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ASKEY–WILSON OPERATOR ON ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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ABSTRACT. In this paper, we first establish a series representation formula for the Askey–Wilson operator applied on entire functions of exponential type and then demonstrate its power in discovering summation formulas, some known and some new.

1. Introduction

Let $\mathscr{B}_\sigma$ denote the set of entire functions of exponential type σ. That is, $f \in \mathscr{B}_\sigma$ if f is an entire function, and for any $\varepsilon > 0$, there is an $A_\varepsilon > 0$ such that

$$|f(z)| \leq A_\varepsilon e^{(\sigma+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$. Boas (see [5], [6, pp. 210–211]), in providing a simpler proof of a Bernstein inequality given in [4], established an interesting interpolating formula for the derivatives of functions in $\mathscr{B}_\sigma$, known as Boas's formula, which is a generalization of an interpolating formula of M. Riesz [13] for trigonometric polynomials: if $f \in \mathscr{B}_\sigma$ and if f is bounded on the real line $\mathbb{R}$,

$$f'(x) = \frac{4\sigma}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^2}\, f\left(x + \frac{\pi}{2\sigma} + \frac{n\pi}{\sigma}\right). \tag{1.1}$$

In this paper, we want to extend Boas's formula by replacing the differentiation with a special divided difference operator of Askey-Wilson, and then show the power of the Askey–Wilson operator in discovering summation formulas.

2. Boas' formula extended

We first establish an extension of Boas's formula for the Askey–Wilson operator.

2.1. **Askey–Wilson operator.** In 1985, Askey and Wilson introduced the Askey–Wilson operator in their study of a class of orthogonal polynomials, the Askey–Wilson polynomials (see [2]).

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Definition 2.1. Let $q \in (0,1)$. The *Askey–Wilson operator*, $\mathscr{D}_q$, is defined by

$$(\mathscr{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{\breve{h}(q^{1/2}z) - \breve{h}(q^{-1/2}z)}, \ (x \in [-1,1]), \tag{2.1}$$

where

$$\breve{h}(z) = \frac{1}{2}\left(z + \frac{1}{z}\right), \ \breve{f}(z) = f\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right), \ z = e^{i\theta}, \ x = \cos\theta.$$

Note that $\breve{h}(q^{1/2}z) - \breve{h}(q^{-1/2}z) = i\sin\theta \cdot (q^{1/2} - q^{-1/2})$, and thus (2.1) can be written as

$$(\mathscr{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{i\sin\theta \cdot (q^{1/2} - q^{-1/2})}. \tag{2.2}$$

Since

$$\lim_{q \to 1^-} (\mathscr{D}_q f)(x) = f'(x)$$

at any point x where $f'(x)$ exists, $\mathscr{D}_q f$ can be considered as a discrete version of the derivative of f.

The definition of $\mathscr{D}_q$ given in [2] was mainly used to act on polynomials f, and it uses values of f at points in $\mathbb{C} \setminus [-1,1]$. To extend the domain of the operator to act on more general functions, Brown and Ismail [7] proposed an approach to define $\mathscr{D}_q$ on a dense subset of $L^2[(1-x^2)^{-1/2}, [-1,1]]$. In this paper, we will consider only entire functions, and so $\mathscr{D}_q$ is well defined as in (2.1), even for complex $x \in \mathbb{C}$.

2.2. **Main result.** For convenience, with $x = \cos\theta$, we write

$$\alpha := \frac{1}{2}(q^{1/2} + q^{-1/2}) \ \cos(\theta) \text{ and } \beta := (q^{1/2} - q^{-1/2}) \ \sin(\theta). \tag{2.3}$$

Note that, when $(x,q) \in [-1,1] \times (0,1)$, we have $\alpha, \beta \in \mathbb{R}$. Now, we are ready to state our main result.

Theorem 2.2. *Assume that $f \in \mathscr{B}_\sigma$ and the restriction of f on $\mathbb{R}$ is bounded. Then, for $x \in [-1,1]$,*

$$(\mathscr{D}_q f)(x) = \frac{4}{\sigma} \sum_{k=-\infty}^{\infty} f\left(\alpha + \frac{\pi}{2\sigma}(2k+1)\right) \frac{(-1)^k \cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2 \pi^2/\sigma^2}. \tag{2.4}$$

Remark 2.3. When $q \to 1^-$, we have $\alpha \to x$ and $\beta \to 0$, and thus, the limiting case of (2.4) becomes the classical *Boas's formula* (1.1).

