EXPLICIT FORMULAS AND VANISHING CONDITIONS FOR CERTAIN COEFFICIENTS OF DRINFELD-GOSS HECKE EIGENFORMS

AHMAD EL-GUINDY

(Communicated by Matthew A. Papanikolas)

Dedicated to the memory of David Goss

ABSTRACT. We obtain a closed form polynomial expression for certain coefficients of Drinfeld-Goss double-cuspidal modular forms which are eigenforms for the degree one Hecke operators with power eigenvalues, and we use those formulas to prove vanishing results for an infinite family of those coefficients.

1. INTRODUCTION

In the classical theory of modular forms, a central role is played by Hecke operators and their eigenforms. The Fourier coefficients of those Hecke eigenforms satisfy elegant relations which encode important number theoretic information. In the setting of Drinfeld modules, there is a parallel theory of modular forms and Hecke operators which was introduced by Goss [8, 9] and expanded on by Gekeler [6] and subsequently studied by many authors. While the basic definitions mirror the classical case in a natural way, there are some significant differences between the nature of the Hecke action in the classical vs. Drinfeld setting, and it is of great interest to unravel the mysteries of the Hecke action in the latter case.

One example of a phenomena that exists exclusively in the Drinfeld setting is the fact that the same system of eigenvalues might appear for different eigenforms. In order to formulate this more precisely, we need to recall some basic definitions and fix some notation. Let $q := p^r$ be a power of a prime p, and consider the finite field $\mathbb{F}_q$. Let $A := \mathbb{F}_q[\theta]$, $K := \mathbb{F}_q(\theta)$. We write A_+ for the set of monic polynomials in A.

For $a \in A$, we define $|a| := q^{\deg a}$ and extend this absolute value to K. The completion of K with respect to this absolute value, denoted by K_∞, is given by $K_\infty = K((\theta^{-1}))$. Let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of K_∞. The field $\mathbb{C}_\infty$ is complete and algebraically closed. The rigid analytic space $\Omega := \mathbb{C}_\infty \setminus K_\infty$ is the function field analogue of the complex upper half-plane, and we shall refer to it as the *Drinfeld upper half-plane*. A holomorphic function $f : \Omega \to \mathbb{C}_\infty$ is called a (Drinfeld-Goss) modular form of weight k (a positive integer) and type m (a residue class in $\mathbb{Z}/(q-1)$) if for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathbf{GL}_2(A)$ we have

$$f(\gamma z) = (cz + d)^k (\det \gamma)^{-m} f(z),$$

in addition to a holomorphicity condition at the 'infinite cusp' (see [6] for the precise definitions). This is clearly strongly analogous to the classical setting, and similar to that case the 'holomorphicity at the cusp' implies a series expansion in powers of a certain 'uniformizer at the cusp'. In the classical setting the uniformizer is $\exp(2\pi i z)$, whereas in the Drinfeld setting that role is played by $t(z)$ given by

$$t(z) := e_C(\tilde{\pi} z)^{-1} = \frac{1}{\tilde{\pi}} \sum_{a \in A} \frac{1}{z + a},$$

where $\tilde{\pi}$ is a fixed choice of a fundamental period of the Carlitz module and e_C is the Carlitz exponential. Let $M_{k,m}$ denote the vector space of modular forms of weight k and type m. For any $f \in M_{k,m}$ and $z \in \mathbb{C}_\infty$ with $|z|_i := \inf_{a \in K_\infty} |z - a|$ sufficiently large ($|z|_i$ is a function field analogue of the 'imaginary distance'), we have

$$(1) \qquad\qquad f(z) = \sum_{n=0}^{\infty} a_n t(z)^n, \qquad\qquad a_n \in \mathbb{C}_\infty,$$

and this expansion determines f uniquely. If $f \in M_{k,m}$ and $a_0 = 0$, then f is called *cuspidal*, and if both $a_0 = 0$ and $a_1 = 0$ we call f *double-cuspidal*. We shall denote the subspace of cuspidal forms inside $M_{k,m}$ by $S_{k,m}$ and the space of double-cuspidal forms by $M_{k,m}^2$. Some examples of modular forms are the weight $k(q-1)$ type 0 Eisenstein series $E_{k(q-1)}$ ($k \geq 1$) and the weight $q+1$ type 1 cusp form h, which are given respectively by

$$(2) \qquad E_{k(q-1)}(z) = \sum_{(a,b) \in A^2 - (0,0)} \frac{1}{(az+b)^{k(q-1)}}, \qquad h(z) = \sum_{a \in A_+} a^q t(az).$$

It is well known that the algebra of modular forms of any weight and type is generated by polynomials in E_{q-1} and h. Furthermore, the subalgebra of type 0 forms is generated by E_{q-1} and the Delta function (normalized to have leading coefficient 1) $\Delta := h^{q-1}$.

