

ON AN EFFECTIVE VARIATION OF KRONECKER'S APPROXIMATION THEOREM AVOIDING ALGEBRAIC SETS

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ABSTRACT. Let $\Lambda \subset \mathbb{R}^n$ be an algebraic lattice coming from a projective module over the ring of integers of a number field K . Let $\mathcal{Z} \subset \mathbb{R}^n$ be the zero locus of a finite collection of polynomials such that $\Lambda \not\subseteq \mathcal{Z}$ or a finite union of proper full-rank sublattices of Λ . Let K_1 be the number field generated over K by coordinates of vectors in Λ , and let L_1, \dots, L_t be linear forms in n variables with algebraic coefficients satisfying an appropriate linear independence condition over K_1 . For each $\varepsilon > 0$ and $\mathbf{a} \in \mathbb{R}^n$, we prove the existence of a vector $\mathbf{x} \in \Lambda \setminus \mathcal{Z}$ of explicitly bounded sup-norm such that

$$\|L_i(\mathbf{x}) - a_i\| < \varepsilon$$

for each $1 \leq i \leq t$, where $\|\cdot\|$ stands for the distance to the nearest integer. The bound on sup-norm of \mathbf{x} depends on ε , as well as on Λ , K , \mathcal{Z} , and heights of linear forms. This presents a generalization of Kronecker's approximation theorem, establishing an effective result on density of the image of $\Lambda \setminus \mathcal{Z}$ under the linear forms L_1, \dots, L_t in the t -torus $\mathbb{R}^t/\mathbb{Z}^t$.

1. INTRODUCTION

Let $1, \theta_1, \dots, \theta_t$ be \mathbb{Q} -linearly independent real numbers. The classical approximation theorem of Kronecker then states that the set of points

$$\{(\{n\theta_1\}, \dots, \{n\theta_t\}) : n \in \mathbb{Z}\}$$

is dense in the t -torus $\mathbb{R}^t/\mathbb{Z}^t$, where $\{\cdot\}$ stands for the fractional part of a real number. This result was originally obtained by Kronecker [23] in 1884 and presents a deep generalization of Dirichlet's 1842 theorem on Diophantine approximation [6]; see, for instance, [19] for a detailed exposition of these classical results.

Kronecker's theorem can also be viewed as a statement on density of the image of the integer lattice under collection of linear forms in the torus $\mathbb{R}^t/\mathbb{Z}^t$ (compare to the famous Oppenheim conjecture for quadratic forms). Specifically, if L_1, \dots, L_t are linear forms in n variables with real coefficients b_{ij} so that the set of numbers 1 and b_{ij} are linearly independent over \mathbb{Q} , then for any $\varepsilon > 0$ and $\mathbf{a} \in \mathbb{R}^t$ there exists $\mathbf{x} \in \mathbb{Z}^n$ such that

$$(1) \quad \|L_i(\mathbf{x}) - a_i\| < \varepsilon \quad \forall 1 \leq i \leq t,$$

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where $\| \cdot \|$ stands for the distance to the nearest integer. A nice survey of a wide variety of results related to Kronecker's theorem is given in [17]. Classical quantitative results in this direction are related to transference theorems for homogeneous and inhomogeneous approximation for the system of linear forms $L_i(\mathbf{x})$ (see [21], [3, Chapter V], and [2]). In particular, these results give effective bounds for the size of the coordinates of the vector \mathbf{x} in (1) under the assumption that there are effective lower bounds for $\max_i \|L_i(\mathbf{x})\|$ in the homogeneous case. Some additional effective results can also be found in [25], [27].

The main goal of this note is somewhat different. We consider linear forms with algebraic coefficients and extend the previously known versions of Kronecker's theorem in three ways:

- (1) allow for the approximating vector \mathbf{x} as in equation (1) above to come from an algebraic lattice Λ ,
- (2) exclude vectors from a prescribed union \mathcal{Z} of projective varieties or sublattices not containing this lattice, that is, we are interested in approximation vectors $\mathbf{x} \in \Lambda \setminus \mathcal{Z}$,
- (3) we obtain effective constants everywhere in our upper bounds.

Effective Diophantine avoidance results, exhibiting solutions to a given problem outside of a prescribed algebraic set, can be viewed as statements on distribution of such solutions: not only do small solutions exist, they are also sufficiently well distributed so that it is not possible to “cut them out” by any finite union of varieties. In recent years, such results were obtained in the general context of Siegel's lemma (also generalizing Faltings' version of Siegel's lemma [8], [22], [7]) in [10], [11], [12], [14], [15], [20], and in the context of Cassels' theorem on small zeros of quadratic forms and its generalizations in [9], [5], [13], [16]. We will extend these investigations to Kronecker's theorem. To obtain effective constants in our bounds we use Liouville-type inequalities (see Remark 3.1 below for stronger non-effective inequalities of similar type, which can be derived from Schmidt's Subspace Theorem). To give precise statements of our results, we need some notation.

