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SIGNATURE FUNCTIONS OF KNOTS

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ABSTRACT. The signature function of a knot is an integer-valued step function on the unit circle in the complex plane. Necessary and sufficient conditions for a function to be the signature function of a knot are presented.

1. Introduction

For a knot $K \subset S^3$, the signature function, $\sigma_K(\omega)$, is an integer-valued step function defined on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. Its discontinuities can occur only at roots of the Alexander polynomial, $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$. The function is *balanced*, in the sense that for all $x \in \mathbb{R}$,

$$\sigma_K(e^{2\pi i x}) = \frac{1}{2} \lim_{\epsilon \to 0^+} \left(\sigma_K(e^{2\pi i (x+\epsilon)}) + \sigma_K(e^{2\pi i (x-\epsilon)}) \right).$$

There is an associated *jump* function,

$$J_K(e^{2\pi i x}) = \frac{1}{2} \lim_{\epsilon \to 0^+} \left(\sigma_K(e^{2\pi i (x+\epsilon)}) - \sigma_K(e^{2\pi i (x-\epsilon)}) \right).$$

Seifert [15] (see also [5]) characterized the set of polynomials that occur as Alexander polynomials of knots: if $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$, there exists a knot K such that $\Delta_K(t) = \Delta(t)$ if and only if $\Delta(1) = \pm 1$ and $\Delta(t) = t^k \Delta(t^{-1})$ for some $k \in \mathbb{Z}$. In general, the Alexander polynomial is well-defined up to multiplication by $\pm t^k$. We refer to any integer polynomial that satisfies these conditions as an *Alexander polynomial*.

Here we characterize the set of signature functions of knots. Recall that two complex numbers are called *Galois conjugate* if they are roots of the same irreducible rational polynomial.

Theorem 1. *Let σ be a balanced integer-valued step function on $\mathbb{S}^1 \subset \mathbb{C}$. Then $\sigma = \sigma_K$ for some knot K if and only if:*

(1) $\sigma(\omega) = \sigma(\overline{\omega})$ *for all* $\omega \in \mathbb{S}^1$.
(2) $\sigma(1) = 0$.
(3) *All discontinuities of σ occur at roots of Alexander polynomials.*
(4) *If $\alpha_1 \in \mathbb{S}^1$ and $\alpha_2 \in \mathbb{S}^1$ are Galois conjugate, then* $\sigma(\alpha_1) \equiv \sigma(\alpha_2) \mod 2$.

Before proceeding to the proof, we briefly present background. The signature of a knot, now viewed as $\sigma_K(-1)$, was first defined by Trotter [18] and Murasugi [12]. The signature function is essentially due to Levine [6] and Tristram [17]. Milnor [11] defined a different set of invariants, now called *Milnor signatures*, and these were proved to be equivalent to the jumps in the signature function by Matumoto [10].

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One can define the knot signature function as we do below, but without taking the two-sided average to make it balanced. This yields a well-defined knot invariant. However, it is not a knot concordance invariant. In particular, it does not vanish for slice knots (knots that bound smooth embedded disks in B^4); specifically, there are slice knots having nonzero (unbalanced) signature functions [2, 7]. In contrast to this, the (balanced) signature function induces a well-defined homomorphism from the knot concordance group to the set of functions on $\mathbb{S}^1$.

2. Definition of the signature function

The signature function of a knot is defined in terms of the Seifert matrix of the knot, V_K. Associated to V_K there is the matrix

$$W_K(t) = (1-t)V_K + (1-t^{-1})V_K^{\mathsf{T}}$$

with entries in the field of fractions, $\mathbb{Q}(t)$. This matrix is hermitian with respect to the involution of $\mathbb{Q}(t)$ induced by $t \to t^{-1}$. Substituting $\omega \in \mathbb{S}^1$ for t yields a complex hermitian matrix having signature which we temporarily denote $s_K(\omega)$. Then σ_K is defined by

$$\sigma_K(e^{2\pi i x}) = \frac{1}{2}\lim_{\epsilon\to 0^+}\left(s_K(e^{2\pi i(x+\epsilon)}) + s_K(e^{2\pi i(x-\epsilon)})\right).$$

For almost all $\omega \in \mathbb{S}^1$, $W_K(\omega)$ is nonsingular. For these ω, $s_K(\omega) \equiv \operatorname{rank}(W_K(\omega)) \equiv 0 \mod 2$. It follows that $\sigma_K(\omega)$ is an integer for all ω. Similarly, the jump function J_K takes on integer values.

