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ALMOST-PERIODIC HOMOGENIZATION OF ELLIPTIC PROBLEMS IN NON-SMOOTH DOMAINS

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Abstract. We consider a family of second-order elliptic operators $\{\mathcal{L}_\varepsilon\}$ in divergence form with rapidly oscillating and almost-periodic coefficients in Lipschitz domains. By using the compactness method, we show that the uniform $W^{1,p}$ estimate of second-order elliptic systems holds for $\frac{2n}{n+1}-\delta<p<\frac{2n}{n-1}+\delta$; the ranges are sharp for $n=2$ or $n=3$. In the scalar case we obtain that the $W^{1,p}$ estimate holds for $\frac{3}{2}-\delta<p<3+\delta$ if $n\geqslant 3$, and $\frac{4}{3}-\delta<p<4+\delta$ if $n=2$; the ranges of p are sharp.

1. Introduction

This paper investigates a family of second-order elliptic operators with rapidly oscillating and almost-periodic coefficients,

$$\mathcal{L}_\varepsilon=-\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}(\frac{x}{\varepsilon})\frac{\partial}{\partial x_j}\right]=-\text{div}\left[A(\frac{x}{\varepsilon})\nabla\right]. \tag{1.1}$$

Suppose that the coefficient matrix $A(y)=a_{ij}^{\alpha\beta}(y)$ $(1\leq i,j\leq n, 1\leq\alpha,\beta\leq m)$ is real and bounded measurable. Here and thereafter we will suppose that $||A||_\infty\leq\mu^{-1}$ and A is elliptic, i.e.,

$$\mu|\xi|^2\leqslant a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \text{ for } \xi=(\xi_i^\alpha)\in\mathbb{R}^{nm},\ y\in\mathbb{R}^n, \tag{1.2}$$

where $\mu>0$.

We shall be interested in the quantitative homogenization of second-order elliptic systems with bounded measurable coefficients that are almost-periodic in the sense of H. Bohr, which means that A is the uniform limit of a sequence of trigonometric polynomials in $\mathbb{R}^n$. Let Trig($\mathbb{R}^n$) denote the set of all trigonometric polynomials. The closure of the set Trig($\mathbb{R}^n$) with respect to the L^∞-norm is called the Bohr space of almost-periodic functions. A useful equivalent description of the almost-periodic functions is given as follows. Let A be bounded and continuous in $\mathbb{R}^n$. Then A is almost-periodic in the sense of Bohr if and only if

$$\limsup_{R\to\infty}\sup_{y\in\mathbb{R}^n}\inf_{\substack{z\in\mathbb{R}^n\\|z|\leq R}}||A(\cdot+y)-A(\cdot+z)||_{L^\infty(\mathbb{R}^n)}=0. \tag{1.3}$$

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Set

$$\rho(R) := \sup_{y\in\mathbb{R}^n} \inf_{\substack{z\in\mathbb{R}^n\\ |z|\le R}} ||A(\cdot + y) - A(\cdot + z)||_{L^\infty(\mathbb{R}^n)}. \tag{1.4}$$

Notice that $\rho(R) = 0$ if A is periodic.

In this paper we study uniform $W^{1,p}$ estimates on Lipschitz domain for second-order elliptic systems with almost-periodic coefficients subject to the Dirichlet boundary condition. Suppose that $F \in L^p(\Omega)$ and $g \in B^{1-\frac{1}{p},p}(\partial\Omega)$, where $B^{\alpha,p}$ denotes the Besov space, and $B^{-1/p,p}(\partial\Omega)$ is defined to mean the dual of Besov space $B^{1/p,p'}(\partial\Omega)$ on $\partial\Omega$ for $1 \le p < \infty$ and $0 < \alpha < 1$. Let u_ε be a weak solution of the Dirichlet problem,

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div} F \text{ in } \Omega \quad \text{ and } \quad u_\varepsilon = g \text{ on } \partial\Omega. \tag{D_p}$$

Theorem 1.1. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$. Assume that A is continuous, symmetric (i.e.,$A = A^*$), and satisfies (1.2) and (1.3) and*

$$\rho(R) \le C[\log R]^{-N} \tag{1.5}$$

for some $N > 5/2$ and any $R \ge 2$. Let $u_\varepsilon \in H^1(\Omega)$ be a weak solution of $(\mathrm{D})_p$ with $F \in L^p(\Omega)$, $g \in B^{1-\frac{1}{p},p}(\partial\Omega)$, where $\frac{2n}{n+1} - \delta < p < \frac{2n}{n-1} + \delta$. Then

$$||\nabla u_\varepsilon||_{L^p(\Omega)} \leqslant C\left\{||F||_{L^p(\Omega)} + ||g||_{B^{1-\frac{1}{p},p}(\partial\Omega)}\right\}, \tag{1.6}$$

where constants $\delta, C > 0$ are independent of ε.

The next theorem is concerned with the scalar case ($m = 1$). The ranges of p are sharp.

Theorem 1.2. *Let $m = 1$. Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$. Assume that A is continuous, symmetric (i.e., $A = A^*$), and satisfies (1.2) and (1.3) and*

$$\rho(R) \le C[\mathrm{logR}]^{-\mathrm{N}} \tag{1.7}$$

for some $N > 5/2$ and any $R \ge 2$. Let $u_\varepsilon \in H^1(\Omega)$ be a weak solution of $(\mathrm{D})_p$ with $F \in L^p(\Omega)$ and $g \in B^{1-\frac{1}{p},p}(\partial\Omega)$, where $\frac{3}{2} - \delta < p < 3 + \delta$ if $n \geqslant 3$, and $\frac{4}{3} - \delta < p < 4 + \delta$ if $n = 2$. Then

$$||\nabla u_\varepsilon||_{L^p(\Omega)} \leqslant C\left\{||F||_{L^p(\Omega)} + ||g||_{B^{1-\frac{1}{p},p}(\partial\Omega)}\right\}, \tag{1.8}$$

where constants $\delta, C > 0$ are independent of ε.