1. *The lattice.* Let $n \geq 1$ be an integer, and for each vector $\mathbf{x} \in \mathbb{R}^n$ define the sup-norm

$$|\mathbf{x}| := \max_{1 \leq i \leq n} |x_i|.$$

Let K be a number field of degree $d = r_1 + 2r_2$ over \mathbb{Q} , where r_1 and r_2 are numbers of its real and complex places, respectively, and write \mathcal{O}_K for its ring of integers. Let $1 \leq s \leq w$ be integers, and let $\mathcal{M} \subset K^w$ be an \mathcal{O}_K -module such that $\mathcal{M} \otimes_K K \cong K^s$. Write $\mathcal{D}_K(\mathcal{M})$ for the discriminant of \mathcal{M} . Define $\mathfrak{U}_K(\mathcal{M})$, a fractional \mathcal{O}_K -ideal in K , to be

$$(2) \quad \mathfrak{U}_K(\mathcal{M}) = \{\alpha \in K : \alpha \mathcal{M} \subseteq \mathcal{O}_K^w\}.$$

We let $\Lambda_K(\mathcal{M}) \subset \mathbb{R}^{wd}$ be the lattice of rank sd , which is the image of \mathcal{M} under the standard Minkowski embedding.

2. *The projective varieties.* Let $m \geq 1$ be an integer. For each $1 \leq i \leq m$, let \mathcal{S}_i be a finite set of homogeneous polynomials in $\mathbb{R}[x_1, \dots, x_{wd}]$ and let $Z(\mathcal{S}_i)$ be its zero set in \mathbb{R}^{wd} , that is,

$$Z(\mathcal{S}_i) = \{\mathbf{x} \in \mathbb{R}^{wd} : P(\mathbf{x}) = 0 \text{ for all } P \in \mathcal{S}_i\}.$$

For the collection $\mathcal{S} := \{\mathcal{S}_1, \dots, \mathcal{S}_m\}$ of finite sets of homogeneous polynomials, define

$$(3) \quad \mathcal{Z}_{\mathcal{S}} := \bigcup_{i=1}^m \mathcal{Z}(\mathcal{S}_i)$$

and

$$(4) \quad M_{\mathcal{S}} := \sum_{i=1}^m \max\{\deg P : P \in \mathcal{S}_i\}.$$

We allow for the possibility that $\mathcal{Z}_{\mathcal{S}} = \{\mathbf{0}\}$, in which case we instead take $M_{\mathcal{S}} = 1$. Notice that $\mathcal{Z}_{\mathcal{S}}$ is an algebraic set, which is a union of a finite collection of projective varieties. Assume that the lattice $\Lambda_K(\mathcal{M})$ is not contained in the set $\mathcal{Z}_{\mathcal{S}}$.

3. *The linear forms.* Let $K_1 = K(\Lambda_K(\mathcal{M}))$, i.e., K_1 is the number field generated over K by the entries of any basis matrix of the lattice $\Lambda_K(\mathcal{M})$. Let $B := (b_{ij})_{1 \leq i \leq t, 1 \leq j \leq wd}$ be a $t \times wd$ matrix with real algebraic entries so that $1, b_{11}, \dots, b_{t(wd)}$ are linearly independent over K_1 , and let $\ell = [E : \mathbb{Q}]$ where $E = K_1(b_{11}, \dots, b_{t(wd)})$. We will also write $\ell_v = [E_v : \mathbb{Q}_v]$ for the local degree of E at every place $v \in M(E)$. Define t linear forms in wd variables:

$$(5) \quad L_i(x_1, \dots, x_{wd}) = \sum_{j=1}^{wd} b_{ij} x_j \in \mathbb{R}[x_1, \dots, x_{wd}] \quad \forall 1 \leq i \leq t.$$

Our first goal here is to prove the following effective result on density of the image of the set $\Lambda_K(\mathcal{M}) \setminus \mathcal{Z}_{\mathcal{S}}$ under the linear forms L_1, \dots, L_t in the torus $\mathbb{R}^t / \mathbb{Z}^t$. Let h denote the usual Weil height on algebraic numbers, as well as its extension to vectors with algebraic coordinates; we recall the definition of height along with other necessary notation in Section 2.

Theorem 1.1. *Let $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$, and let $\varepsilon > 0$. There exist $\mathbf{x} \in \Lambda_K(\mathcal{M}) \setminus \mathcal{Z}_{\mathcal{S}}$ and $\mathbf{p} \in \mathbb{Z}^t$ such that*

$$|L_i(\mathbf{x}) - a_i - p_i| < \varepsilon$$

and

$$|\mathbf{x}| \leq \mathbf{a}_K(t, \ell, s) \left(sd M_{\mathcal{S}} |\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}} \right)^{\mathfrak{K}+1} \left((wd)^{\frac{3}{2}} h(B) \right)^{\mathfrak{K}} \mathbf{c}_K(\mathcal{M}, \ell, t) \varepsilon^{-\ell+1},$$

where the exponent $\mathfrak{K} = \ell^2(t+1) - \ell$ and the constants are

$$\mathbf{a}_K(t, \ell, s) = 2^{\ell t(\ell-1) + sr_1 \mathfrak{K} + \frac{s(d-1)}{2}} (t+1)^{3\ell-1} (t!)^{2\ell}$$

and

$$\mathbf{c}_K(\mathcal{M}, \ell, t) = \min \left\{ h(\alpha)^{(\mathfrak{K}+1)sd-1} h(\alpha^{-1})^{\mathfrak{K}} : \alpha \in \mathfrak{U}_K(\mathcal{M}) \right\}.$$