Let $\mathfrak{p}$ be a monic prime in A. The $\mathfrak{p}$th Hecke operator of weight k on $M_{k,m}$, denoted by $\mathcal{T}_{\mathfrak{p},k}$, is defined by (see [6, §7])

$$\mathcal{T}_{\mathfrak{p},k} f(z) = \mathfrak{p}^k f(\mathfrak{p}z) + \sum_{b \in A, \deg(b) < \deg(\mathfrak{p})} f\left(\frac{z+b}{\mathfrak{p}}\right).$$

This again is reasonably analogous to the definition of Hecke operators in the classical theory. The Hecke action preserves $M_{k,m}$ as well as the subspaces of cuspidal and double-cuspidal forms. The repeated eigensystem phenomena mentioned above becomes evident with the first few computations of eigenforms, as it was shown by Goss [8] that $\mathcal{T}_{\mathfrak{p}}(E_{q-1}) = \mathfrak{p}^{q-1} E_{q-1}$ and $\mathcal{T}_{\mathfrak{p}}(\Delta) = \mathfrak{p}^{q-1}\Delta$ for all primes $\mathfrak{p} \in A_+$ (we shall customarily drop the weight k from the notation when it is clear from the context). Indeed, using the concept of A-expansion to define cusp forms, Petrov was able to construct infinite families of Hecke eigenforms with the same eigensystem. Namely, it follows from [12, Theorems 1.3 and 2.3] that if k and n are positive integers for which $k - 2n$ is a positive multiple of $(q-1)$ and also $n \leq p^{\mathrm{val}_p(k-n)}$ and if we let G_n denote the nth Goss polynomial for the lattice $\tilde{\pi} A$, then

$$(3) \qquad\qquad f_{k,n}(z) := \sum_{a \in A_+} a^{k-n} G_n(t(az))$$

satisfies $\mathcal{T}_{\mathfrak{p}}(f_{k,n}) = \mathfrak{p}^n f_{k,n}$ for all monic prime $\mathfrak{p}$. It thus follows that for any $n \geq 1$ there are infinitely many eigenforms (in different weights) with eigensystem $\mathfrak{p}^n$. One natural question which we aim to address in this work is to identify common features of forms with such common eigensystems, in particular in connection with their t-expansions. The tools and results developed here naturally lead to interesting new results on the vanishing and non-vanishing of coefficients of eigenforms.

Such questions of vanishing and non-vanishing have been an important topic of study in the classical theory of modular forms (see [3, 10, 11] for example). They were addressed in the Drinfeld setting for certain special modular forms [1, 2, 5, 6]. For instance, Gekeler [6] shows that if $\sum a_i s^i$ is the expansion of the forms Δ or g_{q^k-1} (where g_{q^k-1} is the constant multiple of E_{q^k-1} normalized to have leading coefficient 1 with respect to $s = t^{q-1}$ then $a_i \neq 0$ implies that $i \equiv 0, 1 \pmod{q}$. In [5], Gekeler also shows (among other things, as well as similar results for g_{q^k-1} and h) that for Δ one has $\deg(a_{1+i}) = i$ if and only if $i \equiv 0, q \pmod{q^2}$, which of course implies non-vanishing for the coefficients in those classes. The proofs rely on special properties of Δ (such as its product expansion and explicit representation in terms of Eisenstein series). Those properties don't always carry over to general Hecke eigenforms (cf. Section 5 below); nonetheless our results provide information about an infinite family of non-trivial t-expansion coefficients of any Hecke eigenforms with power eigensystems. This includes all the eigenforms with A-expansions, but it extends beyond that as well (again see Section 5 below for illustrations).

In [1] Armana provided formulas for type 0 and type 1 cusp forms (with level) in terms of power sums of coefficients of the Carlitz module. Those results were extended by Baca and Lopez in [2] by explicitly evaluating those sums (together with some of Gekeler's results from [5,6]) to obtain explict formulas for some of the coefficients of the forms h, Δ, and g_{q^k-1}. Work of the author and Petrov [4] provides another angle on Armana's work, as it was shown that for a level 1 eigenform with power eigensystem of any type, there is an infinite family of coefficients (coinciding with the ones given by Armana for the type 0 and type 1 cases) which are completely determined by the corresponding eigenvalues. One of the purposes of the present work is to make those results more precise by providing explicit formulas for those coefficients and to use that to concretely answer questions of their vanishing and non-vanishing.

In the next section we shall state our main results, throughout assuming $q = p$ is prime for simplicity (this was also assumed in [1, 4], although some results hold in more generality). In Section 3 we study a certain recurrence relation relevant to the Hecke action and prove existence and uniqueness results for its solutions possessing certain natural symmetries. We use those results to prove the main theorem in Section 4 and also completely determine the cases in which the coefficients under study vanish. We conclude with some concrete examples in Section 5.