Uniform regularity estimates play an essential role in the study of the convergence problems in homogenization. We refer the reader to [16], [1], [17], and [9]. In periodic setting, the uniform $W^{1,p}$ estimate (1.6) with $1 < p < \infty$ for the Dirichlet boundary problem $(\mathrm{D})_p$ on $C^{1,\alpha}$ domain was obtained in [2] under the assumption that A is Hölder continuous,

$$|A(x) - A(y)| \le C|x - y|^\gamma \text{ for any } \gamma \in (0, 1]. \tag{1.9}$$

The non-tangential maximal function estimates and Lipschitz estimates were also obtained there via an elegant three-step compactness argument. In [11] Kenig and Shen solved the L^2 Dirichlet, Neumann, and Regularity problems in Lipschitz domain for elliptic systems with periodic, symmetric and Hölder continuous coefficients by the method of layer potentials. Using this result, the non-tangential maximal function estimates, boundary Lipschitz estimates, and uniform $W^{1,p}$ estimate (1.6) of the Neumann problem were established by C. Kenig, F. Lin, and Z. Shen in [10] for $1 < p < \infty$ in $C^{1,\alpha}$ domain. The symmetric condition on A was removed in [1] later by using a convergence rate method. In the case of second-order elliptic systems subject to Dirichlet boundary conditions in Lipschitz domains, in a recent paper [5], the authors were able to show that the uniform $W^{1,p}$ estimate (1.6) holds on Lipschitz domains for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n} + \delta$ under the assumption that $A^* = A$ is periodic and satisfies (1.9). Similar results for the linear elasticity problem are also proved in [5] by a different approach. In the case of scalar equation ($m = 1$) on Lipschitz domain, the $W^{1,p}$ estimate (1.8) for the elliptic homogenization problem $\mathcal{L}_\varepsilon u_\varepsilon = \mathrm{divF}$ in Ω was proved in [15] for $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$ if $n = 2$ and for $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ if $n \geqslant 3$; and the ranges of p are sharp.

The study of elliptic homogenization with almost periodic coefficients started from S. M. Kozlov ([12]) and G. C. Papanicolaou and S. R. S. Varadhan [13]. In contrast to the periodic setting, it was proved in [12] that one of the main difficulties in the almost-periodic setting was caused by the lack of the solvability of the corrector equation. Precisely, let $P_j^\beta = y_j(0, ..., 1, ..., 0)$ with 1 in the β^{th} position, the corrector equation

$$\mathcal{L}_1(\chi_j^\beta) = -\mathcal{L}_1(P_j^\beta) \quad \text{in} \quad \mathbb{R}^n, \tag{1.10}$$

which corresponds to the homogenization problem $\mathcal{L}_\varepsilon u_\varepsilon = 0$ may not be solvable directly, unless under some extra assumptions. In [16], by introducing the auxiliary approximate corrector equation

$$\mathcal{L}_1(\chi_{T,j}^\beta) + T^{-2}\chi_{T,j}^\beta = -\mathcal{L}_1(P_j^\beta) \quad \text{in} \quad \mathbb{R}^n, \tag{1.11}$$

the uniform Hölder estimates and convergence rates of elliptic systems with rapidly oscillating almost-periodic coefficients were established by Shen on $C^{1,\alpha}$ domain. Moreover, in a recent paper [1], Armstrong and Shen prove the full boundary Lipschitz estimates for second-order elliptic systems with almost-periodic and Hölder continuous coefficients, the boundary $W^{1,p}$ estimates were also obtained there in $C^{1,\alpha}$ domains for $1 < p < \infty$. For almost periodic operators with complex coefficients, the interior Hölder estimate was obtained in [3] by using the compactness argument.

Our approach is to reduce the $W^{1,p}$ estimate to a weak reverse Hölder inequality via a real-variable argument. However, in the almost-periodic setting, if u_ε is a weak solution to $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in Ω and $u_\varepsilon = g$ n.t. on $\partial\Omega$ with A satisfying (1.3), due to the lack of the Rellich estimate

$$\int_{\partial\Omega} |(\nabla u_\varepsilon)^*|^2 \leqslant C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^2,$$

the method used in [10] or [5] is not applicable. To overcome this difficulty, for some $N > 5/2$ and any $R \geq 2$, under some growth assumption on $\rho(R)$,

$$\rho(R) \leq C[\mathrm{logR}]^{-\mathrm{N}},$$

follow the method in [15], let u_ε be a weak solution of $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in Z_{2r} and $u_\varepsilon = 0$ on S_{2r}, we instead seek the following decay estimate:

$$\begin{aligned}&\int_0^t \int_{|x'|<r} |u_\varepsilon(x', \psi(x'+s))|^{p_n} dx' ds \\ &\leqslant C\left(\frac{t}{r}\right)^{p_n+\alpha_0} \int_0^{2r} \int_{|x'|<2r} |u_\varepsilon(x', \psi(x'+s))|^{p_n} dx' ds,\end{aligned} \tag{1.12}$$

where $\frac{\varepsilon}{\varepsilon_0} < \frac{t}{r} < 1$ and $p_n = \frac{2n}{n-1}$. Notice that

$$u_\varepsilon \to u_0 \text{ strongly in } L^2(\Omega), \tag{1.13}$$

$$\nabla u_\varepsilon \to \nabla u_0 \text{ weakly in } L^2(\Omega), \tag{1.14}$$

as $\varepsilon \to 0$, we then have the homogenization result (see Theorem 2.1 or [8]),

$$A(x/\varepsilon)\nabla u_\varepsilon \to \hat{A}\nabla u_0 \text{ weakly in } L^2(\Omega), \tag{1.15}$$

where $\hat{A}$ is a constant matrix and $-\mathrm{div}(\hat{\mathrm{A}}\nabla \mathrm{u}_0) = \mathrm{F}_0$. This, together with (1.12), by using the compactness argument, will yield the desired weak reverse Hölder inequality, and thus the $W^{1,p}$ estimates.