3. Proof of necessity

The necessity of the first three conditions is well-known, with many references. The fourth condition is also known, but is not stated explicitly in the literature. Summary proofs are included for (1), (2), and (3); more details are provided for (4).

Property (1). The necessity of Property (1) follows from the fact that a hermitian matrix and its complex conjugate have the same signature.

Property (2). If $\omega \in \mathbb{S}^1$ is close to 0 with positive argument, we can use a Taylor approximation to write $\omega = 1 + \nu i + \nu^2 g(\nu)$, where $\nu \in \mathbb{R}_+$ is close to 0 and g is a real-valued differentiable function defined near 0. In terms of ν,

$$W_K(\omega) = \nu i(-V_K + V_K^{\mathsf{T}}) - \nu^2 g(\nu)(V_K + V_K^{\mathsf{T}}).$$

The signature of this matrix is the same as that of

$$i(-V_K + V_K^{\mathsf{T}}) - \nu g(\nu)(V_K + V_K^{\mathsf{T}}).$$

For a knot, the matrix $i(-V_K + V_K^{\mathsf{T}})$ is congruent to the direct sum of 2×2 matrices, each of the form

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

(This is false for links.) This is nonsingular with signature 0. A small perturbation leaves the signature unchanged.

Property (3). We can rewrite

$$W_K = (1-t)(V_K - t^{-1}V_K^{\mathsf{T}}).$$

This matrix is nonsingular except at roots of $\det(V_K - t^{-1}V_K^{\mathsf{T}})$ and at $t = 1$, and hence the signature is locally constant away from such roots. We have just seen that $t = 1$ is not a singular point: $\sigma_K(\omega) = 0$ for ω near 1. The Alexander polynomial can be defined as $\Delta_K(t) = \det(V_K - tV_K^{\mathsf{T}})$. Replacing t with t^{-1} does not change this determinant, modulo multiplication by $\pm t^k$ for some $k \in \mathbb{Z}$. Thus, all singularities of the signature function occur at roots of $\Delta_K(t)$.

Property (4). Since $\mathbb{Q}(t)$ is a field of characteristic 0, the form W_K can be diagonalized. Thus, we can prove the necessity of Property (4) by verifying it for 1×1 forms and applying the additivity of signature. In general, a 1×1 hermitian matrix is given as $(h(t))$ for some rational function $h(t)$ that is invariant under the involution $t \to t^{-1}$. A change of basis permits us to clear the denominator, and we can assume that $h(t)$ is in the Laurent polynomial ring $\mathbb{Q}[t, t^{-1}]$.

The ring $\mathbb{Q}[t, t^{-1}]$ is a principal ideal domain, and thus $h(t)$ factors into irreducible factors which are well-defined up to multiplication by a unit. (Units are elements of the form $u = \alpha t^k$ where $\alpha \neq 0 \in \mathbb{Q}$ and $k \in \mathbb{Z}$.) The proof of these properties of $\mathbb{Q}[t, t^{-1}]$ follows readily from the same result for $\mathbb{Q}[t]$, where the units are nonzero rational numbers.

If the irreducible factorization of $h(t)$ includes a factor $f(t)$ where $f(t) \neq uf(t^{-1})$ for some unit u, then the symmetry of $h(t)$ implies that $f(t^{-1})$ is also a factor. A change of basis for the 1×1 form can then eliminate a term $f(t)f(t^{-1})$. Continuing in this way, we can assume that the irreducible factorization of $h(t)$ is of the form $h(t) = f_1(t)f_2(t)\cdots f_n(t)$, with the property that $f_i(t) = u_i f_i(t^{-1})$ for some unit u_i for all i.