2. Statement of results

In order to motivate the choice of coefficients we shall focus on, we start by recalling the formula for the Hecke action on the t-expansion from [6],

$$(4) \qquad \mathcal{T}_{\mathfrak{p},k}\left(\sum_{n=0}^{\infty} a_n t^n\right) = \mathfrak{p}^k \sum_{n=0}^{\infty} a_n t_{\mathfrak{p}}^n + \sum_{n=0}^{\infty} a_n G_{n,\mathfrak{p}}(\mathfrak{p}t),$$

where $G_{n,\mathfrak{p}}(X)$ is the nth Goss polynomial of the finite lattice formed by the $\mathfrak{p}$-torsion of the Carlitz module (see [6, (3.4)] for the definition of Goss polynomials of a lattice). For a monic prime $\mathfrak{p}$ of degree 1, Gekeler uses (4) to obtain

$$
\begin{aligned}
(5) \qquad a_n(\mathcal{T}_{\mathfrak{p},k}f) \;&= \mathfrak{p}^k \left(\sum_{j,s \geq 0, jq+s(q-1)=n} (-1)^s \binom{j+s-1}{s} \mathfrak{p}^s a_j \right) \\
&+ \sum_{i=0}^{n-1} \binom{n-1}{i} \mathfrak{p}^{n-i} a_{n+i(q-1)}.
\end{aligned}
$$

There is a clear difference in complexity between (5) and the corresponding formula in the classical setting (namely (slightly abusing notation) $a_n(\mathcal{T}_{p,k}f) = a_{pn} + p^{k-1}a_{n/p}$, where the latter term is 0 if $p \nmid n$). That classical formula implies the well-known fact that the pth coefficient of the Fourier expansion of a normalized Hecke eigenform is actually given by the eigenvalue corresponding to the action of $\mathcal{T}_p$ on that form, whereas in the Drinfeld setting there is no general clear connection between the eigenvalues and the coefficients of the cuspidal expansions. Nonetheless, our results will enable precise computation of a certain family of coefficients of Drinfeld-Goss Hecke eigenforms. Given any integer $n \geq 2$ we can write

$$
(6) \qquad n = 1 + q^{\nu_1} + q^{\nu_2} + \cdots + q^{\nu_\ell}
$$

with integers $\nu_i \geq 0$. In general there can be more than one such representation. However, if we assume that no more than $q-1$ of the ν_i are the same, then the correspondence between n and the length ℓ multiset (i.e., elements may repeat) $\nu = \{\nu_1, \ldots, \nu_\ell\}$ is unique. In this paper our interest is in the cases $\ell \leq q-1$; hence all of the cases we consider will have unique correspondence between n and ν. Let $V_\ell := \{\nu = \{\nu_1, \ldots, \nu_\ell\} : \nu_i \geq 0\}$ be the (infinite) set of all multisets as above. For $\nu \in V_\ell$ we set q^ν to be the integer $q^{\nu_1} + \cdots + q^{\nu_\ell}$. Also, given a set $I \subset I_\ell := \{1, 2, \ldots, \ell\}$ with complement $I^c \subset I_\ell$, we write $\widehat{\nu}(I) := \{\nu_j : j \in I^c\}$ and

$$
\nu^+(I) := \{1 + \nu_i : i \in I\} \cup \widehat{\nu}(I).
$$

(Both $\widehat{\nu}(I)$ and $\nu^+(I)$ are multisets in general.) Note that $\nu^+(\emptyset) = \widehat{\nu}(\emptyset) = \nu$ and that $|\widehat{\nu}(I)| = \ell - |I|$, whereas $\nu^+(I) \in V_\ell$ whenever $\nu \in V_\ell$. It turns out that for $\ell \leq q-1$, the following useful simplification of (5) can be obtained for the family of coefficients indexed by V_ℓ.

Lemma 2.1 ([4, Lemma 5.1]). *For $f = \sum a_i t^i \in M^2_{k,m}$ and $\nu = \{\nu_1, \ldots, \nu_\ell\}$ a multiset of non-negative integers of length $\ell \leq q-1$ we have*

$$
\begin{aligned}
(7) \qquad a_{1+q^\nu}(\mathcal{T}_{\theta,k}f) \;&= \sum_{i=0}^{q^\nu} \binom{q^\nu}{i} \theta^{1+q^\nu-i} a_{1+q^\nu+i(q-1)} \\
&= \sum_{I \subset \{1,\ldots,\ell\}} \theta^{1+q^{\widehat{\nu}(I)}} a_{1+q^{\nu^+(I)}}.
\end{aligned}
$$

Remark 2.2. We take this opportunity to correct a typographical error in the statement of the above lemma in [4], where a factor of $\binom{q^\nu}{q^{\widehat{\nu}(I)}}$ erroneously appeared in the second summand. The binomial coefficient in the first summand is correct however, and, when needed, it is accounted for in the second summand not by a binomial coefficient but rather by repetitions in some of the entries of ν resulting in a number of subsets of I_ℓ yielding the same $\nu^+(I)$. Other than the need to remove the factors

$\left(\begin{smallmatrix} q^\nu \\ q^{\nu(I)} \end{smallmatrix}\right)$ from equations (22) and (23) of [4], that error has no consequences on the results of that paper (which were qualitative in nature).

We are now ready to state the main result of this paper, which provides explicit formulas of some coefficients of eigenforms. The proof will be presented in §4 after gathering some necessary preliminary results in §3.