Let ψ be a Lipschitz mapping $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ for $r > 0$. Set

$$Z_r = \{(x', x_n) \in \mathbb{R}^n : |x'| < r \text{ and } \psi(x') < x_n < \psi(x') + (M + 10n)r\}, \tag{1.16}$$

$$S_r = \{(x', \psi(x')) \in \mathbb{R}^n : |x'| < r\}, \tag{1.17}$$

denote the Lipschitz cylinder and its surface.

2. Almost-periodic homogenization

In this section we introduce some preliminaries of the homogenization theory of elliptic systems with almost-periodic coefficients. A detailed presentation may be found in [8].

A function $f(x) \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is called almost-periodic in the sense of Bezikovich if there is a sequence of trigonometric polynomials converging to f in the Bezikovich norm

$$||f||_{B^2} = \limsup_{R\to\infty} \left\{ \fint_{B(0,R)} |f|^2 \right\}^{\frac{1}{2}}. \tag{2.1}$$

The space of such functions is denoted by $B^2(\mathbb{R}^n)$. For any f with finite Bezikovich norm, define its mean value $\langle f \rangle$ by

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(\varepsilon^{-1}x)\phi(x) = \langle f \rangle \int_{\mathbb{R}^n} \phi(x) \text{ for any } \phi \in \mathrm{C}_0^\infty(\mathbb{R}^\mathrm{n}). \tag{2.2}$$

A function $f = f_i^\alpha \in \mathrm{Trig}(\mathbb{R}^\mathrm{n})(1 \le \alpha \le \mathrm{m})$ is called potential if there exists $g = g^\alpha \in \mathrm{Trig}(\mathbb{R}^\mathrm{n})$ such that $f = \nabla g$, $g \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$. A function $f = f_i^\alpha \in \mathrm{Trig}(\mathbb{R}^\mathrm{n})$ is said to be solenoidal if $\mathrm{div f} = 0$. Let

$$V^2_{\mathrm{pot}} = \text{the closure of } \{f \text{ is potential}, \ \langle f \rangle = 0\}, \tag{2.3}$$

$$V^2_{\mathrm{sol}} = \text{the closure of } \{f \text{ is solenoidal}, \ \langle f \rangle = 0\}. \tag{2.4}$$

Then

$$B^2(\mathbb{R}^n) = V^2_{\mathrm{pot}} \oplus V^2_{\mathrm{sol}} \oplus \mathbb{R}^{nm}. \tag{2.5}$$

It follows from the Lax-Milgram theorem and the ellipticity condition (1.2), there exists a unique $\psi_{\ell j}^{\gamma\beta} \in V_{\mathrm{pot}}^2$ such that for any $\phi = (\phi_i^\alpha) \in V_{\mathrm{pot}}^2$,

$$\langle a_{ij}^{\alpha\beta} \phi_i^\alpha \rangle + \langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \phi_i^\alpha \rangle = 0. \tag{2.6}$$

Also, denote $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ by

$$\hat{A} = \langle a_{ij}^{\alpha\beta} \rangle + \langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \rangle. \tag{2.7}$$

Then

$$\mu|\xi|^2 \le \hat{a}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \le \widetilde{\mu}|\xi|^2 \text{ for any } \xi \in \mathbb{R}^{nm}, \tag{2.8}$$

where $\widetilde{\mu}$ depends only on m, n, and μ. Let A^* denote the adjoint of A; then it is known that $\hat{A^*} = (\hat{A})^*$.

The next theorem shows that $\mathcal{L}_0 = -\mathrm{div}(\hat{\mathrm{A}}\nabla)$ is the homogenized operator of $\mathcal{L}_\varepsilon$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lispschitz domain and $F_0 \in H^{-1}(\Omega)$. Assume that $A(y)$ is continuous and satisfies* (1.2) *and* (1.3)*. Let u_k be a weak solution of $-\operatorname{div}(A_k(x/\varepsilon_k)\nabla u_k) = F_k$ in Ω. Suppose that $u_k \to u_0$ strongly in $L^2(\Omega)$ and $\nabla u_k \to \nabla u_0$ weakly in $L^2(\Omega)$ as $\varepsilon_k \to 0$ as well as $F_k \to F_0$ strongly in $H^{-1}(\Omega)$. Then $A_k(x/\varepsilon_k)\nabla u_k \to \hat{A}\nabla u_0$ weakly in $L^2(\Omega)$ with $u_0 \in H^1(\Omega)$ is a weak solution of $-\operatorname{div}(\hat{A}\nabla u_0) = F_0$.*

Proof. We use Tartar's test function method to prove it. If A_k is independent of k, this is also a classical result in the theory of homogenization. See [8] for the scalar case ($m = 1$). The proof for the case $m > 1$ is the same and we give a proof here for the sake of completeness.

Denote $p_k = A_k(x/\varepsilon_k)\nabla u_k$ and assume $p_k \to p_0$ weakly in $L^2(\Omega)$ as $k \to \infty$. Set $\psi \in C_0^\infty(\Omega)$ and let $\chi_{T_k,j}^{k*\beta}(x/\varepsilon_k)$ be the approximate correctors for the adjoint matrix $(A_k)^*$. We then have

$$\begin{aligned}
&\langle F_k, (P_j^\beta + \varepsilon_k \chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))\psi \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \\
&\quad = \int_\Omega A_k(x/\varepsilon_k)\nabla u_k \cdot \nabla\{(P_j^\beta + \varepsilon_k \chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))\psi\} \\
&\quad = \int_\Omega A_k(x/\varepsilon_k)\nabla u_k \cdot \nabla(P_j^\beta + \varepsilon_k \chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))\psi \\
&\quad\quad + \int_\Omega A_k(x/\varepsilon_k)\nabla u_k \cdot (P_j^\beta + \varepsilon_k \chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))\nabla\psi.
\end{aligned} \tag{2.9}$$

