{\Omega}$, the set $X^\infty(\omega)$ consists of exactly one point, $\xi^\infty(\omega)$. Replacing $\tilde{\Omega}$ by $\bigcap_{k=-\infty}^{+\infty}(T^k\tilde{\Omega})$, we may assume that $\tilde{\Omega}$ is invariant.

For every ω, fix any point $\tilde{y}(\omega)$ in the nonempty set $Y(\omega)$ and put $\xi(\omega) = \xi^{\infty}(\omega)$ for $\omega \in \tilde{\Omega}$ and $\xi(\omega) = \tilde{y}(\omega)$ for $\omega \in \Omega \setminus \tilde{\Omega}$. Then for any $\omega \in \tilde{\Omega} \subseteq \Omega^*$ we have $T\omega \in \tilde{\Omega} \subseteq \Omega^*$, and so

$$\begin{aligned}\{\xi(T\omega)\} = \{\xi^{\infty}(T\omega)\} = X^{\infty}(T\omega) = f(\omega, X^{\infty}(\omega))\\ = f(\omega, \{\xi^{\infty}(\omega)\}) = f(\omega, \{\xi(\omega)\}\}\end{aligned}$$

by virtue of (4.1). Consequently, $\xi(\omega)$ satisfies (3.1) for all ω in the set $\tilde{\Omega} \subseteq \Omega^* \subseteq \bar{\Omega}$ of measure one.

Consider the functions $\rho^{(m)}(\omega)$ defined by (4.4). For each $\omega \in \tilde{\Omega}$ and $m \geq m(\omega)$ we have $\omega \in \Omega_m$, and so

$$\sup_{x \in X_{-m-1}(\omega)} \rho(\omega, \xi(\omega), f_0 \cdots f_{-m}(x)) \leq \operatorname{diam} X^{(m)}(\omega) = \rho^{(m)}(\omega) \tag{4.13}$$

because $\{\xi(\omega)\} = X^{\infty}(\omega) \subseteq X^{(m)}(\omega)$. This implies (3.5) since $\lim \rho^{(m)}(\omega) = \rho(\omega) = 0$ on the set $\tilde{\Omega}$ of full measure.

6*th step.* To complete the proof of (i) it is sufficient to show that the mapping ξ constructed above coincides a.s. with some $\mathcal{F}_0$-measurable mapping ζ. Then ζ will be the sought-after solution to (3.1) possessing the properties listed in (i).

Consider some $\mathcal{F}_0$-measurable mappings $y_0(\omega)$, $x_{-m-1}(\omega)$, $m = 0, 1, \ldots$, with values in V such that $y_0(\omega) \in Y(\omega)$ and $x_{-m-1}(\omega) \in X_{-m-1}(\omega)$ for all ω. The existence of these mappings follows from the measurable selection theorem (see [5]) because the graphs of the set-valued mappings $\omega \mapsto Y(\omega)$ and $X_{-m-1}(\omega)$ are measurable with respect to $\mathcal{F}_0 \times \mathcal{V}$ and $\mathcal{F}_{-m-1} \times \mathcal{V} \subseteq \mathcal{F}_0 \times \mathcal{V}$, respectively. Define the mappings $\zeta^m(\omega)$ of Ω into V $(m = 0, 1, \ldots)$ by the formula

$$\zeta^m(\omega) = \begin{cases} f_0(\omega) \cdots f_{-m}(\omega)(x_{-m-1}(\omega)) & \text{if } \omega \in \Omega_m, \\ y_0(\omega)\,, & \text{otherwise.} \end{cases}$$