Theorem 2.3. *Assume $f = \sum_{i=2}^{\infty} a_i t^i \in M_{k,m}^2$ satisfies $\mathcal{T}_{\theta,k}(f) = \theta^{1+q^{N_1}+\cdots+q^{N_\ell}} f$ with $1 \le \ell \le q - 1$ and $N_i \ge 0$. For each $\nu \in V_\ell$ let*

$$(8) \qquad B(\nu) := \prod_{i=1}^{\ell} \prod_{j=0}^{\nu_i - 1} (\theta^{q^{N_i}} - \theta^{q^j}),$$

where we use the standard convention that empty products, as will be the case for the product on j when $\nu_i = 0$, are equal to 1. Consider the action of the symmetric group S_ℓ on $B(\nu)$ given by permuting the N_i; namely

$$(9) \qquad B^\sigma(\nu) := \prod_{i=1}^{\ell} \prod_{j=0}^{\nu_i - 1} (\theta^{q^{N_{\sigma(i)}}} - \theta^{q^j}).$$

Then for all $\nu \in V_\ell$ we have

$$(10) \qquad \ell! a_{1+q^\nu} = a_{1+\ell} \sum_{\sigma \in S_\ell} B^\sigma(\nu).$$

We list a few examples to illustrate the theorem. Note that (8) implies that $B(\nu) = 0$ whenever $\nu_i > N_i$ for some i.

Example 2.4.

(1) If $\mathcal{T}_{\theta,k}(f) = \theta^{1+q^{N_1}} f$, then we are in the case $\ell = 1$ of Theorem 2.3. Thus either $a_{1+q^i} = 0$ for all $i \ge 0$ or we might normalize to have $a_2 = 1$ and

$$a_{1+q^i} = \begin{cases} (\theta^{q^{N_1}} - \theta)(\theta^{q^{N_1}} - \theta^q) \cdots (\theta^{q^{N_1}} - \theta^{q^{(i-1)}}) & \text{if } 1 \le i \le N_1, \\ 0 & \text{if } i > N_1. \end{cases}$$

(2) If $\mathcal{T}_{\theta,k}(f) = \theta^{1+q^{N_1}+q^{N_2}} f$, then we are in the case $\ell = 2$ of Theorem 2.3. Thus either $a_{1+q^{i_1}+q^{i_2}} = 0$ for all $i_1, i_2 \ge 0$ or we might normalize to have $a_3 = 1$ and

$$2a_{2+q} \quad = (\theta^{q^{N_1}} - \theta) + (\theta^{q^{N_2}} - \theta),$$

$$a_{1+2q} \quad = (\theta^{q^{N_1}} - \theta)(\theta^{q^{N_2}} - \theta),$$

$$2a_{2+q^2} \quad = (\theta^{q^{N_1}} - \theta)(\theta^{q^{N_1}} - \theta^q) + (\theta^{q^{N_2}} - \theta)(\theta^{q^{N_2}} - \theta^q),$$

$$2a_{1+q+q^2} = (\theta^{q^{N_1}} - \theta)(\theta^{q^{N_1}} - \theta^q)(\theta^{q^{N_2}} - \theta) + (\theta^{q^{N_2}} - \theta)(\theta^{q^{N_2}} - \theta^q)(\theta^{q^{N_1}} - \theta),$$

$$a_{1+2q^2} \quad = (\theta^{q^{N_1}} - \theta)(\theta^{q^{N_1}} - \theta^q)(\theta^{q^{N_2}} - \theta)(\theta^{q^{N_2}} - \theta^q),$$

$$\vdots$$

$a_{1+q^{i_1}+q^{i_2}} = 0$ whenever $\max(i_1, i_2) > \max(N_1, N_2)$ or $\min(i_1, i_2) > \min(N_1, N_2)$.

3. Existence and uniqueness of universal recurrence solutions

In this section, we study a "universal" form of the recurrence (7) in which we replace the power eigenvalue on the left hand side by a product of generic variables. Namely let ℓ be an integer satisfying $1 \leq \ell \leq q-1$ and let $x_1, \ldots, x_\ell$ be a collection of variables. Consider the recurrence

$$(11) \qquad b_{q^\nu} \prod_{i=1}^{\ell} x_i = \sum_{I \subset \{1,\ldots,\ell\}} b_{q^{\nu^+(I)}} \prod_{j \in I^c} \theta^{q^{\nu_j}},$$

where I^c denotes the set-theoretic complement of I in $\{1, \ldots, \ell\}$ and ν runs over all multisets in V_ℓ. In general such a recurrence could have many solutions, but we will show that when a certain natural condition is satisfied, there is a unique solution with a given initial value, and we also exhibit explicit formulas for that solution. Our interest will be in solutions to the recurrence with a certain symmetry. Namely, set $Y := \mathbb{F}_q[x_1, \ldots, x_\ell, \theta]$ and consider the operator D_1 on Y defined by

$$(12) \quad D_1 f(x_1, \ldots, x_\ell, \theta) = f(x_1+1, x_2+1, \ldots, x_\ell+1, \theta+1) - f(x_1, x_2, \ldots, x_\ell, \theta).$$

Furthermore, let $D_j := D_1^j$ for $0 \leq j \leq q-1$. We shall call an element y of Y *translation invariant* if $D_1(y) = 0$. The set of translation invariant elements forms an $\mathbb{F}_q$ sub-algebra of Y. We shall call a sequence of elements in Y translation invariant if each member is translation invariant.