Remark 2.4. Note that, with (2.3), we have

$$(\mathscr{D}_q f)(x) = \frac{f\left(\alpha + \frac{i\beta}{2}\right) - f\left(\alpha - \frac{i\beta}{2}\right)}{i\beta}. \tag{2.5}$$

We will use $(\mathscr{D} f)(\alpha, \beta)$ to denote the right-hand side to emphasize the dependence of $(\mathscr{D}_q f)(x)$ on α and β.

Remark 2.5. Theorem 2.2 is under the restriction on $(x,q) \in [-1,1] \times (0,1)$. Indeed, the theorem holds for complex x and q. Our next result is such an example by using variables α and β.

Corollary 2.6. *Under the same assumptions on f as in Theorem 2.2, we have that*

$$(\mathscr{D}f)(\alpha,\beta) = \frac{4}{\sigma}\sum_{k=-\infty}^{\infty} f\left(\alpha + \frac{\pi}{2\sigma}(2k+1)\right) \frac{(-1)^k \cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2\,\pi^2/\sigma^2} \tag{2.6}$$

holds for all $\alpha, \beta \in \mathbb{C}$ and that the convergence is locally uniform for $\alpha, \beta \in \mathbb{C}$.

2.3. **Classical sampling theorem.** Our proof of Theorem 2.2 is based on a sampling theorem. There are many forms of sampling theorems (see Butzer et al. [8]). We will use the following form.

Theorem 2.7. *If $f \in \mathscr{B}_\sigma$ and f is bounded on $\mathbb{R}$, then*

$$f(x) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\frac{\sigma}{\pi}(x - \frac{k\pi}{\sigma})\ (x \in \mathbb{C}), \tag{2.7}$$

the convergence being absolute and uniform on compact subsets of $\mathbb{C}$.

Recall that the function sinc is defined as

$$\operatorname{sinc} x := \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & \text{for } x \in \mathbb{C}\backslash\{0\}, \\ 1 & \text{for } x = 0. \end{cases}$$

3. Proof of Theorem 2.2

We first establish a lemma that gives us the action of the Askey–Wilson operator on the sinc function. To indicate that the operator $\mathscr{D}_q$ is applied with respect to x, we will use the notation $\mathscr{D}_{q,x}$.

Lemma 3.1. *For $x \in \mathbb{C}$ and any integer k, we have*

$$\left(\mathscr{D}_{q,x}(\operatorname{sinc}\frac{\sigma}{\pi}(x - y - \frac{k\pi}{\sigma}))\right)\Bigg|_{y=\alpha+\pi/(2\sigma)} = \frac{4}{\sigma}\cdot\frac{(-1)^k\cdot\cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2\,\frac{\pi^2}{\sigma^2}}. \tag{3.1}$$

Remark 3.2. The key feature of the lemma is the evaluation of y at a point $\alpha + \frac{\pi}{2\sigma}$ that is independent of k.

Proof. Let $g(x) := \operatorname{sinc}\frac{\sigma}{\pi}(x - y - \frac{k\pi}{\sigma})$. Then, by the definition of a sinc function, we have

$$g(x) = \frac{\sin\left(\sigma(x - y - \frac{k\pi}{\sigma})\right)}{\sigma(x - y - \frac{k\pi}{\sigma})}.$$

Note that

$$\breve{g}(q^{1/2}z) = \frac{\sin\left(\frac{\sigma}{2}\cdot\left(q^{1/2}z + q^{-1/2}z^{-1}\right) - (\sigma y + k\pi)\right)}{\frac{\sigma}{2}\cdot\left(q^{1/2}z + q^{-1/2}z^{-1}\right) - (\sigma y + k\pi)} \tag{3.2}$$

and

$$\breve{g}(q^{-1/2}z) = \frac{\sin\left(\frac{\sigma}{2}\cdot\left(q^{-1/2}z + q^{1/2}z^{-1}\right) - (\sigma y + k\pi)\right)}{\frac{\sigma}{2}\cdot\left(q^{-1/2}z + q^{1/2}z^{-1}\right) - (\sigma y + k\pi)}. \tag{3.3}$$