It follows from integration by parts we obtain

$$\begin{aligned}
&\int_\Omega A_k(x/\varepsilon_k)\nabla u_k \cdot \nabla(P_j^\beta + \varepsilon_k \chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))\psi \\
&= -\int_\Omega u_k \cdot (A_k)^*(x/\varepsilon_k) \cdot \nabla(P_j^\beta + \varepsilon_k \chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))(\nabla\psi) \\
&\quad - \int_\Omega u_k \cdot (A_k)^* \varepsilon_k \chi_{T_k,j}^{k*\beta}(x/\varepsilon_k)\psi,
\end{aligned} \tag{2.10}$$

where the approximate corrector equation (1.11) with $T_k = \varepsilon_k^{-1/2}$

$$\operatorname{div}\big\{(A_k)^*(x/\varepsilon_k)\cdot\nabla(P_j^\beta + \varepsilon_k\chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))\big\} = -\varepsilon_k\chi_{T_k,j}^{k*\beta}(x/\varepsilon_k) \tag{2.11}$$

was used in (2.10).

By (2.9)-(2.10) we obtain that

$$\begin{aligned}
&\langle F_k,\, (P_j^\beta + \varepsilon_k\chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))\psi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\\
&\quad = -\int_\Omega u_k\cdot(A_k)^*(x/\varepsilon_k)\cdot\nabla(P_j^\beta + \varepsilon_k\chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))(\nabla\psi)\\
&\quad\quad - \int_\Omega u_k\cdot(A_k)^*\varepsilon_k\chi_{T_k,j}^{k*\beta}(x/\varepsilon_k)\psi\\
&\quad\quad + \int_\Omega A_k(x/\varepsilon_k)\nabla u_k\cdot(P_j^\beta + \varepsilon_k\chi_{T_k,j}^{k*\beta}(x/\varepsilon_k))\nabla\psi.
\end{aligned}\tag{2.12}$$

Notice that $\varepsilon_k\chi_{T_k}(x/\varepsilon_k)\to 0$ weakly in $W^{1,2}(\Omega)$ and

$$a_{ik}^{\alpha\gamma}(x/\varepsilon_k)\frac{\partial}{\partial x_k}\{x_j\delta^{\gamma\beta} + \varepsilon_k\chi_{T_k,j}^{\gamma\beta}(x/\varepsilon_k)\}\to\hat a_{ij}^{\alpha\beta}$$

weakly in $L^2(\Omega)$ (see [18]). Using this and taking weak limits on both sides of (2.12), we have that the l.h.s. of (2.9) converges to $\langle F_0, P_j^\beta\psi\rangle$ and the r.h.s. converges to $\int_\Omega \hat a_{ji}^{\beta\alpha}\frac{\partial u_0^\alpha}{\partial x_i}\psi + \int_\Omega p_0 P_j^\beta(\nabla\psi)$, where the fact $(\hat A)^* = \hat A^*$ was used.

Next, we take $P_j^\beta\psi$ as the test function to have

$$\langle F_k, P_j^\beta\psi\rangle = \int_\Omega p_k\nabla(P_j^\beta\psi). \tag{2.13}$$

Let $k\to\infty$. We have

$$\langle F_0, P_j^\beta\psi\rangle = \int_\Omega p_0\nabla P_j^\beta\psi + \int_\Omega p_0 P_j^\beta(\nabla\psi). \tag{2.14}$$

In view of the arbitrariness of ψ, compare with (2.12) and we obtain that $p_0 = \hat A\nabla u_0$. □

3. A sufficient condition and proof of Theorem 1.1 and Theorem 1.2

It was proved in [14] (see also [4], [5]) that the weak reverse Hölder inequality implies the $W^{1,p}$ estimates for second-order elliptic systems with bounded, measurable coefficients, as follows.

Theorem 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n, n\geq 2$ and $p>2$. Let $\mathcal{L} = \operatorname{div}(A(x)\nabla)$ with A satisfying (1.2). Let $v\in H^1(Z_{2r})$ be a weak solution of $\mathcal{L}(v) = 0$ in Z_{2r} and $v = 0$ on S_{2r}. Assume that the weak reverse Hölder inequality*

$$\left(\fint_{Z_r}|\nabla v|^p\right)^{\frac1p}\leqslant C_0\left(\fint_{Z_{2r}}|\nabla v|^2\right)^{\frac12} \tag{3.1}$$

holds. Let $u\in H_0^1(\Omega)$ be a solution of $(D)_2$ with $F\in L^p(\Omega)$. Then $u\in W^{1,p}(\Omega)$ and

$$||\nabla u||_{L^p(\Omega)}\leq C||F||_{L^p(\Omega)} \tag{3.2}$$

with constant $C>0$ depending only on n, p, μ, C_0, and the Lipschitz character of Ω.

The following theorem is concerned with the interior $W^{1,p}$ estimate.

Theorem 3.2. *Let $f \in L^p(2B)$ for some $2 < p < \infty$ and $\rho(R)$ be defined as* (1.4). *Suppose that $u_\varepsilon \in H^1(2B)$ is a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathrm{divF}$ in $2B$ for some ball $B \subset \mathbb{R}^n$. Assume that A is continuous and satisfies* (1.2) *and* (1.3) *with*

$$\rho(R) \leq C[\mathrm{logR}]^{-\mathrm{N}} \tag{3.3}$$

for some $N > 5/2$ and any $R \geq 2$. Then we have

$$\left\{ ⨍_B |\nabla u_\varepsilon|^p \right\}^{1/p} \leq C \left\{ \left(⨍_{2B} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(⨍_{2B} |F|^p \right)^{1/p} \right\}, \tag{3.4}$$

where C depends only on p, μ.

Proof. See [1]. □

We have the following reverse Hardy type estimate.

Lemma 3.3. *Let $u_\varepsilon \in H^1(Z_{3r})$ be a solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Z_{3r} and $u_\varepsilon = 0$ on S_{3r}. Let A satisfy the same assumptions as in Theorem* 3.2. *Then for any $p > 1$,*

$$\int_0^{cr} \int_{|x'|<r} |\nabla u_\varepsilon(x', \psi(x') + s)|^p \, dx' ds \leq C \int_0^{2cr} \int_{|x'|<2r} \left| \frac{u_\varepsilon(x', \psi(x') + s)}{s} \right|^p dx' ds, \tag{3.5}$$

where $c = 10\sqrt{n}$ and $C > 0$ are independent on ε.