If $f_i(t) = uf_j(t)$ for some unit u with $i \neq j$, then a change of basis can eliminate $f_i(t)f_j(t)$ from the product. Continuing to eliminate factors in this way reduces us to the setting in which $h(t)$ has the following property: if $\delta(t) \in \mathbb{Q}[t, t^{-1}]$ is irreducible and satisfies $\delta(t^{-1}) = u\delta(t)$, then $\delta(t)$ has exponent 0 or 1 in the irreducible factorization of $h(t)$.

We can now proceed with the proof of Property (4). Suppose that $\alpha_1 \in \mathbb{S}^1$ and $\alpha_2 \in \mathbb{S}^1$ are roots of the same irreducible polynomial $\delta(t) \in \mathbb{Q}[t]$. If $\delta(t)$ does not divide $h(t)$, then $h(\alpha_1) \neq 0 \neq h(\alpha_2)$, and the signature of the 1×1 form $(h(t))$ evaluated at α_i is ± 1; in particular $\sigma(\alpha_1) \equiv \sigma(\alpha_2) \mod 2$, as desired.

On the other hand, if $\delta(t)$ does divide $h(t)$, then $h(\alpha_1) = 0 = h(\alpha_2)$. Notice that $h(t)$ is an analytic function of the complex variable t near α_1 and α_2 and has a *simple* root at both of these points (since $\delta(t)$ has exponent 1 in the irreducible factorization of $h(t)$). Thus, the complex derivative of $h(t)$ is nonzero at both α_1 and α_2. If we restrict our attention to $\mathbb{S}^1$ by considering the real-valued function $s(x) = h(e^{2\pi i x})$, the chain rule implies that the derivative $s'(x_i) \neq 0$, where x_i is chosen so that $e^{2\pi i x_i} = \alpha_i$, $i = 1, 2$. It follows that the signature of $(h(t))$ when restricted to $\mathbb{S}^1$ changes sign at both α_1 and α_2, going from ± 1 to ∓ 1. The (averaged) signature at both points is thus 0.

4. Proof of sufficiency

4.1. **Background.** The proof of sufficiency depends on some previously known facts, which we collect here as a series of lemmas.

Lemma 1. *If $\alpha \in \mathbb{S}^1$ is the root of an Alexander polynomial $\Delta(t)$, then it is the root of an irreducible Alexander polynomial.*

Proof. Suppose that $\delta_1(\alpha) = 0$, where $\delta_1(t) \in \mathbb{Q}[t, t^{-1}]$ is an irreducible factor of $\Delta(t)$. Since $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$, by Gauss's Lemma we can assume that $\delta_1(t) \in \mathbb{Z}[t, t^{-1}]$ and $\Delta(t)/\delta_1(t) \in \mathbb{Z}[t, t^{-1}]$. Thus, the fact that $\Delta(1) = \pm 1$ implies that $\delta_1(1) = \pm 1$. It remains to prove the symmetry of $\delta_1(t)$.

Normalize $\delta_1(t)$ so that $\delta_1(t) \in \mathbb{Z}[t]$ with nonzero constant coefficient. Let k denote the degree of $\delta_1(t)$ and let $\delta_2(t) = t^k \delta_1(t^{-1})$. We have $\delta_2(\alpha^{-1}) = 0$. But $\alpha^{-1} = \overline{\alpha}$, so $\delta_2(\overline{\alpha}) = 0$. On the other hand, δ_1 has real coefficients, so $\delta_1(\overline{\alpha}) = 0$. Thus, $\delta_1(t) = a\delta_2(t)$ for some $a \in \mathbb{Q}$; that is, $\delta_1(t) = at^k\delta_1(t^{-1})$. Letting $t = 1$, we have $\delta_1(1) = a\delta_1(1)$. Since $\delta_1(1) \neq 0$, we have $a = 1$, implying the symmetry of $\delta_1(t)$. □

Lemma 2. *If $\alpha \in \mathbb{S}^1$ is the root of a symmetric irreducible polynomial $\delta(t)$ that has odd exponent as a factor of $\Delta_K(t)$, then $J_K(\alpha) \equiv 1 \mod 2$.*