Clearly $\zeta^m(\omega) \in Y(\omega)$ for all ω. The mappings $\zeta^m(\omega)$ are $\mathcal{F}_0$-measurable because $\Omega_m \in \mathcal{F}_0$ and the mappings $\bar{f}_{-m}(\omega, x)$ are measurable with respect to $\mathcal{F}_{-m-1} \times \mathcal{V} \subseteq \mathcal{F}_0 \times \mathcal{V}$. For each $\omega \in \tilde{\Omega}$ and $m \geq m(\omega)$, we have $\omega \in \Omega_m$ and

$$\rho(\omega, \xi(\omega), \zeta^m(\omega)) = \rho(\omega, \xi(\omega), f_0(\omega) \cdots f_{-m}(\omega)(x_{-m-1}(\omega)) \leq \rho^{(m)}(\omega),$$

where $\rho^{(m)}(\omega) \to 0$ as $m \to \infty$ (see (4.6), (4.12), and (4.13)). Thus $\zeta^m(\omega) \to \xi(\omega)$ on a set $\tilde{\Omega} \in \mathcal{F}_0$, where $P(\tilde{\Omega}) = 1$. Thus $\xi(\omega)$ is an a.s. limit of $\mathcal{F}_0$-measurable functions, and consequently it is $\mathcal{F}_0$-measurable since $\mathcal{F}_0$ is complete.

We know that $\xi(T\omega) = f(\omega, \xi(\omega))$ (a.s.) and since $\xi(\omega)$ coincides a.s. with some $\mathcal{F}_0$-measurable mapping $\xi'(\omega)$, we obtain

$$\xi'(T\omega) = \xi(T\omega) = f(\omega, \xi(\omega)) = f(\omega, \xi'(\omega)) \text{ (a.s.)},$$

where the first equality is valid because the transformation T preserves the measure P.

7*th step.* It remains to prove (ii). If $\eta : \Omega \to V$ is a mapping for which $\eta(\omega) \in X(\omega)$ and equation (3.1) holds, then

$$\eta(\omega) = f(T^{-1}\omega, \eta(T^{-1}\omega)) = f(T^{-1}\omega)(\eta(T^{-1}\omega))$$

$$= f(T^{-1}\omega) f(T^{-2}\omega)(\eta(T^{-2}\omega)) = \cdots = f(T^{-1}\omega) \cdots f(T^{-m-1}\omega)(\eta(T^{-m-1}\omega)) \text{ (a.s.)},$$

which yields

$$\eta(\omega) = f_0(\omega) f_{-1}(\omega) \cdots f_{-m}(\omega)(\eta(T^{-m-1}\omega)) \text{ (a.s.)}. \tag{4.14}$$

By combining (4.14) and (3.5), we get

$$\rho(\omega, \xi(\omega), \eta(\omega)) \leq \sup_{x \in X_{-m-1}(\omega)} \rho(\omega, \xi(\omega), f_0(\omega) \cdots f_{-m}(\omega)(x)) \to 0 \ \text{(a.s.)},$$

and so $\xi(\omega) = \eta(\omega)$ (a.s.). The proof is complete. □

5. Nonlinear Perron–Frobenius theorem: Proof

In this section we prove Theorem 1. The proof is based on a lemma.

Lemma 1. *There exists a sequence of $\mathcal{F}_0$-measurable sets $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Omega$ such that $P(\Gamma_m) \to 1$, and for each $m = 1, 2, \ldots$ and $\omega \in \Gamma_m$, the mapping $C(m, T^{-m}\omega)$ from the cone $K_{-m}(\omega)$ to the cone $K_0(\omega)$ is strictly monotone.*

Proof. For each $m \geq 1$, consider the set Δ_m of those ω for which the mapping $C(m, \omega, x) = C(m, \omega)x$ of the cone $K_0(\omega)$ into the cone $K_m(\omega)$ is strictly monotone in x. Let us show that $\Delta_m \in \mathcal{F}_m$. Denote by $H_m(\omega)$ the closed set $V \setminus K_m^\circ(\omega)$ and by $\delta(z, H_m(\omega))$ the distance (defined in terms of the norm $\|\cdot\|$) between the point $z \in V$ and $H_m(\omega)$. Clearly, $z \in K_m^\circ(\omega)$ if and only if $\delta(z, H_m(\omega)) > 0$.