So, we have

$$\left(\mathscr{D}_{q,x}(\operatorname{sinc}\frac{\sigma}{\pi}(x - y - \frac{k\pi}{\sigma}))\right)(x) = \frac{\breve{g}(q^{1/2}z) - \breve{g}(q^{-1/2}z)}{i\sin\theta\cdot\left(q^{1/2} - q^{-1/2}\right)} \tag{3.4}$$

with

$$\begin{aligned}
&\breve{g}(q^{1/2}z) - \breve{g}(q^{-1/2}z)\\
&= \frac{\sin\left(\frac{\sigma}{2}\left(q^{1/2}z + \frac{1}{q^{1/2}z}\right) - \sigma y - k\pi\right)}{\frac{\sigma}{2}\left(q^{1/2}z + \frac{1}{q^{1/2}z}\right) - \sigma y - k\pi} - \frac{\sin\left(\frac{\sigma}{2}\left(\frac{z}{q^{1/2}} + \frac{q^{1/2}}{z}\right) - \sigma y - k\pi\right)}{\frac{\sigma}{2}\left(\frac{z}{q^{1/2}} + \frac{q^{1/2}}{z}\right) - \sigma y - k\pi}\\
&= (-1)^k \left[\frac{\sin\left(-\frac{\sigma}{2}\left(q^{1/2}z + \frac{1}{q^{1/2}z}\right) + \sigma y\right)}{-\frac{\sigma}{2}\left(q^{1/2}z + \frac{1}{q^{1/2}z}\right) + \sigma y + k\pi} - \frac{\sin\left(\frac{\sigma}{2}\left(\frac{z}{q^{1/2}} + \frac{q^{1/2}}{z}\right) - \sigma y\right)}{\frac{\sigma}{2}\left(\frac{z}{q^{1/2}} + \frac{q^{1/2}}{z}\right) - \sigma y - k\pi}\right].
\end{aligned}$$

Note that, when $y = \alpha + \frac{\pi}{2\sigma}$, we have $y = \frac{1}{4}(q^{1/2} + q^{-1/2})(z + z^{-1}) + \frac{\pi}{2\sigma}$ and

$$-\frac{1}{2}\left(q^{1/2}z + \frac{1}{q^{1/2}z}\right) + y - \frac{\pi}{2\sigma} = \frac{1}{2}\left(\frac{z}{q^{1/2}} + \frac{q^{1/2}}{z}\right) - y + \frac{\pi}{2\sigma}. \tag{3.5}$$

Call the common value of the two sides in (3.5) as w. Then

$$w = \frac{1}{4}(q^{1/2} - q^{-1/2})(z^{-1} - z) = \frac{1}{4}(q^{1/2} - q^{-1/2})(-2i)\sin\theta = -\frac{1}{2}i\beta. \tag{3.6}$$

Thus, we can write

$$\begin{aligned}
\breve{g}(q^{1/2}z) - \breve{g}(q^{-1/2}z) &= (-1)^k \left[\frac{\sin\left(\sigma w + \frac{\pi}{2}\right)}{\sigma w + \frac{\pi}{2} + k\pi} - \frac{\sin\left(\sigma w - \frac{\pi}{2}\right)}{\sigma w - \frac{\pi}{2} - k\pi}\right]\\
&= \frac{(-1)^k \cdot 2w\cos(\sigma w)}{\sigma \cdot \left\{w^2 - \left(k + \frac{1}{2}\right)^2 \frac{\pi^2}{\sigma^2}\right\}}.
\end{aligned}$$

From this and (3.6), (3.4) yields

$$\left.\left(\mathscr{D}_{q,x}(\operatorname{sinc}\frac{\sigma}{\pi}(x - y - \frac{k\pi}{\sigma}))\right)\right|_{y=\alpha+\pi/(2\sigma)} = \frac{4}{\sigma} \cdot \frac{(-1)^k \cdot \cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2 \frac{\pi^2}{\sigma^2}},$$

which is the desired result, (3.1). □

Proof of Theorem 2.2. We start by introducing a translation parameter in the sampling theorem, Theorem 2.7: Fix $y \in \mathbb{R}$ and apply Theorem 2.7 to $g_y(x) := f(x+y)$ to obtain

$$g_y(x) = \sum_{k=-\infty}^{\infty} f(y + \frac{k\pi}{\sigma}) \operatorname{sinc}\frac{\sigma}{\pi}(x - \frac{k\pi}{\sigma}).$$