Proof. The diagonalization process used in the proof of the necessity of Property (4) does not change the exponent of $\delta(t)$ as a factor of the determinant, modulo 2. Thus, after diagonalizing and removing repeated factors, there are an odd number of diagonal entries of the form $f(t)\delta(t)$ where $\delta(t)$ does not divide $f(t)$; each such entry contributes ± 1 to the jump function. □

Lemma 3. *For every Alexander polynomial $\Delta(t)$, there is an unknotting number one knot K with $\Delta_K(t) = \Delta(t)$.*

Proof. This is a theorem of Kondo [4] and Sakai [14]. □

Lemma 4. *There is a dense subset $\{\beta_i\}$ of $\mathbb{S}^1$, each element of which is the root of a quartic Alexander polynomial having precisely two roots on $\mathbb{S}^1$. Consequently, for every $\omega \in \mathbb{S}^1$, there is an ω' arbitrarily close to ω and a knot K such that σ_K has nontrivial jumps only at ω' and $\overline{\omega'}$.*

Proof. In [2, Section 2] it is shown that for any $\omega \in \mathbb{S}^1$ and for any $\epsilon > 0$, there is a quartic Alexander polynomial $\Delta(t)$ having precisely two real roots and two complex roots, $\omega', \overline{\omega'} \in \mathbb{S}^1$, with $|\omega - \omega'| < \epsilon$. Let K be a knot satisfying $\Delta_K(t) = \Delta(t)$. By Lemma 2, the signature function of K has nontrivial jumps at ω' and $\overline{\omega'}$. Since these are the only roots of $\Delta_K(t)$ on $\mathbb{S}^1$, these are the only nontrivial jumps of σ_K. □

4.2. **Orientations and crossing changes.** In the arguments that follow, it will be important to keep track of signs and orientations. We summarize one approach here, which is consistent with references such as [1, 8, 13].

First, we use the formal definition of a knot to be a smooth oriented submanifold of S^3 that is diffeomorphic to S^1. Two knots K_1 and K_2 are equivalent if there is an orientation preserving diffeomorphism F of S^3 to itself with $F(K_1) = K_2$.

Let K be a knot and let D be an embedded disk in S^3 such that $D \cap K = A \cup p$, where A is a closed arc in ∂D and p is a point in the interior of D; we assume that the intersection of K and D is transverse at p. A new knot, K', can be constructed from K by removing A, replacing it with the closure of the complementary arc on ∂D, and then smoothing. The orientation of K' is chosen to agree with that of K on their intersection.

The orientation of K determines one on A, and thus on D. (In a moment we will discuss conventions.) There is then an orientated intersection number of K and D, which we call the sign of the crossing change. If the sign is positive, we say the crossing is from positive to negative.

In this process, a number of choices in convention are needed. These are made so that the standard right-handed trefoil, $T(2,3)$, can be unknotted by a positive crossing change, that is, by changing a crossing from positive to negative.

Let $m\colon S^3 \to S^3$ be an orientation reversing diffeomorphism. For a knot K, the knot $m(K)$ is called the mirror image of K. If K can be unknotted with a single crossing change from positive to negative, then $m(K)$ can be unknotted with a crossing change from negative to positive. In general, for all $\omega \in \mathbb{S}^1$, $\sigma_{m(K)}(\omega) = -\sigma_K(\omega)$.

We should note now that initially there is another choice of sign involved in defining the signature function. It is done so that the signature function of the right-handed trefoil knot T satisfies $\sigma_T(-1) = -2$.

Lemma 5. *If K can be unknotted by changing one crossing from negative to positive, then $0 \le \sigma_K(\omega) \le 2$ for all $\omega \in \mathbb{S}^1$.*

Proof. A proof of this for the classical signature $\sigma_K(-1)$ appears in [3, Proposition 2.1]. That proof can be generalized by considering p-fold covers rather than 2-fold covers. Here is a brief argument, similar to one given in [9] for the Heegaard Floer τ-invariant.