Theorem 3.1 (Uniqueness). *Let b_{q^ν} and c_{q^ν} be two translation invariant sequences in Y indexed by V_ℓ such that both satisfy* (11). *If $b_\ell = c_\ell$ (which corresponds to $\nu = \{0, 0, \ldots, 0\} \in V_\ell$), then $b_{q^\nu} = c_{q^\nu}$ for all $\nu \in V_\ell$.*

Proof. Without loss of generality we can order the entries of any $\nu \in V_\ell$ in non-ascending order. Furthermore, we can define lexicographical order on V_ℓ in the usual way. It is easy to verify that this defines a total order on V_ℓ. If the theorem is not true, then there must be a minimal multiset ν for which $b_{q^\nu} \neq c_{q^\nu}$. We can't have $\nu = \{0, \ldots, 0\}$ since $b_\ell = c_\ell$ is given. Thus at least one entry of ν is non-zero. Define a multiset $\mu \in V_\ell$ by

$$\mu_j = \begin{cases} \nu_j - 1 & \text{if } \nu_j > 0, \\ 0 & \text{if } \nu_j = 0. \end{cases}$$

Let e and e' denote the number of 0 entries in ν and μ, respectively. Clearly $e' \geq e$ and $\nu = \mu^+(\{1, 2, \ldots, \ell - e\})$. In fact, we have $\mu^+(I) = \nu$ exactly when $I = \{1, 2, \ldots, \ell - e'\} \cup J$, where J is any subset with exactly $e' - e$ elements of $\{\ell - e' + 1, \ell - e' + 2, \ldots, \ell\}$. Furthermore if $I \subset \{1, 2, \ldots, \ell\}$ satisfies $|I| \leq \ell - e$, then $\mu^+(I) \leq \nu$ in lexicographical order. (This is clear for $I \subset \{1, 2, \ldots, \ell - e\}$. If I contains entries j larger than $\ell - e$, then it will have to miss an equal number of entries $i_j \leq \ell - e$ and we have $1 = \mu_j + 1 \leq \nu_{i_j}$). It is easy to verify that if $|I^c| < e$, then $D_e(\prod_{j \in I^c} \theta^{q^{\nu_j}}) = 0$, whereas for $|I^c| = e$ we have $D_e(\prod_{j \in I^c} \theta^{q^{\nu_j}}) = e!$. Thus applying D_e to (11) annihilates all terms with $|I| > \ell - e$. Among the remaining terms there are exactly $\binom{e'}{e}$ terms with $\mu^+(I) = \nu$ (all with $|I| = \ell - e$). We thus have

$$(13)$$

$$b_{q^\mu} D_e \left(\prod_{i=1}^{\ell} x_i \right) - \sum_{I \subset \{1,\ldots,\ell\},\, |I| \leq \ell - e,\, \mu^+(I) \neq \nu} b_{q^{\mu^+(I)}} D_e \left(\prod_{j \in I^c} \theta^{q^{\nu_j}} \right) = \binom{e'}{e} e! b_{q^\nu}.$$

By the minimality of ν, we have $b_{\mu^+(I)} = c_{\mu^+(I)}$ for all the subsets on the left hand side (including $\mu = \mu^+(\emptyset)$), and since $\binom{e'}{e}e! \neq 0$ we obtain $b_{q^\nu} = c_{q^\nu}$, a contradiction which proves the theorem. $\qquad\square$

Our next goal is to present explicit solutions to the recurrence (11). To achieve that, consider the action of the symmetric group S_ℓ on $Y := \mathbb{F}_q[x_1, \ldots, x_\ell, \theta]$ given by $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(\theta) = \theta$, and extended naturally to all of Y. We shall also need the following lemma.

Lemma 3.2. *Let $x_1, \ldots, x_\ell$ and $t_1, \ldots, t_\ell$ be any collections of variables. Then the following identity holds:*

$$(14) \qquad \prod_{i=1}^{\ell} x_i = \sum_{I \subset \{1,2,\ldots,\ell\}} \left[\prod_{i \in I}(x_i - t_i) \prod_{j \in I^c} t_j \right].$$