Then

$$f(x) = g_y(x - y) = \sum_{k=-\infty}^{\infty} f(y + \frac{k\pi}{\sigma}) \operatorname{sinc}\frac{\sigma}{\pi}(x - y - \frac{k\pi}{\sigma}). \tag{3.7}$$

Now, apply $\mathscr{D}_q$ with respect to x on both sides of (3.7) to obtain

$$(\mathscr{D}_q f)(x) = \sum_{k=-\infty}^{\infty} f(y + \frac{k\pi}{\sigma})\ \mathscr{D}_{q,x}\left(\operatorname{sinc}\frac{\sigma}{\pi}(x - y - \frac{k\pi}{\sigma})\right).$$

The left-hand side is independent of y, and so we can take a special value of y on the right-hand side. Letting $y = \alpha + \frac{\pi}{2\sigma}$ and using (3.1) of Lemma 3.1, for $x \in [-1, 1]$, we get

$$(\mathscr{D}_q f)(x) = \sum_{k=-\infty}^{\infty} f\left(\alpha + (k+\frac{1}{2})\frac{\pi}{\sigma}\right)\ \frac{4}{\sigma} \cdot \frac{(-1)^k \cdot \cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2 \frac{\pi^2}{\sigma^2}},$$

which is (2.4). ☐

Proof of Corollary 2.6. Note that both sides of (2.6) are entire functions of α and β (of exponential type σ and $\sigma/2$, respectively) and they are equal when $\alpha, \beta \in \mathbb{R}$ by Theorem 2.2 (as (x, q) runs through $[-1,1]\times(0,1)$). Thus, by the Identity Theorem, (2.6) holds for all $\alpha, \beta \in \mathbb{C}$. Finally, we prove the local uniform convergence of the series. To this end, we need an estimate of Boas ([6, p. 84]): for $\varepsilon > 0$, there is an $A_\varepsilon > 0$ such that

$$|f(z)| \leq A_\varepsilon e^{(\sigma+\varepsilon)|\mathrm{Im}(z)|} \text{ for } z \in \mathbb{C}.$$

Applying this to the series in (2.6), we can establish the local uniform convergence by using the Weierstrass M-test. ☐

4. Identities of infinite series

As applications of the extended Boas formula in Theorem 2.2 and its corollary, we derive identities on infinite series, some new and some known. We begin with two general remarks.

(i) As a direct consequence from the locally uniform convergence in (2.6), convergence in series below is locally uniform in α and β.

(ii) The extra parameter q introduced by the Askey–Wilson operator in (2.4), which is not available in Boas's formula, will be seen as a desirable feature.

First, we apply (2.4) with $f(x) = 1$. Then $\mathscr{D}_q f(x) = 0$, and so

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\beta^2 + (2k+1)^2} = 0, \tag{4.1}$$

which can also be verified directly.

Another interesting identity can be obtained by taking $f(x) = \sin \sigma x$. We have

$$(\mathscr{D} f)(\alpha, \beta) = \frac{2\cos(\sigma\alpha)\sinh(\frac{\sigma}{2}\beta)}{\beta}, \tag{4.2}$$

which by (2.6) equals

$$\begin{aligned}
&\frac{4}{\sigma}\cosh\left(\frac{\sigma}{2}\beta\right) \sum_{k=-\infty}^{\infty} \sin(\sigma(\alpha + \frac{\pi}{2\sigma}(2k+1)))\ \frac{(-1)^k}{\beta^2 + (2k+1)^2 \pi^2/\sigma^2} \\
&= \frac{4}{\sigma}\cosh\left(\frac{\sigma}{2}\beta\right) \sum_{k=-\infty}^{\infty} \cos(\sigma\alpha)\ \frac{1}{\beta^2 + (2k+1)^2 \pi^2/\sigma^2}.
\end{aligned}$$

Combining this with (4.2), we get

$$\frac{\sigma \tanh(\frac{\sigma}{2}\beta)}{2\beta} = \sum_{k=-\infty}^{\infty} \frac{1}{\beta^2 + (2k+1)^2 \pi^2/\sigma^2}. \tag{4.3}$$

Note that, by Corollary 2.6, (4.3) holds for all $\beta \in \mathbb{C} \setminus \{\pm(2k+1)\frac{i\pi}{\sigma}\}_{-\infty}^{\infty}$. This is equivalent to a known result; see, e.g., [15, p. 136] or [10, 1.421.2].