One special case of Theorem 1.1 is when $\mathcal{Z}_{\mathcal{S}}$ is a union of linear spaces, which means that the point \mathbf{x} in question is in $\Lambda_K(\mathcal{M})$ but outside of a union of sublattices of smaller rank than $\Lambda_K(\mathcal{M})$. What if the rank of such sublattices is equal to the rank of $\Lambda_K(\mathcal{M})$? The next theorem addresses this situation.

Theorem 1.2. *Let $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$, and let $\varepsilon > 0$. Let $m > 0$, let $\Gamma_1, \dots, \Gamma_m \subset \Lambda_K(\mathcal{M})$ be proper sublattices of full rank and respective determinants $\mathcal{D}_1, \dots, \mathcal{D}_m$, and let $\mathcal{D} = \mathcal{D}_1 \cdots \mathcal{D}_m$. Then for every $\alpha \in \mathfrak{U}_K(\mathcal{M})$ there exist $\mathbf{x} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$ and $\mathbf{p} \in \mathbb{Z}^t$ such that*

$$|L_i(\mathbf{x}) - a_i - p_i| < \varepsilon$$

and

$$|\mathbf{x}| \leq \left(\mathbf{b}_K(t, \ell, s, w) (h(\alpha)h(\alpha^{-1})h(B)\mathcal{E}_\alpha)^{\mathfrak{K}} \frac{\mathcal{D} \varepsilon^{-\ell+1}}{|\mathcal{D}_K(\mathcal{M})|^{\frac{sm}{2}}} + 1 \right) \mathcal{E}_\alpha,$$

where the exponent $\mathfrak{K} = \ell^2(t+1) - \ell$, as in Theorem 1.1, the constant

$$\mathbf{b}_K(t, \ell, s, w) = 2^{\ell t(\ell-1) + \frac{\mathfrak{K}}{2} + smr_2} (t+1)^{3\ell-1} (t!)^{2\ell} (wd)^{\frac{3\mathfrak{K}}{2}},$$

and where \mathcal{E}_α equals

$$(6) \quad \mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m) := 2^{\frac{sr_1-1}{2}} h(\alpha)^{sd-1} |\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}} \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}}.$$

Remark 1.1. The bounds of Theorems 1.1 and 1.2 can be recorded in a slightly weaker simplified form as

$$|\mathbf{x}| \ll (\det \Lambda_K(\mathcal{M}))^{\mathfrak{K}+1} h(B)^{\mathfrak{K}} \mathfrak{c}_K(\mathcal{M}, \ell, t) \varepsilon^{-\ell+1}$$

and

$$|\mathbf{x}| \ll \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} \right)^{\mathfrak{K}+2} (\det \Lambda_K(\mathcal{M}))^{\mathfrak{K}-m+1} h(B)^{\mathfrak{K}} \mathfrak{c}_K(\mathcal{M}, \ell, t) \varepsilon^{-\ell+1},$$

respectively, where the constants in the \ll Vinogradov notation depend on the number field K and the integer parameters t, ℓ, s, m . The expression $\mathfrak{c}_K(\mathcal{M}, \ell, t)$ can be viewed as a certain measure of arithmetic complexity of \mathcal{M} ; in particular, if $\mathcal{M} \subseteq \mathcal{O}_K^w$, then $\mathfrak{c}_K(\mathcal{M}, \ell, t) = 1$.

Here is a sketch of the proofs of Theorems 1.1 and 1.2. We first construct a point $\mathbf{y} \in \Lambda_K(\mathcal{M})$ of controlled sup-norm, which is outside of \mathcal{Z}_S or $\bigcup_{i=1}^m \Gamma_i$, respectively: in the first case, we use the classical Minkowski's Successive Minima Theorem and a version of Alon's *Combinatorial Nullstellensatz* [1] (we use the convenient formulation developed in [12]), while in the second we employ a recent result of Henk and Thiel [20] on points of small norm in a lattice outside of a union of full-rank sublattices. We use \mathbf{y} to construct an infinite sequence of points $n\mathbf{y}$ satisfying the above conditions, and use an effective version of Kronecker's original theorem to obtain a value of the index n (depending on $\varepsilon > 0$) for which the required inequalities on values of linear forms are satisfied. In other words, our avoidance strategy is to follow the line $n\mathbf{y}$ until a necessary point is found. One may wish to use a similar strategy, but following a higher-dimensional subspace of the ambient space in the hope of a better bound, however, it is difficult to guarantee avoiding our fixed algebraic set with such strategy. A convenient effective version of Kronecker's theorem that we use is worked out in Section 3. It should be remarked that the most important feature of approximation results such as our Theorems 1.1 and 1.2 is the exponent on ε in the bounds for $|\mathbf{x}|$. As we show, this exponent is the same as in the corresponding bound of the effective version of Kronecker's theorem that we use.