Fix the value of ω on the upper half circle $\mathbb{S}^1_+$. Use Lemma 4 to choose an $\omega' \in \mathbb{S}^1_+$ with smaller argument than ω and such that there is a quartic Alexander polynomial $\Delta(t)$ having a unique root ω' on $\mathbb{S}^1_+$. By Lemma 3, there is an unknotting number one knot J having $\Delta(t)$ as its Alexander polymomial. By replacing J with its mirror if need be, we will assume the unknotting operation for J changes a positive crossing to negative.

According to [3, Proposition 2.1], $\sigma_J(-1)$ is 0 or -2, and since there is a unique nontrivial jump in the signature function on the upper half circle, we have that $\sigma_J(-1) = -2$. Furthermore, $\sigma_J(e^{2\pi i x})$ is constant for x between the argument of ω' and $1/2$, and hence $\sigma_J(\omega) = -2$.

Since K can be unknotted with a single crossing change, it bounds a disk D in B^4 with one double point. Two small disks on D near the double point can be removed and replaced with an embedded annulus, yielding a punctured torus bounded by K in B^4. According to [16], this implies $|\sigma_K(\omega)| \le 2$.

Next, observe that $K\#J$ bounds a disk D in B^4 with two double points of opposite sign. Thus, a pair of disks on D, one containing each double point, can be removed and replaced with an annulus having interior in the complement of D to show that $K\#J$ bounds a punctured torus in B^4. We now have $|\sigma_{K\#J}(\omega)| \le 2$. The three results $\sigma_J(\omega) = -2$, $-2 \le \sigma_K(\omega) \le 2$, and $-2 \le \sigma_K(\omega) + \sigma_J(\omega) \le 2$ easily yield the desired result. □

4.3. **Proof of sufficiency.** There is a jump function $J(\omega)$ associated to $\sigma(\omega)$, as defined in the introduction. We begin with the observation that for all $\alpha \in \mathbb{S}^1$, $J(\alpha) \equiv \sigma(\alpha) \mod 2$. To see this, first note that at each discontinuity the change in signature is twice the jump. This implies that the signature is even away from the discontinuities. The jumps are also even (actually 0) away from the discontinuities, so we can focus on a point of discontinuity $\omega_0 = e^{2\pi i x_0}$. Here we have $J(e^{2\pi i x_0}) + \sigma(e^{2\pi i x_0}) = \sigma(e^{2\pi i(x_0+\epsilon)})$ for any small $\epsilon > 0$. Since for almost all ϵ, the right-hand side is even, we have $J(\alpha) \equiv \sigma(\alpha) \mod 2$ as desired.

The jump function (along with the fact that $\sigma(1) = 0$) permits us to restate the main theorem in terms of jump functions. Notice that we can eliminate the condition concerning the value at 0, since for any Alexander polynomial $\Delta(t)$, $\Delta(1) = \pm 1 \neq 0$.

Theorem 2. *Let J be an integer-valued function on $\mathbb{S}^1 \subset \mathbb{C}$ with finite support. Then $J = J_K$ for some knot K if and only if:*

(1) *$J(\omega) = J(\overline{\omega})$ for all $\omega \in \mathbb{S}^1$.*
(2) *$J(\omega) = 0$ if ω is not a root of an Alexander polynomial.*
(3) *If $\alpha_1 \in \mathbb{S}^1$ and $\alpha_2 \in \mathbb{S}^1$ are Galois conjugate, then $J(\alpha_1) \equiv J(\alpha_2) \mod 2$.*

We will use the following lemma in the construction of knots with specified jump functions.

Lemma 6. *Let $\Delta(t)$ be an Alexander polynomial having only simple roots on $\mathbb{S}^1$. There exists a knot K with $\Delta_K(t) = \Delta(t)$ such that the $J_K(\omega)$ is zero except at the unit roots of $\Delta(t)$, where its value alternates between 1 and -1, beginning with 1.*

Proof. According to Lemma 3, there is an unknotting number one knot K with $\Delta_K(t) = \Delta(t)$, and we can assume the unknotting crossing change is from negative to positive. The jumps in the signature function can occur only at the roots of $\Delta_K(t)$ on $\mathbb{S}^1$, and by Lemma 2, these jumps are all odd. Lemma 5 then implies that the jumps are all ± 1. Lemma 5 also implies that the first jump must be positive and also rules out the possibility of two successive jumps being the same sign. $\square$