Here is one more identity that can be derived directly from (2.6): Let $f(x) = \operatorname{sinc} x$. Note that

$$(\mathscr{D}f)(\alpha,\beta) = \frac{-4\beta\sin(\pi\alpha)\cosh(\frac{\pi}{2}\beta) + 8\alpha\cos(\pi\alpha)\sinh(\frac{\pi}{2}\beta)}{\pi\beta(4\alpha^2+\beta^2)}.$$

Using this in (2.4), we have

$$\begin{aligned}
&\frac{-4\beta\sin(\pi\alpha)\cosh(\frac{\pi}{2}\beta) + 8\alpha\cos(\pi\alpha)\sinh(\frac{\pi}{2}\beta)}{\pi\beta(4\alpha^2+\beta^2)} \\
&= \frac{4}{\pi}\cosh(\frac{\pi}{2}\beta)\sum_{k=-\infty}^{\infty}\frac{(-1)^k\sin(\pi(\alpha+k+\frac{1}{2}))}{\pi(\alpha+k+\frac{1}{2})(\beta^2+(2k+1)^2)} \\
&= \frac{4}{\pi}\cosh(\frac{\pi}{2}\beta)\sum_{k=-\infty}^{\infty}\frac{\cos(\pi\alpha)}{\pi(\alpha+k+\frac{1}{2})(\beta^2+(2k+1)^2)}.
\end{aligned}$$

So, dividing by $\frac{4}{\pi}\cosh(\frac{\pi}{2}\beta)\cos(\pi\alpha)$, we get

$$\frac{-\pi\beta\tan(\pi\alpha) + 2\pi\alpha\tanh(\frac{\pi}{2}\beta)}{\beta(4\alpha^2+\beta^2)} = \sum_{k=-\infty}^{\infty}\frac{1}{(\alpha+k+\frac{1}{2})(\beta^2+(2k+1)^2)},$$

which implies several known identities as special cases.

Next, consider $f(x) = x$. Then $f \in \mathscr{B}_0 \subset \mathscr{B}_\pi$. Since $\mathscr{D}_q f(x) = 1$, it is tempting to let $f(x) = x$ in (2.4) to get

$$1 = \frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}\left(\alpha + \frac{1}{2}(2k+1)\right)\frac{(-1)^k}{\beta^2+(2k+1)^2}. \tag{4.4}$$

But there is one serious problem: when $f(x) = x$, the assumption that f is bounded on $\mathbb{R}$ is not satisfied, and so we could not apply Theorem 2.2 directly to $f(x) = x$. Fortunately, we can prove (4.4) through a limiting process which we will present later. Assuming that (4.4) is valid, then we can put it in an equivalent form:

$$\frac{\pi}{4\cosh(\frac{\pi}{2}\beta)} = \sum_{k=-\infty}^{\infty}\left(\alpha + \frac{1}{2}(2k+1)\right)\frac{(-1)^k}{\beta^2+(2k+1)^2}. \tag{4.5}$$

Using (4.1) (or setting $\alpha = 0$), we get, for $\beta \in \mathbb{C}$,

$$\frac{\pi}{2\cosh\left(\frac{\pi}{2}\beta\right)} = \sum_{k=-\infty}^{\infty}\frac{(-1)^k(2k+1)}{\beta^2+(2k+1)^2}. \tag{4.6}$$

This is another known identity (see [15, p. 136] or [10]). Note that the series in (4.6) converges locally uniformly for $\beta \in \mathbb{C} \setminus \{\pm(2k+1)i\}_{-\infty}^{\infty}$. So by integrating both sides of (4.6) from $\beta = 0$ to $\beta = x$, with term-by-term integration on the right-hand side, Berndt used this identity to obtain an identity of Ramanujan ([3, p. 457]).

Now, we verify (4.4) by applying a trick motivated by the ones used in [6, p. 211] and [11, Lemmas 1 and 2].