In Section 2 we introduce the necessary notation and provide all the details of our setup. We derive an effective version of Kronecker's theorem in Section 3. We then prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5.

2. NOTATION AND SETUP

Let the notation be as in Section 1. Here we introduce some additional notation needed for our algebraic setup. Let the number field K have discriminant \mathcal{D}_K , r_1 real embeddings $\sigma_1, \dots, \sigma_{r_1}$ of K , and r_2 conjugate pairs of complex embeddings $\tau_1, \bar{\tau}_1, \dots, \tau_{r_2}, \bar{\tau}_{r_2}$; then $d = r_1 + 2r_2$. For each τ_k , write $\Re(\tau_k)$ for its real part and $\Im(\tau_k)$ for its imaginary part. Let us write $M(K)$ for the set of all places of K . Then the archimedean places of K are in correspondence with the embeddings of K , and we choose the absolute values $|\cdot|_{v_1}, \dots, |\cdot|_{v_{r_1+r_2}}$ so that for each $a \in K$

$$|a|_{v_k} = |\sigma_k(a)| \quad \forall 1 \leq k \leq r_1$$

and

$$|a|_{v_{r_1+k}} = |\tau_k(a)| = \sqrt{\Re(\tau_k(a))^2 + \Im(\tau_k(a))^2} \quad \forall 1 \leq k \leq r_2,$$

where $|\cdot|$ stands for the usual absolute value on \mathbb{R} or \mathbb{C} , respectively. For each $v \in M(K)$, we write K_v for the completion of K at v , and for each $n \geq 1$ we define a local norm $|\cdot|_v : K_v^n \rightarrow \mathbb{R}$ by

$$|\mathbf{a}|_v := \max_{1 \leq j \leq n} |a_j|_v,$$

for each $\mathbf{a} = (a_1, \dots, a_n) \in K_v^n$. Then the extended Weil height on K^n is given by

$$h(\mathbf{a}) = \prod_{v \in M(K)} \max\{1, |\mathbf{a}|_v\}^{d_v/d},$$

where $d_v = [K_v : \mathbb{Q}_v]$ is the local degree of K at v so that $\sum_{v|u} d_v = d$ for each $u \in M(\mathbb{Q})$.

For each integer $n \geq 1$, define the standard Minkowski embedding $\rho_K^n : K^n \rightarrow \mathbb{R}^{nd}$ by

$$\rho_K^n(\mathbf{a}) := (\sigma_1^n(\mathbf{a}), \dots, \sigma_{r_1}^n(\mathbf{a}), \Re(\tau_1^n(\mathbf{a})), \Im(\tau_1^n(\mathbf{a})), \dots, \Re(\tau_{r_2}^n(\mathbf{a})), \Im(\tau_{r_2}^n(\mathbf{a}))).$$

We will now use Minkowski's embedding to construct lattices from \mathcal{O}_K -modules and outline some of their main properties; see [13] for further details. Let $1 \leq s \leq w$ be integers, and let $\mathcal{M} \subset K^w$ be an \mathcal{O}_K -module such that $\mathcal{M} \otimes_K K \cong K^s$. By the structure theorem for finitely generated projective modules over Dedekind domains (see, for instance, [24]),

$$\mathcal{M} = \left\{ \sum_{j=1}^s \beta_j \mathbf{y}_j : \mathbf{y}_j \in \mathcal{O}_K^w, \beta_j \in \mathcal{I}_j \right\}$$

for some \mathcal{O}_K -fractional ideals $\mathcal{I}_1, \dots, \mathcal{I}_s$ in K . By Proposition 13 on p. 66 of [24], the discriminant of \mathcal{M} is then

$$(7) \quad \mathcal{D}_K(\mathcal{M}) := \mathcal{D}_K \prod_{j=1}^s \mathbb{N}(\mathcal{I}_j)^2,$$

where $\mathbb{N}(\mathcal{I}_j)$ is the norm of the fractional ideal \mathcal{I}_j .

Let $\Lambda_K(\mathcal{M}) := \rho_K^w(\mathcal{M})$ be an algebraic lattice of rank sd in \mathbb{R}^{wd} . Then a direct adaptation of Lemma 2 on p. 115 of [24] implies that the determinant of $\Lambda_K(\mathcal{M})$ is

$$(8) \quad \det(\Lambda_K(\mathcal{M})) = 2^{-sr_2} |\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}} = 2^{-sr_2} |\mathcal{D}_K|^{\frac{s}{2}} \prod_{j=1}^s \mathbb{N}(\mathcal{I}_j),$$

where the last identity follows by (7) above. Let $\mathbf{x} \in \Lambda_K(\mathcal{M})$; then $\mathbf{x} = \rho_K^w(\mathbf{a})$ for some $\mathbf{a} \in \mathcal{M}$ and