For each $i, j = 1, 2, \ldots$ denote by $\Lambda_{ij}(\omega)$ the set of those $(x, y) \in K_0(\omega) \times K_0(\omega)$ for which

$$y - x \in K_0(\omega), \ \|y - x\| \geq 1/i, \ \max\{\|x\|, \|y\|\} \leq j.$$

The set-valued mapping $\omega \mapsto \Lambda_{ij}(\omega)$ is $\mathcal{F}_0$-measurable, and so it possesses a countable dense set of $\mathcal{F}_0$-measurable selections $(x_{ij}^l(\omega), y_{ij}^l(\omega))$, $l = 1, 2, \ldots$ (see [5, Theorems III.22 and III.30]). In view of the compactness of $\Lambda_{ij}(\omega)$ and continuity of $C(m, \omega, \cdot)$, we have

$$\Delta_m = \bigcup_{i,j=1}^{\infty} \{\omega : \inf_l [\delta(C(m, \omega, y_{ij}^l(\omega)) - C(m, \omega, x_{ij}^l(\omega)), H_m(\omega))] > 0\}.$$

Since the set-valued mapping $\omega \mapsto H_m(\omega)$ is $\mathcal{F}_m$-measurable, the set Δ_m is a union of a countable family of $\mathcal{F}_m$-measurable sets and is thus $\mathcal{F}_m$-measurable.

If $\omega \in \Delta_m$ and $y \succ_{K_0(\omega)} x$, we have $C(m, \omega)y >_{K_m(\omega)} C(m, \omega)x$. Furthermore, $D(T^m\omega)$ is a completely monotone mapping from $K_0(T^m\omega) = K_m(\omega)$ into $K_1(T^m\omega) = K_{m+1}(\omega)$. Therefore

$$C(m+1, \omega)y = D(T^m\omega)C(m, \omega)y >_{K_{m+1}(\omega)} D(T^m\omega)C(m, \omega)x = C(m+1, \omega)x,$$

and so $\omega \in \Delta_{m+1}$. Consequently, $\Delta_m \subseteq \Delta_{m+1}$. By virtue of assumption (**C**), we have $P(\bigcup_{m=1}^{\infty} \Delta_m) = 1$. By virtue of the inclusion $\Delta_m \subseteq \Delta_{m+1}$, this implies $P(\Delta_m) \to 1$. Define $\Gamma_m = T^m\Delta_m$. Then $\omega \in \Gamma_m$ if and only if the mapping $C(m, T^{-m}\omega)$ of the cone $K_0(T^{-m}\omega) = K_{-m}(\omega)$ into the cone $K_m(T^{-m}\omega) = K_0(\omega)$ is strictly monotone. Furthermore, $\Gamma_m \in \mathcal{F}_0$ for every m because $T^m\Delta_m \in \mathcal{F}_0$ if and only if $\Delta_m \in \mathcal{F}_m$. If $\omega \in \Gamma_m$, then $\omega \in \Gamma_{m+1}$ because the mapping

$$C(m+1, T^{-m-1}\omega) = C(m, T^{-m}\omega)D(T^{-m-1}\omega) \tag{5.1}$$

of $K_{-m-1}(\omega)$ into $K_0(\omega)$ is strictly monotone as the product of two mappings, one of which is completely monotone (from $K_{-m-1}(\omega)$ to $K_{-m}(\omega)$) and the other strictly monotone (from $K_{-m}(\omega)$ to $K_0(\omega)$). Thus, $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$, where $P(\Gamma_m) = P(\Delta_m) \to 1$, which completes the proof. □

Proof of Theorem 1. Put

$$X(\omega) = \hat{K}(\omega),\ \phi_t(\omega) = \phi(T^t\omega),\ \hat{K}_t(\omega) = \hat{K}(T^t\omega),\ X_t(\omega) = X(T^t\omega)$$

for any $t = 0, \pm 1, \pm 2, \ldots$. We will apply Theorem 3 to the mapping

$$f(\omega, x) = \frac{D(\omega, x)}{\langle \phi_1(\omega), D(\omega, x)\rangle},\ x \in X_0(\omega). \tag{5.2}$$

The mapping $f(\omega, x)$ is well defined because $D(\omega, x)$ is a completely monotone mapping from $K_0(\omega)$ into $K_1(\omega)$. This implies that $\langle \phi_1(\omega), D(\omega, x)\rangle > 0$ because $D(\omega, x) \neq 0$ for $x \neq 0$. Furthermore, $\langle \phi_1(\omega), f(\omega, x)\rangle = 1$, which means that $f(\omega, x) \in X_1(\omega) = \hat{K}_1(\omega)$, and so $f(\omega, \cdot)$ is a mapping of $X_0(\omega)$ into $X_1(\omega)$.