Proof. Set $\rho(x) = \mathrm{dist}(x, \partial Z_{4r})$. It follows from Theorem 3.2 that we obtain

$$\begin{aligned}
\int_{B(x, c\rho(x))} |\nabla u_\varepsilon(y)|^p dy &\leq C\rho(x)^{n-p} \left(⨍_{B(x, 2c\rho(x))} |u_\varepsilon(y)|^2 dy \right)^{p/2} \\
&\leq C\rho(x)^{n-p} \left(⨍_{B(x, 2c\rho(x))} |u_\varepsilon(y)|^p dy \right) \\
&\leq C \int_{B(x, 2c\rho(x))} \left| \frac{u_\varepsilon(y)}{\rho(y)} \right|^p dy, \qquad (3.6)
\end{aligned}$$

where we used the Cacciopoli's inequality in the first inequality and Hölder's inequality in the second one. Next we multiply both sides of (3.6) by $\rho(x)^{-n}$ and then integrate on Z_r. The proof is similar to that of Lemma 3.2 in [15] and thus omitted. □

To utilize the compactness argument, we need to recall the regularity result for second-order elliptic systems and equations with constant coefficients.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let u be a weak solution of $Lu = 0$ in Z_{2r} and $u = 0$ on S_{2r}, where $L = -\mathrm{div}(A\nabla)$ with A is a constant matrix and $A = A^*$. Then*

1) *if $m > 1$, then* (3.1) *holds for $\frac{2n}{n+1} - \delta < p = p_n < \frac{2n}{n-1} + \delta$;*

2) *if $m = 1$, then* (3.1) *holds for $\frac{3}{2} - \delta < p = p_n < 3 + \delta$ if $n \geqslant 3$ ($\frac{4}{3} - \delta < p = p_2 < 4 + \delta$ if $n = 2$); the ranges of p are sharp.*

Proof. See [7]. □

Lemma 3.5. *Let $L = -\mathrm{div}(A\nabla)$ and let A be a constant matrix with $A = A^*$. Suppose that $u_0 \in W^{1,2}(Z_{3/2})$, $L(u_0) = 0$ in $Z_{3/2}$ and $u_0 = 0$ on $S_{3/2}$. Let p_n be the same as in Lemma* 3.4. *Then*

$$\begin{aligned}\int_0^t \int_{|x'|<1} |u_0(x',\psi(x')+s)|^{p_n}\,dx'ds \\ \le C\,t^{p_n+2\sigma}\int_0^{3/2}\int_{|x'|<\frac{3}{2}} |u_0(x',\psi(x')+s)|^{p_n}\,dx'ds\end{aligned} \tag{3.7}$$

for any $0<t<1$, where $C,\sigma>0$ depending only on n, m, μ, and M.

Proof. The proof follows from Lemma 3.4 by taking $p = p_n = \frac{2n}{n-1}$. See [5]. □

Next, we prove a homogenization result on a sequence of domains.

Lemma 3.6. *Let $A_k(y)$ be a sequence of matrices and $\{\psi_k\}$ a sequence of Lipschitz functions. Suppose that A_k are symmetric, continuous, and satisfy* (1.2) *and* (1.3). *Assume that*

$$\begin{cases}\mathrm{div}(A_k(\frac{x}{\varepsilon_k})\nabla u_{\varepsilon_k}) = 0 \ \ in\ Z_r(\psi_k),\\ \ u_{\varepsilon_k} = 0 \quad on\ S_r(\psi_k),\end{cases} \tag{3.8}$$

where $\varepsilon_k \to 0$ and

$$||u_{\varepsilon_k}||_{H^1(Z_r(\psi_k))} \le C. \tag{3.9}$$

Then there exist subsequences of $\{\psi_k\}$ and $\{u_{\varepsilon_k}\}$, which we still denote by the same notation, and a Lipschitz function ψ, $u \in L^2(Z_r(\psi))$, and a constant matrix $\hat{A}$ such that

$$\begin{cases}\psi_k \to \psi \ \ in\ \ \{x\in\mathbb{R}^n : |x'|<5\},\\ u_{\varepsilon_k}(x',x_n-\psi_k(x')) \to u(x',x_n-\psi(x'))\ strongly\ in\ L^2(E_r),\end{cases} \tag{3.10}$$

where $E_r = \{(x',x_n) : |x'|<r$ and $0<x_n<10(M+1)r\}$, and u is a solution of

$$\begin{cases}\mathrm{div}(\hat{A}\nabla u) = 0 \ \ in\ \ Z_r(\psi),\\ u = 0 \ \ on\ S_r(\psi).\end{cases} \tag{3.11}$$

Proof. We first note that (3.10) follows from (3.9) by the Arzelá-Ascoli theorem. To show (3.11), we let

$$v_{\varepsilon_k}(x',x_n) = u_{\varepsilon_k}(x',x_n+\psi_k(x')).$$

Note that $||v_{\varepsilon_k}||_{W^{1,2}(E_r)} \le C$, by passing to a subsequence, we have

$$\begin{aligned} v_{\varepsilon_k} &\to v \text{ strongly in } L^2(E_r),\\ \nabla v_{\varepsilon_k} &\to \nabla v \text{ weakly in } L^2(E_r).\end{aligned}$$

It follows from Theorem 2.1 that u is a weak solution of $\mathrm{div}(\hat{A}\nabla u) = 0$ in $Z_r(\psi)$. Finally, set

$$u_{\varepsilon_k} = v_{\varepsilon_k}(x',x_n-\psi_k(x')) \quad \text{and} \quad u = v(x',x_n-\psi(x')).$$