$$(9) \quad |\mathbf{x}| \geq \frac{1}{\sqrt{2}} h(\alpha)^{-1}$$

for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$ by inequality (54) of [13]. Let $v \in M(K)$ be an archimedean place, and assume first that it corresponds to a real embedding σ_j for some $1 \leq j \leq r_1$; then $|\mathbf{a}|_v = |\mathbf{x}|$. On the other hand, if v corresponds to a complex embedding τ_j for some $1 \leq j \leq r_2$, then $|\mathbf{a}|_v \leq \left(\sum_{j=1}^{wd} x_j^2 \right)^{1/2} \leq \sqrt{wd} |\mathbf{x}|$. Hence for each $v \mid \infty$,

$$(10) \quad |\mathbf{x}| \leq |\mathbf{a}|_v \leq \sqrt{wd} |\mathbf{x}|.$$

Let L_1, \dots, L_t be the linear forms defined in (5). For each $1 \leq i \leq t$, we define

$$|L_i|_v = \max_{1 \leq j \leq wd} |b_{ij}|_v$$

for each place $v \in M(E)$, and define the height of L_i to be

$$h(L_i) = h(b_{i1}, \dots, b_{i(wd)}) = \prod_{v \in M(E)} \max\{1, |L_i|_v\}^{\ell_v/\ell}.$$

We similarly define the height of the matrix B to be

$$h(B) = h(b_{11}, \dots, b_{t(wd)});$$

then $h(L_i) \leq h(B)$ for all $1 \leq i \leq t$. We are now ready to proceed.

3. AN EFFECTIVE VERSION OF KRONECKER'S THEOREM

In this section we derive an effective version of Kronecker's theorem, which we then use to prove Theorems 1.1 and 1.2. Similar to the setup in the beginning of Section 1, let $1, \theta_1, \dots, \theta_t$ be \mathbb{Q} -linearly independent real algebraic numbers. For each $1 \leq j \leq t$, let $f_j(x) \in \mathbb{Z}[x]$ be the minimal polynomial of θ_j of degree d_j , let $|f_j|$ be the maximum of absolute values of the coefficients of f_j , and let A_j be the leading coefficient of f_j , so $A_j \leq |f_j|$. By Lemma 3.11 of [28],

$$\frac{1}{2^{d_j}} |f_j| \leq h(\theta_j)^{d_j} \leq \sqrt{d_j + 1} |f_j|$$

for every $1 \leq j \leq t$. Define A to be the least common multiple of A_1, \dots, A_t , so

$$(11) \quad A \leq \prod_{j=1}^t |f_j| \leq \prod_{j=1}^t (2h(\theta_j))^{d_j}.$$

Let $F = \mathbb{Q}(\theta_1, \dots, \theta_t)$ be a number field of degree $e \geq t + 1$; then $e \leq \prod_{j=1}^t d_j$. Let $\theta_{t+1}, \dots, \theta_{e-1} \in F$ be such that

$$1 = \theta_0, \theta_1, \dots, \theta_t, \theta_{t+1}, \dots, \theta_{e-1}$$

form a \mathbb{Q} -basis for F . Let $\sigma_1, \dots, \sigma_e$ be the embeddings of F into \mathbb{C} . We recall Liouville inequality. For any $\mathbf{m} = (m_0, \dots, m_t, 0, \dots, 0) \in \mathbb{Z}^e$,

$$(12) \quad A^e \prod_{i=1}^e \left| \sum_{j=0}^{e-1} \sigma_i(\theta_j) m_j \right| \geq 1,$$

and so

$$(13) \quad A^e \left((t+1) \max_{1 \leq i \leq e, 0 \leq j \leq t} |\sigma_i(\theta_j)| \right)^{e-1} |\mathbf{m}|^{e-1} \|m_1\theta_1 + \cdots + m_t\theta_t\| \geq 1.$$

Now observe that

$$\max_{1 \leq i \leq e, 0 \leq j \leq t} |\sigma_i(\theta_j)| \leq \max_{1 \leq j \leq t} h(\theta_j)^{d_j},$$

and so define

$$(14) \quad C_1 = C_1(\theta_1, \dots, \theta_t) := \left((t+1) \max_{1 \leq j \leq t} h(\theta_j)^{d_j} \right)^{e-1} \prod_{j=1}^t (2h(\theta_j))^{ed_j}.$$

Then for any $\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^t$,

$$(15) \quad \|m_1\theta_1 + \cdots + m_t\theta_t\| \geq C_1^{-1} |\mathbf{m}|^{-e+1}.$$

We will now apply a transference homogeneous-inhomogeneous argument. A transference principle of this sort was first described in Chapter V, §4 of [3]; the particular stronger result we are applying here is obtained in [2]. Let us write

$$M(\mathbf{y}) = \sum_{i=1}^t \theta_i y_i$$

for $\mathbf{y} = (y_1, \dots, y_t) \in \mathbb{Z}^t$, and let

$$L_j(x) = \theta_j x, \quad 1 \leq j \leq t,$$

for $x \in \mathbb{Z}$. Then (15) guarantees that for any $\mathbf{0} \neq \mathbf{y} \in \mathbb{Z}^t$ with $|\mathbf{y}| \leq Y$,