Denote by $\mathcal{V}$ the Borel σ-algebra on V induced by the Euclidean topology on V. Then the measurable space $(V, \mathcal{V})$ is standard. Let us verify the assumptions of Theorem 3 for the mappings $\omega \mapsto X(\omega)$ and $f(\omega, x) : X_0(\omega) \to X_1(\omega)$.

We have

$$\{(\omega, x) : x \in X(\omega)\} = \{(\omega, x) : x \in K(\omega),\ \langle \phi(\omega), x\rangle = 1\} \in \mathcal{F}_0 \times \mathcal{V}$$

because $\phi(\omega)$ and $\omega \mapsto K(\omega)$ are $\mathcal{F}_0$-measurable.

To check (**A1**) we need to show that the mapping $\bar{f}(\omega, x)$, which is equal to $f(\omega, x)$ if $x \in X_0(\omega)$ and ∞ otherwise, is $\mathcal{F}_1 \times \mathcal{V}$-measurable. This follows from the fact that the set $\Gamma := \{(\omega, x) : \omega \in \Omega,\ \ x \in K_0(\omega),\ \ \langle \phi_0(\omega), x\rangle = 1\}$ is $\mathcal{F}_0 \times \mathcal{V}$-measurable and the mapping $f(\omega, x)$ (see (5.2)) restricted to Γ is $\mathcal{F}_1 \times \mathcal{V}$-measurable by virtue of (**D1**) and the $\mathcal{F}_1$-measurability of $\phi_1(\omega)$.

For each ω, we define $Y(\omega)$ as $\hat{K}(\omega) \cap K^\circ(\omega)$ (which corresponds to our previous notation) and consider the Hilbert-Birkhoff metric $d(\omega) = d(\omega, x, y)$ on $Y(\omega)$. For every $k = 0, \pm 1, \pm 2, \ldots$ define

$$Y_t(\omega) = Y(T^t\omega),\ d_k(\omega) = d_k(\omega, x, y) = d(T^k\omega, x, y).$$

Let us verify the assumptions in (**A2**). To check (a) observe that the set-valued mapping $\omega \mapsto Y(\omega)$ is $\mathcal{F}_0$-measurable because its graph is the intersection of the $\mathcal{F}_0 \times \mathcal{V}$-measurable sets $\{(\omega, x) : \langle \phi(\omega), x\rangle = 1\}$ and $\{(\omega, x) : x \in K^\circ(\omega)\}$ (as regards the second set, its measurability can be proved by using Theorems III.22 and III.30 in [5]).

To verify (b) consider a real number r and the set

$$Q = \{(\omega, x, y) : x, y \in Y(\omega),\ d(\omega, x, y) > r\}.$$

We have to show that $Q \in \mathcal{F}_0 \times \mathcal{V} \times \mathcal{V}$. To this end observe that $d(\omega, x, y) > r$ if and only if

$$\inf_j \{\beta_j : x \leq_{K(\omega)} \beta_j y\} > e^r \sup_j \{\alpha_j : \alpha_i y \leq_{K(\omega)} x\},$$

where $\alpha_j > 0$ and $\beta_j > 0$ are rational numbers. By combining this observation with the fact that $\{(\omega, x, y) : x \leq_{K(\omega)} y\} \in \mathcal{F}_0 \times \mathcal{V} \times \mathcal{V}$ (following from the $\mathcal{F}_0$-measurability of $\omega \mapsto K(\omega)$), we obtain (b).

As we have noticed in Section 2, $(Y(\omega), d(\omega))$ is a complete separable metric space and the topology generated by the metric $d(\omega)$ on $Y(\omega)$ coincides with the Euclidean topology on $Y(\omega)$. From the fact that $D(\omega, x)$ is completely monotone, it follows that the map $f(\omega, x)$ transforms $Y(\omega)$ into $Y_1(\omega)$. Furthermore, $f(\omega, \cdot)$ is continuous in the Euclidean topology and hence with respect to the metric $d(\omega)$ on $Y(\omega)$ and the metric $d_1(\omega)$ on $Y_1(\omega)$. Therefore, conditions (c) and (d) hold.