Proof of (4.4). For $\delta \in (0, \frac{1}{2})$, define $g_\delta(x) = \sin(\delta x)$. Then $g_\delta \in \mathscr{B}_\delta \subseteq \mathscr{B}_\pi$ and g_δ is also bounded on $\mathbb{R}$. Note that

$$(\mathscr{D}_q g_\delta)(x) = \frac{2\cos(\delta\alpha)\sinh(\frac{\delta}{2}\beta)}{\beta}.$$

Thus, we can apply Theorem 2.2 to g_δ with $\sigma = \pi$ to obtain

$$\frac{2\cos(\delta\alpha)\sinh(\frac{\delta}{2}\beta)}{\delta\beta} = \frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}(-1)^k\frac{\sin(\delta(\alpha+k+\frac{1}{2}))}{\delta(\beta^2+(2k+1)^2)}. \tag{4.7}$$

Next, note that the partial sum

$$\sum_{k=-K}^{K}(-1)^k\frac{\sin(\delta(\alpha+k+\frac{1}{2}))}{\delta}$$

is $\mathcal{O}(K)$ uniformly in $\delta > 0$. So, by using Abel's partial summation formula, it is not hard to verify that the series on the right-hand side of (4.7) converges uniformly in $\delta > 0$. Thus, by taking limits as $\delta \to 0^+$ on both sides of (4.7), we obtain

$$1 = \frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}(\alpha+k+\frac{1}{2})\ \frac{(-1)^k}{\beta^2+(2k+1)^2},$$

which is (4.4). □

We can apply the same idea to obtain extensions to yet another known identity due to Gosper, Ismail, and Zhang [9, (1.3)]. For $b \in \mathbb{R}$, consider the function

$$f_{\delta,b}(x) = \frac{\sin(\delta x)\sin\sqrt{b^2+(\pi-\delta)^2x^2}}{\sqrt{b^2+(\pi-\delta)^2x^2}}.$$

Then $f_{\delta,b} \in \mathscr{B}_\pi$ and $f_{\delta,b}$ is bounded on the real line. So, we can apply (2.4) to $f_{\delta,b}$ to get

$$\frac{\pi(\mathscr{D}_q f_{\delta,b})(x)}{4\delta\cosh(\frac{\pi}{2}\beta)} = \sum_{k=-\infty}^{\infty}\frac{(-1)^k\sin(\delta(\alpha+k+\frac{1}{2}))\sin\sqrt{b^2+(\pi-\delta)^2(\alpha+k+\frac{1}{2})^2}}{\delta(\beta^2+(2k+1)^2)\sqrt{b^2+(\pi-\delta)^2(\alpha+k+\frac{1}{2})^2}}.$$

As above, it can be verified that the series on the right-hand side is uniformly convergent in $\delta \in (0, \frac{1}{2})$. Taking $\delta \to 0^+$ yields

$$\lim_{\delta\to 0^+}\frac{\pi(\mathscr{D}_q f_{\delta,b})(x)}{4\delta\cosh(\frac{\pi}{2}\beta)} = \sum_{k=-\infty}^{\infty}\frac{(-1)^k(\alpha+k+\frac{1}{2})\sin\sqrt{b^2+\pi^2(\alpha+k+\frac{1}{2})^2}}{(\beta^2+(2k+1)^2)\sqrt{b^2+\pi^2(\alpha+k+\frac{1}{2})^2}}. \tag{4.8}$$

We need to work out the left-hand side of (4.8). Note that

$$\begin{aligned}&\lim_{\delta\to 0^+}\frac{(\mathscr{D}_q f_{\delta,b})(x)}{\delta}\\ &= \frac{(\alpha+\frac{1}{2}i\beta)\sin\sqrt{b^2+\pi^2(\alpha+\frac{i}{2}\beta)^2}}{i\beta\sqrt{b^2+\pi^2(\alpha+\frac{i}{2}\beta)^2}} - \frac{(\alpha-\frac{1}{2}i\beta)\sin\sqrt{b^2+\pi^2(\alpha-\frac{i}{2}\beta)^2}}{i\beta\sqrt{b^2+\pi^2(\alpha-\frac{i}{2}\beta)^2}}.\end{aligned}$$

Using this in (4.8), we obtain the following new identity: For any $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\frac{(-1)^k(\alpha+k+\frac{1}{2})\sin\sqrt{b^2+\pi^2(\alpha+k+\frac{1}{2})^2}}{(\beta^2+(2k+1)^2)\sqrt{b^2+\pi^2(\alpha+k+\frac{1}{2})^2}}\\ &= \frac{\pi(\alpha+\frac{1}{2}i\beta)\sin\sqrt{b^2+\pi^2(\alpha+\frac{i}{2}\beta)^2}}{4i\beta\cosh(\frac{\pi}{2}\beta)\sqrt{b^2+\pi^2(\alpha+\frac{i}{2}\beta)^2}} - \frac{\pi(\alpha-\frac{1}{2}i\beta)\sin\sqrt{b^2+\pi^2(\alpha-\frac{i}{2}\beta)^2}}{4i\beta\cosh(\frac{\pi}{2}\beta)\sqrt{b^2+\pi^2(\alpha-\frac{i}{2}\beta)^2}}.\end{aligned} \tag{4.9}$$