$$\|M(\mathbf{y})\| \geq C_1^{-1} Y^{-(e-1)}.$$

Now applying the transference Lemma 3 of [2] to these linear forms, we have that for every $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ there exists $x \in \mathbb{Z}$ such that $|x| \leq 2^{-t}((t+1)!)^2 C_1 Y^{e-1}$ and

$$\max_{1 \leq j \leq t} \|L_j(x) - a_j\| \leq 2^{-t}((t+1)!)^2 Y^{-1}.$$

Letting $Q = (2^t((t+1)!)^{-2} Y)^{e-1}$, we obtain that

$$\max_{1 \leq j \leq t} \|L_j(x) - \alpha_j\| \leq Q^{-\frac{1}{e-1}}$$

for some $0 \neq x \in \mathbb{Z}$ with $|x| \leq 2^{-et}((t+1)!)^{2e} C_1 Q$. Taking $\varepsilon = Q^{-\frac{1}{e-1}}$ immediately yields the following effective version of Kronecker's theorem.

Theorem 3.1. *Let $1, \theta_1, \dots, \theta_t$ be \mathbb{Q} -linearly independent real algebraic numbers, and let $e = [\mathbb{Q}(\theta_1, \dots, \theta_t) : \mathbb{Q}]$. Let C_1 be given by (14) above, and let $\varepsilon > 0$. Then for any $(a_1, \dots, a_t) \in \mathbb{R}^t$ there exists $q \in \mathbb{Z} \setminus \{0\}$ such that*

$$(16) \quad \|q\theta_j - a_j\| \leq \varepsilon, \quad 1 \leq j \leq t,$$

and

$$|q| \leq 2^{-et}((t+1)!)^{2e} C_1 \varepsilon^{-e+1}.$$

In particular, if $h(\theta_j) \leq H$ for all $1 \leq j \leq t$ and $\max\{e, d_1, \dots, d_t\} \leq \ell$, then

$$|q| \leq \left(2^{\ell t(\ell-1)} (t+1)^{3\ell-1} (t!)^{2\ell} H^{\ell^2(t+1)-\ell} \right) \varepsilon^{-\ell+1}.$$

Remark 3.1. Stronger noneffective results can be derived as corollaries of Schmidt's Subspace Theorem. For instance, results discussed in Chapter 6, §2 of [26] together with the transference principles of Chapter V, §4 of [3] and [2] imply, for any $\varepsilon > 0$ and $\mathbf{a} \in \mathbb{R}^t$ under the assumptions of Theorem 3.1, the existence of $q \in \mathbb{Z}$ satisfying 16 such that

$$|q| \leq \mathcal{C}'(\delta)\varepsilon^{-t-\delta},$$

for any $\delta > 0$, where the constant $\mathcal{C}'(\delta)$ is noneffective. This would result in the same exponent on ε in the bounds for $|q|$ in Theorems 1.1 and 1.2, but with noneffective constants.

4. PROOF OF THEOREM 1.1

Here we present the proof of our first result. Since $\Lambda_K(\mathcal{M}) \not\subseteq \mathcal{Z}_{\mathcal{S}}$, $\Lambda_K(\mathcal{M}) \not\subseteq Z(\mathcal{S}_i)$ for all $1 \leq i \leq m$, and so for each i at least one polynomial P_i in \mathcal{S}_i is not identically zero on $\Lambda_K(\mathcal{M})$. Clearly for each $1 \leq i \leq m$,

$$Z(\mathcal{S}_i) \subseteq Z(P_i) := \{\mathbf{x} \in \mathbb{R}^{wd} : P_i(\mathbf{x}) = 0\}.$$

Define

$$P(\mathbf{x}) = \prod_{i=1}^m P_i(\mathbf{x})$$

so that $\mathcal{Z}_{\mathcal{S}} \subseteq Z(P)$ and $\deg(P) \leq M_{\mathcal{S}}$, while $\Lambda_K(\mathcal{M}) \not\subseteq Z(P)$. Indeed, $Z(P)$ is the union of hypersurfaces $Z(P_1), \dots, Z(P_m)$, and a lattice cannot be covered by a finite union of hypersurfaces unless it is contained in one of them. We will next construct a point $\mathbf{y} \in \Lambda_K(\mathcal{M})$ of controlled sup-norm such that $P(\mathbf{y}) \neq 0$.

Let $V = \text{span}_{\mathbb{R}} \Lambda_K(\mathcal{M})$ be the sd -dimensional subspace of \mathbb{R}^{wd} spanned by the lattice $\Lambda_K(\mathcal{M})$. For a positive real number μ , let us write

$$C_V(\mu) := \{\mathbf{x} \in V : |\mathbf{x}| \leq \mu\}$$

for the sd -dimensional cube with side-length 2μ centered at the origin in V , so $C_V(\mu) = \mu C_V(1)$. Let $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{sd}$ be the successive minima of $\Lambda_K(\mathcal{M})$ with respect to the cube $C_V(1)$. In other words, for each $1 \leq i \leq sd$,

$$\lambda_i := \min \{\mu \in \mathbb{R}_{>0} : \dim_{\mathbb{R}} \text{span}_{\mathbb{R}} (\Lambda_K(\mathcal{M}) \cap C_V(\mu)) \geq i\}.$$