Proof. We shall proceed by induction on ℓ. For $\ell = 1$ we clearly have $x_1 = (x_1 - t_1) + t_1$, where the first summand corresponds to $I = \{1\}$ and the second to $I = \emptyset$. Assuming the truth of the statement for ℓ, write

$$\prod_{i=1}^{\ell+1} x_i = [(x_{\ell+1} - t_{\ell+1}) + t_{\ell+1}] \prod_{i=1}^{\ell} x_i$$

$$= \sum_{I \subset \{1,\ldots,\ell+1\}, \ell+1 \in I} \left[\prod_{i \in I}(x_i - t_i) \prod_{j \in I^c} t_j \right] + \sum_{I \subset \{1,\ldots,\ell+1\}, \ell+1 \in I^c} \left[\prod_{i \in I}(x_i - t_i) \prod_{j \in I^c} t_j \right]$$

$$= \sum_{I \subset \{1,\ldots,\ell+1\} \in I} \left[\prod_{i \in I}(x_i - t_i) \prod_{j \in I^c} t_j \right].$$

The last equality, which completes the induction and establishes the result, follows since the conditions $\ell + 1 \in I$ and $\ell + 1 \in I^c$ partition the subsets of $\{1, \ldots, \ell + 1\}$ into two non-overlapping collections. $\qquad\square$

Theorem 3.3 (Explicit formulas)**.** *Assume the notation above, and for any multiset $\nu \in V_\ell$ set*

$$(15) \qquad P(\nu) := \prod_{i=1}^{\ell} \prod_{j=0}^{\nu_i - 1} (x_i - \theta^{q^j}).$$

Then the sequence given for any $\nu \in V_\ell$ by

$$(16) \qquad b_{q^\nu} = \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \sigma(P(\nu))$$

is a solution to (11), which satisfies $b_\ell = 1$.

Proof. We start by noting that for any $I \subset \{1, \ldots, \ell\}$ we have

$$P(\nu^+(I)) = P(\nu) \prod_{i \in I}(x_i - \theta^{q^{\nu_i}}).$$

The following verification shows that b_{q^ν} satisfies (11):

$$
\begin{aligned}
\ell! \sum_{I \subset \{1,\dots,\ell\}} b_{q^{\nu^+(I)}} \prod_{j \in I^c} \theta^{q^{\nu_j}} \;&=\; \sum_{\sigma \in S_\ell} \sum_{I \subset \{1,\dots,\ell\}} \prod_{j \in I^c} \theta^{q^{\nu_j}} \sigma(P(\nu^+(I))) \\
&=\; \sum_{\sigma \in S_\ell} \sigma \left(\sum_{I \subset \{1,\dots,\ell\}} \prod_{j \in I^c} \theta^{q^{\nu_j}} P(\nu^+(I)) \right) \\
&=\; \sum_{\sigma \in S_\ell} \sigma \left(P(\nu) \sum_{I \subset \{1,\dots,\ell\}} \prod_{j \in I^c} \theta^{q^{\nu_j}} \prod_{i \in I} (x_i - \theta^{q^{\nu_i}}) \right) \\
&=\; \prod_{i=1}^{\ell} x_i \sum_{\sigma \in S_\ell} \sigma\left(P(\nu)\right) = \ell! b_{q^\nu} \prod_{i=1}^{\ell} x_i,
\end{aligned}
$$

where the penultimate equality follows from Lemma 3.2 and the fact that $\sigma\left(P(\nu)\prod_{i=1}^{\ell} x_i\right) = \left(\prod_{i=1}^{\ell} x_i\right)\sigma(P(\nu))$ since $\prod_{i=1}^{\ell} x_i$ is invariant under S_ℓ. This completes the proof. $\qquad\square$

4. Proof of Theorem 2.3 and consequences

Proof of Theorem 2.3. It is clear that (11) and the quantities $P(\nu)$ specialize to (7) and $B(\nu)$, respectively, upon setting $x_i = \theta^{q^{N_i}}$ for $1 \le i \le \ell$ and substituting a_{1+q^ν} for b_{q^ν}. It thus follows that we can obtain (10) from (16) once we can establish the translational invariance property that we used in Theorem 3.1. This can be deduced by noting that if we replace θ by a translate $\tilde\theta = \theta + c$ with $c \in \mathbb{F}_q$ as a generator of A over $\mathbb{F}_q$, then the corresponding parameter $\tilde t$ at the cusp will be equal to t. It also follows that if $\mathcal{T}_{\theta,k} f = \theta^N f$, then $\mathcal{T}_{\tilde\theta,k} f = \tilde\theta^N f$, and hence the coefficients of f must be invariant under $\theta \mapsto \tilde\theta$, which implies that $D_1(a_i) = 0$ for all coefficients of f. $\qquad\square$

Remark 4.1. The details of the proof above show that Theorem 2.3 remains unchanged if the prime θ is replaced by $\theta + c$ with $c \in \mathbb{F}_q$. On the other hand, it would be interesting to generalize the results of this work for primes of degree 2 or higher, the main difficulty being that the analogue of (5) seems daunting for such primes. Nonetheless we would like to express our gratitude to the referees for pointing out the interest in such extensions, as well as other questions of generalizations to cases with $\ell \ge q$ or to more general eigensystems, and we hope to be able to address some of those generalizations in future work.

Using formula (10) as well as the tools developed in §3, we can now state the following vanishing criteria which generalize what was observed in the examples of §2.