If we let $\alpha = 0$, then the above identity becomes

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k (2k+1) \sin\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}}{(\beta^2+(2k+1)^2)\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}} = \frac{\pi \sin\sqrt{b^2-\frac{\pi^2\beta^2}{4}}}{2\cosh(\frac{\pi}{2}\beta)\sqrt{b^2-\frac{\pi^2\beta^2}{4}}}. \tag{4.10}$$

When taking β to be a pure imaginary number, we can recover identity (1.10) of Gosper, Ismail, and Zhang in [9]. If we further take $\beta = 0$, then (4.10) reduces to

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k \sin\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}}{(2k+1)\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}} = \frac{\pi \sin b}{2b}, \tag{4.11}$$

which is the identity (1.3) of Gosper, Ismail, and Zhang in [9]. Thus, identity (4.9) provides a two-parameter family extension of these identities of Gosper, Ismail, and Zhang.

Indeed, our argument above really applies to a much more general family of functions as indicated by the following result.

Theorem 4.1. *Let g be an entire function of exponential type π that is bounded on $\mathbb{R}$. Then, for $\alpha, \beta \in \mathbb{C}$,*

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k (\alpha+k+\frac{1}{2}) g(\alpha+k+\frac{1}{2})}{\beta^2+(2k+1)^2} \tag{4.12}$$

$$= \frac{\pi\alpha\Big[g(\alpha+\frac{i}{2}\beta)-g(\alpha-\frac{i}{2}\beta)\Big]+\frac{i}{2}\pi\beta\Big[g(\alpha+\frac{i}{2}\beta)+g(\alpha-\frac{i}{2}\beta)\Big]}{4i\beta\cosh(\frac{\pi}{2}\beta)}$$

$$= \frac{\pi}{4\cosh(\frac{\pi}{2}\beta)}\Big\{\alpha\left(\mathscr{D}_q g\right)(x) + \left(\mathscr{A}_q g\right)(x)\Big\}, \tag{4.13}$$

where $\mathscr{A}_q g$ is the "average operator" defined in [12, p. 301] *as*

$$\left(\mathscr{A}_q g\right)(x) = \frac{1}{2}\Big\{\breve{g}(q^{1/2}z) + \breve{g}(q^{-1/2}z)\Big\}. \tag{4.14}$$

Proof. Let $g_\delta(x) = g\left(\frac{\pi-\delta}{\pi}x\right)$, for $\delta \in \left(0, \frac{1}{2}\right)$. First, we shall apply Theorem 2.2 to the function $\tilde{g}(x) = \sin(\delta x) g_\delta(x)$ with $\sigma = \pi$ to get

$$\frac{\left(\mathscr{D}_q \tilde{g}\right)(x)}{\delta} = \frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right) \sum_{k=-\infty}^{\infty} (-1)^k \frac{\sin\left(\delta\left(\alpha+k+\frac{1}{2}\right)\right) g_\delta\left(\alpha+k+\frac{1}{2}\right)}{\delta(\beta^2+(2k+1)^2)}.$$

Again, as in the proof of (4.4), we can show that the series is uniformly convergent in $\delta \in \left(0, \frac{1}{2}\right)$. So, we can take the limit as $\delta \to 0^+$ to get

$$\lim_{\delta\to 0^+} \frac{\left(\mathscr{D}_q \tilde{g}\right)(x)}{\delta} = \frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right) \sum_{k=-\infty}^{\infty} \frac{(-1)^k (\alpha+k+\frac{1}{2}) g(\alpha+k+\frac{1}{2})}{\beta^2+(2k+1)^2}. \tag{4.15}$$

Now we shall directly compute $\lim_{\delta\to 0^+} \frac{(\mathscr{D}_q\tilde{g})(x)}{\delta}$ with the use of (2.5):