Let $\mathbf{v}_1, \dots, \mathbf{v}_{sd}$ be a collection of linearly independent vectors in $\Lambda_K(\mathcal{M})$ corresponding to these successive minima; then $|\mathbf{v}_i| = \lambda_i$. Since the volume of the sd -dimensional cube $C_V(1)$ is 2^{sd} , Minkowski's Successive Minima Theorem (see, for instance, [4] or [18]) implies that

$$\frac{\det(\Lambda_K(\mathcal{M}))}{(sd)!} \leq \prod_{i=1}^{sd} |\mathbf{v}_i| \leq \det(\Lambda_K(\mathcal{M})),$$

where $\frac{1}{\sqrt{2}}h(\alpha)^{-1} \leq |\mathbf{v}_1| \leq \dots \leq |\mathbf{v}_{sd}|$, by (9). This means that

$$(17) \quad |\mathbf{v}_1| \leq \dots \leq |\mathbf{v}_{sd}| \leq \left(\sqrt{2}h(\alpha)\right)^{sd-1} \det(\Lambda_K(\mathcal{M})).$$

Let $I(M_{\mathcal{S}}) = \{0, 1, 2, \dots, M_{\mathcal{S}}\}$ be the set of the first $M_{\mathcal{S}} + 1$ nonnegative integers. For each $\boldsymbol{\xi} \in I(M_{\mathcal{S}})^{sd}$, define

$$\mathbf{v}(\boldsymbol{\xi}) = \sum_{i=1}^{sd} \xi_i \mathbf{v}_i.$$

Then

$$(18) \quad |\mathbf{v}(\boldsymbol{\xi})| = \max_{1 \leq j \leq wd} \left| \sum_{i=1}^{sd} \xi_i v_{ij} \right| \leq sd |\boldsymbol{\xi}| |\mathbf{v}_{sd}| \leq sd M_{\mathcal{S}} \left(\sqrt{2} h(\alpha) \right)^{sd-1} \det(\Lambda_K(\mathcal{M})),$$

by (17). Assume that $P(\mathbf{v}(\boldsymbol{\xi})) = 0$ for each $\boldsymbol{\xi} \in I(M_{\mathcal{S}})^{sd}$. Then Theorem 4.2 of [12] implies that $P(\mathbf{x})$ must be identically zero on V , which would contradict the fact that P does not vanish identically on $\Lambda_K(\mathcal{M})$. Hence there must exist some $\boldsymbol{\xi} \in I(M_{\mathcal{S}})^{sd}$ such that P does not vanish at the corresponding $\mathbf{y} := \mathbf{v}(\boldsymbol{\xi})$, and $|\mathbf{y}| \leq sd M_{\mathcal{S}} (\sqrt{2} h(\alpha))^{sd-1} \det(\Lambda_K(\mathcal{M}))$ by (18). Since $P(\mathbf{x})$ is a homogeneous polynomial, it must be true that $P(n\mathbf{y}) \neq 0$ for every $n \in \mathbb{Z}_{>0}$. On the other hand, by our construction

$$n\mathbf{y} = n \sum_{i=1}^{sd} \xi_i \mathbf{v}_i \in \text{span}_{\mathbb{Z}} \{\mathbf{v}_1, \dots, \mathbf{v}_{sd}\} \subseteq \Lambda_K(\mathcal{M}),$$

and so $\{n\mathbf{y}\}_{n \in \mathbb{Z}_{>0}}$ gives an infinite sequence of points in $\Lambda_K(\mathcal{M})$ outside of $\mathcal{Z}_{\mathcal{S}}$. For each such point, we have

$$L_i(n\mathbf{y}) = n L_i(\mathbf{y}) \quad \forall 1 \leq i \leq t.$$

Let us define, for each $1 \leq i \leq t$,

$$(19) \quad \theta_i := L_i(\mathbf{y}) = \sum_{j=1}^{wd} b_{ij} y_j \neq 0,$$

since $y_j \in K_1$, not all zero, and b_{ij} are K_1 -linearly independent. Notice that $\theta_1, \dots, \theta_t \in E$, and hence all of them are algebraic numbers of degree $\leq \ell$.

Let $\alpha \in \mathfrak{U}_K(\mathcal{M})$. Then, by (10), for each archimedean $v \in M(E)$,

$$(20) \quad \begin{aligned} \max\{1, |\theta_i|_v\} &\leq \max\{1, (wd)^{\frac{3}{2}} |L_i|_v |\mathbf{y}|\} \leq (wd)^{\frac{3}{2}} \max\{1, |\mathbf{y}|\} \max\{1, |L_i|_v\} \\ &\leq \sqrt{2} (wd)^{\frac{3}{2}} h(\alpha) |\mathbf{y}| \max\{1, |L_i|_v\}, \end{aligned}$$

by (9). By (18), $|\mathbf{y}| \leq sd M_{\mathcal{S}} (\sqrt{2} h(\alpha))^{sd-1} \det(\Lambda_K(\mathcal{M}))$, and hence

$$(21) \quad \max\{1, |\theta_i|_v\} \leq sd (wd)^{\frac{3}{2}} M_{\mathcal{S}} \left(\sqrt{2} h(\alpha) \right)^{sd} \det(\Lambda_K(\mathcal{M})) \max\{1, |L_i|_v\}.$$