Then $u = 0$ on $S_r(\psi)$ follows from the fact that $v_{\varepsilon_k} \to v$ weakly in $H^1(Z_r(0))$ and $v_{\varepsilon_k} = 0$ on $S_r(0)$. □

Lemma 3.7. *Let $u_\varepsilon \in W^{1,2}(Z_3)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Z_3 and $u_\varepsilon = 0$ on S_3. Suppose that A is continuous, symmetric, and satisfies* (1.2) *and* (1.3). *Then there exists $\varepsilon_0 > 0$, depending only on n, μ, and M, such that for any $0 < \varepsilon \le \varepsilon_0$,*

$$\begin{aligned} &\int_0^{t_0}\int_{|x'|<1} |u_\varepsilon(x', \psi(x') + t)|^{p_n}\, dx'dt \\ &\qquad \le t_0^{p_n+\sigma} \int_0^{3c}\int_{|x'|<3} |u_\varepsilon(x', \psi(x') + t)|^{p_n}\, dx'dt, \end{aligned} \tag{3.12}$$

where $0 < t_0 < 1/2$ and $c = (M + 10n)$.

Proof. We will prove the lemma by contradiction. For any $k \in \mathbb{N}$, denote

$$Z_r^k = \big\{(x', x_n) : \ |x'| < r \text{ and } \psi_k(x') < x_n < \psi_k(x') + (M + 10n)r\big\},$$
$$S_r^k = \big\{(x', x_n) : \ |x'| < r \text{ and } x_n = \psi_k(x')\big\},$$

where $||\nabla\psi_k||_\infty \le M$ and $\psi_k(0) = 0$. Suppose that (3.12) is not true; then there exist $\{\varepsilon_k\}$, $\{\mathcal{L}_{\varepsilon_k}^{(k)}\}$, $\{\psi_k\}$, and $\{u_{\varepsilon_k}\}$ as well as a sequence of uniformly almost-periodic operators $\{A_k\}$ satisfying (1.2) and such that $\varepsilon_k \to 0$ as $k \to \infty$,

$$\mathcal{L}_{\varepsilon_k}^{(k)}(u_{\varepsilon_k}) = -\mathrm{div}\Big(A_k\Big(\frac{x}{\varepsilon_k}\Big)\nabla u_{\varepsilon_k}\Big) = 0 \quad \text{in} \quad Z_3^k \text{ and } u_{\varepsilon_k} = 0 \quad \text{on} \quad S_3^k,$$

$$\int_0^{3c}\int_{|x'|<3} |u_{\varepsilon_k}(x', \psi_k(x') + t)|^{p_n}\, dx'dt = 1, \tag{3.13}$$

and

$$\int_0^{t_0}\int_{|x'|<1} |u_{\varepsilon_k}(x', \psi_k(x') + t)|^{p_n}\, dx'dt > t_0^{p_n+\sigma}. \tag{3.14}$$

Let

$$b_{ij}^{\alpha\beta,k} = \langle a_{ij}^{\alpha\beta,k}\rangle + \langle a_{i\ell}^{\alpha\gamma,k}\psi_{\ell j}^{\gamma\beta}\rangle, \tag{3.15}$$

where $\psi_{\ell j}^{\gamma\beta} \in V_{\text{pot}}^2$ and $b_{ij}^{\alpha\beta,k}$ are bounded. Hence, by passing to a subsequence, we may suppose that

$$b_{ij}^{\alpha\beta} = \lim_{k\to\infty} b_{ij}^{\alpha\beta,k} \tag{3.16}$$

exists for $1 \le i, j \le n$, $1 \le \alpha, \beta \le m$. Thus we have

$$\mu|\xi|^2 \le b_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \le \widetilde{\mu}|\xi|^2 \tag{3.17}$$

for any $\xi \in \mathbb{R}^{nm}$ and $\widetilde{\mu}$ depends only on m, n, and μ (see, e.g., [8]).

Let $v_{\varepsilon_k}(x', t) = u_{\varepsilon_k}(x', \psi_k(x') + t)$ and E_r be defined as in Lemma 3.6. Note that by Cacciopoli's inequality and (3.13), $\{v_{\varepsilon_k}\}$ is uniformly bounded in $W^{1,2}(E_2)$. Thus, $v_{\varepsilon_k} \to v_0$ weakly in $W^{1,2}(E_2)$ and strongly in $L^{p_n}(E_2)$ due to the compact embedding. In view of (3.13) and (3.14) we obtain

$$\begin{aligned} &\int_0^2\int_{|x'|<2} |v_0(x', t)|^{p_n}\, dx'dt \le 1, \\ &\int_0^{t_0}\int_{|x'|<1} |v_0(x', t)|^{p_n}\, dx'dt \ge t_0^{p_n+\sigma}. \end{aligned} \tag{3.18}$$

Next, let $u_0(x', x_n) = v_0(x', x_n - \psi_0(x'))$. Then $u_0 \in W^{1,2}(\widetilde{Z}_2)$ and $u_0 = 0$ on $\widetilde{S}_2$, where

$$\widetilde{Z}_r = \big\{(x', t) : \ |x'| < r \text{ and } \psi_0(x') < t < \psi_0(x') + (M + 10n)r\big\},$$
$$\widetilde{S}_r = \big\{(x', \psi_0(x')) : \ |x'| < r\big\}.$$