Theorem 4.2. *Let* $f \in M^2_{k,m}$ *be as in Theorem 2.3, and further assume that* $N_1 \ge N_2 \ge \cdots \ge N_\ell$. *Consider* $\nu \in V_\ell$, *again with the ordering* $\nu_1 \ge \nu_2 \ge \cdots \ge \nu_\ell$. *Then we have* $a_{1+q^\nu} = 0$ *if and only if* $N_i < \nu_i$ *for some* $1 \le i \le \ell$.

Proof. First, we prove that the condition $\nu_i > N_i$ implies vanishing. For such ν, we clearly have $B(\nu) = 0$. We would like to also show that $B^\sigma(\nu) = 0$ for all $\sigma \in S_\ell$, which would follow if we can show that for any such σ there exists an index $j \in \{1,\dots,\ell\}$ for which $\nu_j > N_{\sigma(j)}$. Indeed we either have that σ permutes the

elements of $\{1, \ldots, i-1\}$ among themselves, in which case we must have $\sigma(i) \geq i$ and hence $N_{\sigma(i)} \leq N_i < \nu_i$, or else there is a value $j \in \{1, \ldots, i-1\}$ with $\sigma(j) \geq i$, again yielding $N_{\sigma(j)} \leq N_i < \nu_i \leq \nu_j$, establishing vanishing. To prove necessity, we note that if $\nu_i \leq N_i$ for all $1 \leq i \leq \ell$, then clearly $B(\nu) \neq 0$. However, we might still have $B^\sigma(\nu) = 0$ for some of the permutations $\sigma \in S_\ell$. Indeed, if we write, for $1 \leq j \leq \ell$, $i_j := \min\{i : N_j \geq \nu_i\}$, and set $U := \{\sigma \in S_\ell : i_j \leq \sigma(j) \leq \ell \text{ for all } j\}$, then it is not hard to see that $B^\sigma(\nu) \neq 0$ if and only if $\sigma \in U$. (Note that $1 \leq i_j \leq j$ and hence the identity permutation is clearly in U). All the polynomials $B^\sigma(\nu)$ for $\sigma \in U$ have the same lowest degree term (namely $(-\theta)^w$ with $w := \sum_{i=1}^\ell \frac{q^{\nu_i}-1}{q-1}$). A simple counting argument gives that

$$|U| = \prod_{j=2}^{\ell} (j - i_j + 1),$$

and hence the size of U is not divisible by p since none of its factors is (we are utilizing $\ell \leq q-1$ and $q = p$ in this step). Thus the lowest degree term of $\sum_{\sigma \in U} B^\sigma(\nu)$ has non-vanishing coefficient $|U|(-1)^w$, and it follows that $\sum_{\sigma \in S_\ell} B^\sigma(\nu) = \sum_{\sigma \in U} B^\sigma(\nu)$ doesn't vanish whenever $N_i \geq \nu_i$ for all i, completing the proof. □

Remark 4.3. We would like to point out that there is a subtle yet important difference between the action on S_ℓ on the polynomials $P(\nu)$ and on $B(\nu)$. The former is in fact an action on all of Y and defines an endomorphism; in particular we have $\sigma(0) = 0$. On the other hand, the action of S_ℓ appearing in (10) is only a permutation action of certain parameters in the definition of $B(\nu)$ and doesn't extend to all of A. As an explicit example, if we take $\nu = \{2, 1\}$ and $\{N_1, N_2\} = \{3, 1\}$, then $B(\nu) = (\theta^{q^3} - \theta)(\theta^{q^3} - \theta^q)(\theta^q - \theta)$, but $B^\sigma(\nu) = 0$ with $\sigma = (1\ 2)$.

5. EXAMPLES

We look more closely at some concrete cases of Theorem 2.3 and in particular of Example 2.4. We work with $q = 3$, $\ell = 1$, and $N_1 = 1$, thus yielding the eigensystem $\mathfrak{p}^4$. Working out the conditions in [12, Theorem 1.3], we have the infinite family of forms with that eigensystem which is given for $n \geq 1$ by

$$f_{18n+4,4} = \sum_{a \in A_+} a^{18n} G_4(t_a),$$

where the Goss polynomial G_4 is given explicitly by $G_4(X) = X^4 + \frac{X^2}{\theta^3 - \theta}$. In particular, we see that we have such eigenforms with A-expansion in the set of weights $\{22, 40, 58, 76, \ldots\}$. In addition, direct computations on lower weights reveal normalized (i.e., leading coefficient $a_2 = 1$) double-cuspidal eigenforms ϕ_{12} and ϕ_{20} both with eigensystem $\mathfrak{p}^4$ in weights 12 and 20 respectively. Neither ϕ_{12} nor ϕ_{20} possesses an A-expansion. Nonetheless, all of these forms fall under the scope of Example 2.4(1), and as the t-expansions illustrate, they indeed *all* have $a_4 = \theta^3 - \theta$