Consider the $\mathcal{F}_0$-measurable sets $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Omega$ constructed in Lemma 1. Let $\Omega_m = \Gamma_{m+1}$, $(m = 0, 1, \ldots)$. We will show that the sets $\Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega$ possess the properties listed in (**A3**). Consider the mappings $f_m(\omega, x)$ and $f^{(m)}(\omega, x)$ defined by (3.2) and (3.3), respectively. By virtue of (3.2) and (5.2), we get

$$f_m(\omega, x) = f(T^{m-1}\omega, x) = \frac{D(T^{m-1}\omega, x)}{\langle \phi_m(\omega), D(T^{m-1}\omega, x)\rangle}, x \in X_{m-1}(\omega).$$

Let us prove by induction with respect to $m = 0, 1, 2, \ldots$ the following formula for every $x \in X_{-m-1}(\omega)$:

$$f^{(m)}(\omega, x) = f_0(\omega) f_{-1}(\omega) \cdots f_{-m}(\omega) x = \frac{C(m+1, T^{-m-1}\omega)x}{\langle \phi(\omega), C(m+1, T^{-m-1}\omega)x\rangle}. \tag{5.3}$$

If $m = 0$, then

$$f_0(\omega, x) = f(T^{-1}\omega, x) = \frac{D(T^{-1}\omega, x)}{\langle \phi(\omega), D(T^{-1}\omega, x)\rangle} = \frac{C(1, T^{-1}\omega)x}{\langle \phi(\omega), C(1, T^{-1}\omega)x\rangle},\ x \in X_{-1}(\omega).$$

Suppose equation (5.3) holds for $m - 1$. To verify it for m we take $x \in X_{-m-1}(\omega)$, put

$$z = \frac{D(T^{-m-1}\omega)x}{\langle \phi_{-m}(\omega), D(T^{-m-1}\omega)x\rangle},$$

and write

$$\begin{aligned}
\frac{C(m+1, T^{-m-1}\omega)x}{\langle \phi(\omega), C(m+1, T^{-m-1}\omega)x\rangle} &= \frac{C(m, T^{-m}\omega)D(T^{-m-1}\omega)x}{\langle \phi(\omega), C(m, T^{-m}\omega)D(T^{-m-1}\omega)x\rangle} \\
&= \frac{C(m, T^{-m}\omega)z}{\langle \phi(\omega), C(m, T^{-m}\omega)z\rangle} = f_0(\omega) f_{-1}(\omega) \cdots f_{-m+1}(\omega) z \\
&= f_0(\omega) f_{-1}(\omega) \cdots f_{-m+1}(\omega) \frac{D(T^{-m-1}\omega)x}{\langle \phi_{-m}(\omega), D(T^{-m-1}\omega)x\rangle} \\
&= f_0(\omega) f_{-1}(\omega) \cdots f_{-m+1}(\omega) f_{-m}(\omega) x.
\end{aligned}$$

In this chain of equalities, the first one follows from (5.1), the second from the definition of z and homogeneity of the mappings under consideration, the third from the assumption of induction, the fourth from the definition of z, and the last from the definition of f_{-m}.

Let m be any nonnegative integer and let $\omega \in \Omega_m = \Gamma_{m+1}$. Then, according to Lemma 1, the mapping $C(m+1, T^{-m-1}\omega)$ is strictly monotone. Consequently (see (5.3)), $f^{(m)}(\omega, x) \in K^\circ(\omega)$ for all $x \in X_{-m-1}(\omega)$, and so $f^{(m)}(\omega, X_{-m-1}(\omega))$ is a subset in $Y(\omega)$. Moreover, $f^{(m)}(\omega, X_{-m-1}(\omega))$ is a compact set in $Y(\omega)$, as a continuous image of the set $X_{-m-1}(\omega)$ which is compact in the Euclidean topology, and hence $f^{(m)}(\omega, X_{-m-1}(\omega))$ is compact with respect to the metric $d(\omega)$. By virtue of Theorem 2, $f^{(m)}(\omega, x) : Y_{-m-1}(\omega) \to Y(\omega)$ is a strictly nonexpansive mapping with respect to the metric $d_{-m-1}(\omega)$ on $Y_{-m-1}(\omega)$ and $d(\omega)$ on $Y(\omega)$. Consequently, condition (**A3**) holds. Thus, all the conditions sufficient for the validity of Theorem 3 are verified.