Let $L = -\mathrm{div}(\bar{A}\nabla)$, where $\bar{A} = (b_{ij}^{\alpha\beta})$. It follows from Lemma 3.6 that $L(u_0) = 0$ in Z_2. In view of Lemma 3.5 and (3.18) we obtain

$$\begin{aligned}
&\int_0^{t_0} \int_{|x'|<1} |u_0(x', \psi_0(x') + t)|^{p_n} \, dx' dt \\
&\qquad \le C_0 t_0^{p_n + 2\sigma} \int_0^2 \int_{|x'|<2} |u_0(x', \psi_0(x') + t)|^{p_n} \, dx' dt \\
&\qquad \le (1/2) t_0^{p_n + \sigma},
\end{aligned} \tag{3.19}$$

which contradicts the second inequality in (3.18). This completes the proof. □

Lemma 3.8. *Let $u_\varepsilon \in W^{1,2}(Z_3)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Z_3 and $u_\varepsilon = 0$ on S_3. Suppose that A and $\varepsilon_0 > 0$ are the same as Lemma 3.7. There exist positive constants δ and C, depending only on n, μ, and M, such that for $(\varepsilon/\varepsilon_0) < t < 1$,*

$$\begin{aligned}
&\int_0^{t} \int_{|x'|<1} |u_\varepsilon(x', \psi(x') + s)|^{p_n} \, dx' ds \\
&\qquad \le C t^{p_n + \delta} \int_0^{3c} \int_{|x'|<3} |u_\varepsilon(x', \psi(x') + s)|^{p_n} \, dx' ds.
\end{aligned} \tag{3.20}$$

Proof. Lemma 3.8 follows from Lemma 3.7 by rescaling and iteration argument. See [15, pp. 2294-2295] for more details. □

Next we give part of the proof of Theorem 1.1 in the case of $g = 0$.

Theorem 3.9. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $F \in L^p(\Omega)$, where $\frac{2n}{n+1} - \delta < p < \frac{2n}{n-1} + \delta$. Let u_ε be a weak solution to $\mathcal{L}_\varepsilon u_\varepsilon = \mathrm{div}\, F$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega$. Assume that A is continuous, symmetric, and satisfies* (1.2) *and* (1.3) *and*

$$\rho(R) \le C[\log R]^{-N}$$

for some $N > 5/2$ and any $R \ge 2$. Then

$$||\nabla u_\varepsilon||_{L^p(\Omega)} \leqslant C||F||_{L^p(\Omega)}, \tag{3.21}$$

where constants $\delta, C > 0$ are independent of ε.

Proof. Without loss of generality, we may assume $r = 1$. It suffices to prove the weak reverse Hölder inequality (3.1) for $2 < p < \frac{2n}{n-1} + \delta$ and the ranges $\frac{2n}{n+1} - \delta < p < 2$ will be obtained by a duality argument. If $\varepsilon \ge \varepsilon_0/4$, estimate (3.1) follows from the standard regularity estimate of second-order elliptic systems with variable coefficients (see [6]).

Hence we suppose that $\varepsilon < \varepsilon_0/4$. For $2^{-j_0-1} \le \varepsilon/\varepsilon_0 \le 2^{-j_0}$, we decompose

$$\begin{aligned}&\int_0^c \int_{|x'|<1} \left|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\right|^{p_n} dx'ds \\ &= \left\{ \int_0^{\varepsilon/\varepsilon_0} \int_{|x'|<1} + \sum_{j=1}^{j_0} \int_{2^{j-1}\varepsilon/\epsilon_0}^{2^j\varepsilon/\varepsilon_0} \int_{|x'|<1} \right. \\ &\quad \left. + \int_{2^{j_0}\varepsilon/\varepsilon_0}^{c} \int_{|x'|<1} \right\} \left|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\right|^{p_n} dx'ds, \\ &= I + II + III.\end{aligned} \tag{3.22}$$

It is easy to see that

$$III \le C \int_0^{3c} \int_{|x'|<3} |u_\varepsilon(x',\psi(x')+s)|^{p_n}\, dx'ds \tag{3.23}$$

and

$$\begin{aligned} II &\le C \sum_{j=1}^{j_0} \left(2^{j-1}\frac{\varepsilon}{\varepsilon_0}\right)^{-p_n} \left(2^j \frac{\varepsilon}{\varepsilon_0}\right)^{p_n+\delta} \int_0^{3c} \int_{|x'|<3} |u_\varepsilon(x',\psi(x')+s)|^{p_n}\, dx'ds \\ &\le C \int_0^{3c} \int_{|x'|<3} |u_\varepsilon(x',\psi(x')+s)|^{p_n}\, dx'ds, \end{aligned} \tag{3.24}$$

where (3.20) was used in the first inequality.

To estimate I, we claim that

$$\begin{aligned}&\int_0^{\varepsilon/\varepsilon_0} \int_{|x'|<1} \left|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\right|^{p_n} dx'ds \\ &\qquad \le C \int_0^{3c} \int_{|x'|<3} \left|u_\varepsilon(x',\psi(x')+s)\right|^{p_n} dx'ds.\end{aligned} \tag{3.25}$$

Assume (3.25) for a moment; then it is easy to see that I is handled by (3.25), that is,

$$I \le C \int_0^{3c} \int_{|x'|<3} |u_\varepsilon(x',\psi(x')+s)|^{p_n}\, dx'ds. \tag{3.26}$$

Therefore, we have shown that

$$\int_0^1 \int_{|x'|<1} \left|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\right|^{p_n} dx'ds \le C \int_{Z_3} |u_\varepsilon(x)|^{p_n}\, dx. \tag{3.27}$$

In view of Lemma 3.3 and Sobolev imbedding, this implies that

$$\int_{Z_1} |\nabla u_\varepsilon|^{p_n}\, dx \le C \int_{Z_3} |u_\varepsilon|^{p_n}\, dx \le C \left\{ \int_{Z_3} |\nabla u_\varepsilon|^2\, dx \right\}^{p_n/2}. \tag{3.28}$$

This completes the proof of Theorem 1.1.