If two irreducible rational polynomials have a root in common, then they have all roots in common. Thus, the given jump function $J(\omega)$ can be written as the sum of functions $J_i(\omega)$, where each $J_i(\omega)$ satisfies the conditions of Theorem 2 and has the added property that it is nonzero away from the roots of an irreducible Alexander polynomial $\Delta_i(t)$. Jump functions for knots are additive under connected sum, so the theorem is proved by constructing knots K_i realizing the jump functions $J_i(\omega)$. It follows that we can restrict to the case that J satisfies the conditions of Theorem 2 and all its nonzero values occur at the roots of a single irreducible Alexander polynomial $\Delta(t)$.

Denote the set of roots of $\Delta(t)$ that lie on the upper half circle $\mathbb{S}^1_+$ by $\{\alpha_1, \ldots, \alpha_k\}$. By Lemma 6 there is a knot whose jump function is nontrivial precisely at the roots of $\Delta(t)$, where it is odd. The jump function of the unknot is 0 everywhere. Any function on the upper half circle satisfying the conditions of Theorem 2 with support contained in $\{\alpha_1, \ldots, \alpha_k\}$ differs from one of these two by a function which is 0 off the set $\{\alpha_1, \ldots, \alpha_k\}$ and which is even everywhere. Thus, the proof of the theorem is reduced to the following result.

Theorem 3. *For each α_m, there exists a knot K with jump function satisfying $J_K(\alpha_m) = 2$ and $J_K(\omega) = 0$ if $\omega \in \mathbb{S}^1_+$ and $\omega \neq \alpha_m$.*

Proof. For notational purposes, we let $\delta_1(t) = \Delta(t)$. Assume the set of numbers $\{\alpha_1, \ldots, \alpha_k\}$ is ordered by increasing argument. We focus on one element of the set, α_m. Choose a $\beta \in \mathbb{S}^1_+$ with argument between that of α_m and α_{m+1}. (In the case $m = k$, choose $\beta \in \mathbb{S}^1_+$ with argument greater than that of α_k.) As in the proof of Lemma 5, we can use Lemma 4 to ensure that β is a root of an irreducible Alexander polynomial, $\delta_2(t)$, having a unique root on the upper unit circle $\mathbb{S}^1_+$.

Let K_1, K_2, and K_3 be the unknotting number one knots having Alexander polynomials $\delta_1(t)$, $\delta_2(t)$, and $\delta_1(t)\delta_2(t)$, respectively, provided by Lemma 6. The signature functions for these knots can have nontrivial jumps only at elements of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta, \alpha_{m+1}, \ldots, \alpha_k\}$. By Lemma 6, the jump functions of each must be as follows. (We write the jump at β in bold to highlight its location in the list in position $(m+1)$; the entire list is an ordered $(k+1)$–tuple.)

- Jumps for σ_{K_1}: $[1, -1, \ldots, (-1)^{m+1}, \mathbf{0}, (-1)^m, \ldots, (-1)^{k+1}]$.
- Jumps for σ_{K_2}: $[0, 0, \ldots, 0, \mathbf{1}, 0, \ldots, 0]$.
- Jumps for σ_{K_3}: $[1, -1, \ldots, (-1)^{m+1}, (-\mathbf{1})^{\mathbf{m}}, (-1)^{m+1}, \ldots, (-1)^k]$.

We now see that the jumps for the connected sum $J_m = K_1 \# (-1)^{m+1} K_2 \# K_3$ are given by

$$[2, -2, \ldots, (-1)^{m+1}2, \mathbf{0}, 0, \ldots, 0].$$

Since the jumps for this knot occur only at α_i, we list the jumps at those points as a k–tuple:

$$[2, -2, \ldots, (-1)^{m+1}2, 0, \ldots, 0].$$

The last nonzero entry is in the m position.

For $m \geq 2$, our desired knot K is $(-1)^{m+1}(J_m \# -J_{m-1})$. For $m = 1$, the knot $K = J_1$ suffices. $\square$

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