(4.16)
$$\lim_{\delta\to 0^+}\frac{(\mathscr{D}_q\tilde{g})(x)}{\delta} = \frac{\left(\alpha+\frac{i}{2}\beta\right)g\left(\alpha+\frac{i}{2}\beta\right)-\left(\alpha-\frac{i}{2}\beta\right)g\left(\alpha-\frac{i}{2}\beta\right)}{i\beta}$$
$$= \frac{\alpha\Big[g\left(\alpha+\frac{i}{2}\beta\right)-g\left(\alpha-\frac{i}{2}\beta\right)\Big]+\frac{i\beta}{2}\Big[g\left(\alpha+\frac{i}{2}\beta\right)+g\left(\alpha-\frac{i}{2}\beta\right)\Big]}{i\beta}.$$

Equating the two sides of (4.15) and (4.16) yields (4.12). Finally, (4.13) follows from the observation that,

$$(\mathscr{A}_q g)(x) = \frac{1}{2}\left\{g\left(\alpha+\frac{i\beta}{2}\right)+g\left(\alpha-\frac{i\beta}{2}\right)\right\}.$$

□

Remark 4.2. When $g(x)=\frac{\sin\sqrt{b^2+\pi^2x^2}}{\sqrt{b^2+\pi^2x^2}}$, (4.12) implies (4.9).

Remark 4.3. Note that the following are additional examples of entire functions of exponential type π that are also bounded on $\mathbb{R}$: For $\nu>\frac{1}{4}$, (i) $\frac{J_\nu(\sqrt{b^2+\pi^2x^2})}{\left(\sqrt{b^2+\pi^2x^2}\right)^\nu}$, and (ii) $J_\nu\left[\frac{\pi}{2}(\sqrt{b^2+x^2}+x)\right]J_\nu\left[\frac{\pi}{2}(\sqrt{b^2+x^2}-x)\right]$. Here J_ν denotes a Bessel function of the first kind of order ν (see [1,14]). Applying Theorem 4.1 to these functions will verify extensions of (4.11), [9, (1.6)], and identities of Zayed [16, p. 702]. To illustrate this we take the function in (ii) above. Applying Theorem 4.1 to this function, we get the following identity: with $\alpha_k := \alpha+k+\frac{1}{2}$,

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^k\alpha_k J_\nu[\frac{\pi}{2}(\sqrt{b^2+\alpha_k^2}+\alpha_k)]J_\nu[\frac{\pi}{2}(\sqrt{b^2+\alpha_k^2}-\alpha_k)]}{\beta^2+(2k+1)^2}$$
$$=\pi\alpha\left\{\frac{J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha+\frac{i\beta}{2})^2}+(\alpha+\frac{i\beta}{2}))]J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha+\frac{i\beta}{2})^2}-(\alpha+\frac{i\beta}{2}))]}{4i\beta\cosh(\frac{\pi}{2}\beta)}\right.$$
$$\left.-\frac{J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha-\frac{i\beta}{2})^2}+(\alpha-\frac{i\beta}{2}))]J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha-\frac{i\beta}{2})^2}-(\alpha-\frac{i\beta}{2}))]}{4i\beta\cosh(\frac{\pi}{2}\beta)}\right\}$$
$$+\frac{i\pi\beta}{2}\left\{\frac{J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha+\frac{i\beta}{2})^2}+(\alpha+\frac{i\beta}{2}))]J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha+\frac{i\beta}{2})^2}-(\alpha+\frac{i\beta}{2}))]}{4i\beta\cosh(\frac{\pi}{2}\beta)}\right.$$
$$\left.+\frac{J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha-\frac{i\beta}{2})^2}+(\alpha-\frac{i\beta}{2}))]J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha-\frac{i\beta}{2})^2}-(\alpha-\frac{i\beta}{2}))]}{4i\beta\cosh(\frac{\pi}{2}\beta)}\right\}.$$

Taking $\alpha=0$ and letting $\beta\to 0$ above, we get

$$\left[J_\nu\left(\frac{\pi b}{2}\right)\right]^2 = 2\sum_{k=0}^{\infty}J_\nu\left[\frac{\pi}{2}\left(\sqrt{b^2+\left(k+\frac{1}{2}\right)^2}+\left(k+\frac{1}{2}\right)\right)\right]$$
$$\times J_\nu\left[\frac{\pi}{2}\left(\sqrt{b^2+\left(k+\frac{1}{2}\right)^2}-\left(k+\frac{1}{2}\right)\right)\right]\frac{(-1)^k}{\pi(k+1/2)},$$

which is an identity of Zayed [16, p. 702].

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