Next, it remains to show that the claim (3.25) holds. Observe that $v(x) = u_\varepsilon(\varepsilon x)$ is a weak solution of $\mathcal{L}_1(v) = 0$. Thus by Hardy's inequality and the boundary

Hölder estimate we obtain that,

$$
\begin{aligned}
&\int_0^{1/\varepsilon_0}\int_{|x'|<1/\varepsilon_0}\Big|\frac{v(x',\psi(x')+s)}{s}\Big|^{p_n}\,dx'ds\\
&\le C\int_0^{2/\varepsilon_0}\int_{|x'|<2/\varepsilon_0}|\nabla v(x',\psi(x')+s)|^{p_n}\,dx'ds\\
&\le C\big(\frac{1}{\varepsilon_0}\big)^{n-\frac{n}{2}p_n}\left\{\int_0^{2/\varepsilon_0}\int_{|x'|<2/\varepsilon_0}|\nabla v(x',\psi(x')+s)|^2\,dx'ds\right\}^{p_n/2}\\
&\le C\big(\frac{1}{\varepsilon_0}\big)^{n-\frac{n}{2}p_n-p_n-n+\frac{n}{2}p_n}\int_0^{2/\varepsilon_0}\int_{|x'|<2/\varepsilon_0}|v(x',\psi(x')+s)|^{p_n}\,dx'ds\\
&= C\big(\frac{1}{\varepsilon_0}\big)^{-p_n}\int_0^{2/\varepsilon_0}\int_{|x'|<2/\varepsilon_0}|v(x',\psi(x')+s)|^{p_n}\,dx'ds,
\end{aligned}
\tag{3.29}
$$

where we have used the weak reverse Hölder inequality and Cacciopoli's inequality in the second and third estimates. A scaling argument yields that

$$
\begin{aligned}
&\int_0^{\varepsilon/\varepsilon_0}\int_{|x'|<\varepsilon/\varepsilon_0}\Big|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\Big|^{p_n}\,dx'ds\\
&\le\frac{C}{(\varepsilon)^{p_n}}\int_0^{c\varepsilon/\varepsilon_0}\int_{|x'|<2\varepsilon/\varepsilon_0}\big|u_\varepsilon(x',\psi(x')+s)\big|^{p_n}\,dx'ds.
\end{aligned}
\tag{3.30}
$$

By covering S_1 with surface balls of radius $\varepsilon/\varepsilon_0$, we can deduce from (3.30) that

$$
\begin{aligned}
&\int_0^{\varepsilon/\varepsilon_0}\int_{|x'|<1}\Big|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\Big|^{p_n}\,dx'ds\\
&\le\frac{C}{\varepsilon^{p_n}}\int_0^{c\varepsilon/\varepsilon_0}\int_{|x'|<2}\big|u_\varepsilon(x',\psi(x')+s)\big|^{p_n}\,dx'ds\\
&\le C\int_0^{3c}\int_{|x'|<3}\big|u_\varepsilon(x',\psi(x')+s)\big|^{p_n}\,dx'ds,
\end{aligned}
\tag{3.31}
$$

where we have used Lemma 3.8 in the last inequality. Thus we finish the claim of (3.25).

In the last, it remains to show the ranges $\frac{2n}{n+1}-\delta<p<2$. To do this, suppose that $u_\varepsilon, v_\varepsilon\in W_0^{1,2}(\Omega)$ satisfy $\mathcal{L}_\varepsilon u_\varepsilon=\mathrm{div}F$ and $\mathcal{L}_\varepsilon^* v_\varepsilon=\mathrm{div}g$ for some $F,g\in L^2(\Omega)$. Notice that $\mathcal{L}_\varepsilon^*=-\mathrm{div}(\mathrm{A}^*\nabla)=\mathcal{L}_\varepsilon$ since A is symmetric. Thus we obtain

$$
\int_\Omega F\cdot\nabla v_\varepsilon=\int_\Omega g\cdot\nabla u_\varepsilon. \tag{3.32}
$$

By duality, the above weak formulation implies that if $||\nabla u_\varepsilon||_{L^p(\Omega)}\le C||F||_{L^p(\Omega)}$ holds for $2<p<\frac{2n}{n-1}+\delta$, then we have $||\nabla v_\varepsilon||_{L^p(\Omega)}\le C||g||_{L^p(\Omega)}$ for any $\frac{2n}{n+1}-\delta<p<2$. Hence we complete the proof. □

The next theorem is concerned with the case of $g\neq 0$.

Theorem 3.10. *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $g\in B^{1-\frac{1}{p},p}(\partial\Omega)$, where $\frac{2n}{n+1}-\delta<p<\frac{2n}{n-1}+\delta$. Assume that A is continuous, symmetric, and satisfies* (1.2) *and* (1.3) *and*

$$
\rho(R)\le C[\log R]^{-N} \tag{3.33}
$$

for some $N > 5/2$ and any $R \geq 2$. Let u_ε be a weak solution to $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in Ω and $u_\varepsilon = g$ on $\partial\Omega$. Then

$$||\nabla u_\varepsilon||_{L^p(\Omega)} \leqslant C||g||_{B^{1-1/p,p}(\partial\Omega)}, \tag{3.34}$$

where constants $\delta, C > 0$ are independent of ε.

Proof. The proof could be reduced to the case $g = 0$. Since $g \in B^{1-\frac{1}{p},p}(\partial\Omega)$, it follows from the trace theorem, that there exists $G \in W^{1,p}(\Omega)$ such that $G = g$ on $\partial\Omega$. Moreover, we have

$$||G||_{W^{1,p}(\Omega)} \leqslant C||g||_{B^{1-1/p,p}(\partial\Omega)}. \tag{3.35}$$

Hence we may reduce the general case to the case $G = 0$ by considering the function $u_\varepsilon - G$. Then the desired estimate (3.34) follows from Lemma 3.9 directly. $\square$

We are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let v_ε be a weak solution of $\mathcal{L}_\varepsilon v_\varepsilon = 0$ in Ω and $v_\varepsilon = g$ on $\partial\Omega$. Let w_ε be a weak solution of $\mathcal{L}_\varepsilon w_\varepsilon = \operatorname{div} F$ in Ω and $w_\varepsilon = 0$ on $\partial\Omega$. Also, by setting $u_\varepsilon = v_\varepsilon + w_\varepsilon$, it follows from Theorems 3.9 and 3.10, we get (1.6), thus complete the proof. $\square$

By a similar manner as that of Theorem 1.1, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof of Theorem 1.2 almost follows from the same argument as Theorem 1.1. In view of Lemma 3.4, together with the duality argument, yields (1.8). $\square$

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