(in $\mathbb{F}_3$) and $a_{10} = a_{28} = a_{82} = 0 = a_{1+3^i}$ for all $i \geq 2$. Explicitly we have (with $g := g_{q-1} = g_2$)

$$\phi_{12} = h^2 g^2 = t^2 + (\theta^3 + 2\theta)t^4 + (\theta^6 + \theta^4 + \theta^2 + 2)t^6 + (\theta^9 + 2\theta^3)t^{12} + t^{14}$$
$$+ (2\theta^3 + \theta)t^{16} + 2t^{18} + (2\theta^9 + 2\theta^3 + 2\theta)t^{20} + (2\theta^{12} + \theta^{10} + 2\theta^6 + \theta^2)t^{22}$$
$$+ (2\theta^{15} + 2\theta^{13} + 2\theta^{11} + \theta^9 + \theta^7 + \theta^5 + \theta^3 + 2\theta)t^{24} + t^{26}$$
$$+ (2\theta^{18} + \theta^{12} + \theta^{10} + 2\theta^4 + 2)t^{30} + O(t^{32}),$$

$$\phi_{20} = t^2 + (\theta^3 + 2\theta)t^4 + (\theta^6 + \theta^4 + \theta^2 + 2)t^6 + (2\theta^9 + \theta)t^8 + (2\theta^9 + 2\theta^3 + 2\theta)t^{12}$$
$$+ (\theta^{18} + \theta^{12} + 2\theta^4 + 2\theta^2 + 1)t^{14} + (2\theta^3 + \theta)t^{16} + (2\theta^{18} + \theta^{12} + \theta^{10} + 2\theta^4 + 2)t^{18}$$
$$+ (\theta^{21} + 2\theta^{19} + 2\theta^{15} + \theta^{11} + \theta^9 + \theta^7 + 2\theta^5 + 2\theta^3)t^{20} + (\theta^{12} + 2\theta^{10} + 2\theta^6 + \theta^4)t^{22}$$
$$+ (\theta^{15} + \theta^{13} + \theta^{11} + 2\theta^9 + 2\theta^7 + 2\theta^5 + 2\theta^3 + \theta)t^{24} + (\theta^{18} + \theta^{10} + \theta^2 + 1)t^{26}$$
$$+ (2\theta^{12} + \theta^{10} + \theta^4 + 2\theta^2 + 2)t^{30} + O(t^{32}).$$

Set $\phi_{22} = (\theta^3 - \theta)f_{22,4}$. We have $\phi_{22} = h^2 g^7 - (\theta^3 - \theta)h^4 g^3$ and

$$\phi_{22} = t^2 + (\theta^3 + 2\theta)t^4 + 2t^6 + (\theta^9 + 2\theta)t^8 + (2\theta^9 + \theta)t^{12}$$
$$+ (\theta^{18} + \theta^{12} + 2\theta^4 + 2\theta^2 + 1)t^{14} + (2\theta^{21} + \theta^{19} + 2\theta^{13} + \theta^{11} + 2\theta^5 + T)t^{16}$$
$$+ (2\theta^{18} + 2\theta^{10} + \theta^6 + \theta^4 + 2)t^{18} + (2\theta^{21} + \theta^{19} + 2\theta^{13} + \theta^{11} + \theta^9 + 2\theta^5 + 2\theta^3 + \theta)t^{20}$$
$$+ (2\theta^{24} + 2\theta^{22} + 2\theta^{20} + 2\theta^{16} + 2\theta^{14} + 2\theta^{10} + 2\theta^8 + 2\theta^4 + 2\theta^2)t^{22} + t^{26}$$
$$+ 2t^{30} + O(t^{32}).$$

Remark 5.1. It is instructive to compare the previous examples with the expansion of $\Delta = \sum a_i s^i \in M_{8,0}^2$ in powers of $s = t^2$ over $\mathbb{F}_3$. As alluded to in the Introduction, Gekeler's general results [5,6] imply the following for $a_i \in \mathbb{F}_3[\theta]$: (i) $a_i \neq 0$ implies that $i \equiv 0, 1 \pmod 3$, (ii) $\deg(a_{1+i}) \leq i$, and (iii) $\deg(a_{1+i}) = i$ if and only if $i \equiv 0, 3 \pmod 9$. The first few instances of those results may indeed be verified from the initial terms of the expansion of Δ below. On the other hand, it is clear that those special properties of Δ are not shared by any of the forms ϕ_{12}, ϕ_{20}, and ϕ_{22} above:

$$\Delta = t^2 + 2t^6 + (\theta^3 + 2\theta)t^8 + t^{14} + 2t^{18} + (2\theta^9 + 2\theta^3 + 2\theta)t^{20} + (\theta^9 + 2\theta^3)t^{24}$$
$$+ (2\theta^{12} + \theta^{10} + \theta^6 + 2\theta^4 + 1)t^{26} + 2t^{30} + O(t^{32}).$$

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CAIRO UNIVERSITY, GIZA, EGYPT 12613

Current address: Science Program, Texas A&M University in Qatar, Doha, Qatar

Email address: a.elguindy@gmail.com