By virtue of assertion (i) of Theorem 3, there exists an $\mathcal{F}_0$-measurable mapping $\xi(\omega) \in Y(\omega)$ for which the equation $\xi(T\omega) = f(\omega, \xi(\omega))$ (a.s.) holds, i.e., we have

$$\xi(T\omega) = \frac{D(\omega, \xi(\omega))}{\langle \phi_1(\omega), D(\omega, \xi(\omega))\rangle} \text{ (a.s.)}.$$

Let $x(\omega) = \xi(\omega)$ and $\alpha(\omega) = \langle \phi_1(\omega), D(\omega, \xi(\omega))\rangle$. Then $\alpha(\omega)x(T\omega) = D(\omega)x(\omega)$ (a.s.), and since $x(\omega) \in Y(\omega)$, we have $x(\omega) \in K^\circ(\omega)$, $\langle \phi(\omega), x(\omega)\rangle = 1$ and $\alpha(\omega) > 0$. Furthermore, $x(\omega)$ is $\mathcal{F}_0$-measurable and $\alpha(\omega)$ is $\mathcal{F}_1$-measurable, which proves assertion (a).

To prove (b), take any $(\alpha'(\omega), x'(\omega))$, where $\alpha'(\omega) \geq 0$, $x'(\omega) \in K(\omega)$, and $\langle \phi(\omega), x'(\omega))\rangle = 1$, satisfying $\alpha'(\omega)x'(T\omega) = D(\omega)x'(\omega)$ (a.s.). Then

$$\alpha'(\omega) = \alpha'(\omega)\langle \phi_1(\omega), x'(T\omega)\rangle = \langle \phi_1(\omega), D(\omega)x'(\omega)\rangle > 0,$$

and so

$$x'(T\omega) = \frac{D(\omega)x'(\omega)}{\alpha'(\omega)} = \frac{D(\omega)x'(\omega)}{\langle \phi_1(\omega), D(\omega)x'(\omega)\rangle} = f(\omega, x'(\omega)) \text{ (a.s.)}.$$

By virtue of assertion (ii) of Theorem 3, we have $x'(\omega) = x(\omega)$ (a.s.), and consequently, $\alpha'(\omega) = \alpha(\omega)$ (a.s.), which proves (b).

To prove (c) we observe that $x := a/\langle \phi_{-m-1}(\omega), a\rangle \in X_{-m-1}(\omega)$ for any $0 \neq a \in K_{-m-1}(\omega)$, and by virtue of (5.3), the following equations hold:

$$\frac{C(m+1, T^{-m-1}\omega)a}{\langle \phi(\omega), C(m+1, T^{-m-1}\omega)a\rangle} = \frac{C(m+1, T^{-m-1}\omega)x}{\langle \phi(\omega), C(m+1, T^{-m-1}\omega)x\rangle} = f^{(m)}(\omega, x).$$

By using (3.5), we get

$$\begin{aligned}&\lim_{m(\omega)\leq m\to\infty} \sup_{0\neq a\in K_{-m-1}(\omega)} d(\omega, \xi(\omega), \frac{C(m+1, T^{-m-1}\omega)a}{\langle \phi(\omega), C(m+1, T^{-m-1}\omega)a\rangle})\\ &= \lim_{m(\omega)\leq m\to\infty} \sup_{x\in X_{-m-1}(\omega)} d(\omega, \xi(\omega), f^{(m)}(\omega, x)) = 0 \text{ (a.s.)}.\end{aligned}$$

In view of (2.2), here we can replace the Hilbert-Birkhoff metric $d(\omega, x, y)$ by the norm $\|x - y\|$, which yields (1.2). This completes the proof. □

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