Now suppose $v \in M(E)$ is nonarchimedean. Then αy_j is an algebraic integer for each $1 \leq j \leq wd$, and hence $|\alpha y_j|_v = |\alpha|_v |y_j|_v \leq 1$, meaning that

$$\max\{1, |y_1|_v, \dots, |y_{wd}|_v\} \leq \max\{1, |\alpha|_v^{-1}\}.$$

Then

$$(22) \quad \begin{aligned} \max\{1, |\theta_i|_v\} &\leq \max\{1, |L_i|_v\} \max\{1, |y_1|_v, \dots, |y_{wd}|_v\} \\ &\leq \max\{1, |\alpha|_v^{-1}\} \max\{1, |L_i|_v\} \end{aligned}$$

for each nonarchimedean $v \in M(E)$. Taking a product over all places of E , we obtain

$$\begin{aligned} h(\theta_i) &= \prod_{v \in M(E)} \max\{1, |\theta_i|_v\}^{\frac{\ell_v}{t}} = \left(\prod_{v|\infty} \max\{1, |\theta_i|_v\}^{\ell_v} \times \prod_{v \nmid \infty} \max\{1, |\theta_i|_v\}^{\ell_v} \right)^{\frac{1}{t}} \\ &\leq sd(wd)^{\frac{3}{2}} M_{\mathcal{S}} \left(\sqrt{2}h(\alpha) \right)^{sd} \det(\Lambda_K(\mathcal{M})) h(L_i) \prod_{v \nmid \infty} \max\{1, |\alpha^{-1}|_v\}^{\frac{\ell_v}{t}} \\ &\leq sd(wd)^{\frac{3}{2}} M_{\mathcal{S}} \left(\sqrt{2}h(\alpha) \right)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(L_i). \end{aligned}$$

Recalling that $h(L_i) \leq h(B)$ for all $1 \leq i \leq t$, we obtain

$$(23) \quad h(\theta_i) \leq 2^{\frac{sd}{2}} sd(wd)^{\frac{3}{2}} M_{\mathcal{S}} h(\alpha)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(B)$$

for each $1 \leq i \leq t$, where the choice of $\alpha \in \mathfrak{U}_K(\mathcal{M})$ is arbitrary.

We will now show that $1, \theta_1, \dots, \theta_t$ are \mathbb{Q} -linearly independent. Suppose not. Then there exist $c_0, c_1, \dots, c_t \in \mathbb{Q}$, not all zero, such that

$$c_0 = \sum_{i=1}^t c_i \theta_i = \sum_{i=1}^t \sum_{j=1}^{wd} c_i y_j b_{ij},$$

where not all $c_i y_j$ are equal to zero. Recall that $\mathbf{y} \in \Lambda_K(\mathcal{M})$, meaning that coordinates of \mathbf{y} are in K_1 , hence all $c_i y_j$ are in K_1 . This contradicts the assumption that $1, b_{11}, \dots, b_{1(wd)}$ are linearly independent over K_1 . Hence $1, \theta_1, \dots, \theta_t$ must be linearly independent over \mathbb{Q} .

Now let $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and let $\varepsilon > 0$, as in the statement of our theorem. Then, by (23) and Theorem 3.1, there exists $q \in \mathbb{Z}$ and $\mathbf{p} \in \mathbb{Z}^t$ such that

$$(24) \quad \begin{aligned} |q| &\leq 2^{\ell t(\ell-1)} (t+1)^{3\ell-1} (t!)^{2\ell} \\ &\times \left(2^{\frac{sd}{2}} sd(wd)^{\frac{3}{2}} M_{\mathcal{S}} h(\alpha)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(B) \right)^{\ell^2(t+1)-\ell} \varepsilon^{-\ell+1} \end{aligned}$$

and

$$|q\theta_i - a_i - p_i| < \varepsilon \quad \forall 1 \leq i \leq t.$$

Letting $\mathbf{x} = q\mathbf{y}$, we see that $q\theta_i = L_i(\mathbf{x})$ for each $1 \leq i \leq t$ and $|\mathbf{x}| = |q||\mathbf{y}|$. Combining these observations with (18), (24), and (8) and taking a minimum over all $\alpha \in \mathfrak{U}_K(\mathcal{M})$ finishes the proof of the theorem.

5. PROOF OF THEOREM 1.2

Let $\Gamma_1, \dots, \Gamma_m$ be full-rank sublattices of $\Lambda_K(\mathcal{M})$ of respective determinants $\mathcal{D}_1, \dots, \mathcal{D}_m$. Let $\Omega = \bigcap_{i=1}^m \Gamma_i$; then Ω also has full rank and

$$\mathcal{D} := \mathcal{D}_1 \cdots \mathcal{D}_m \geq \det \Omega.$$

We write λ_i for the successive minima of $\Lambda_K(\mathcal{M})$ and $\lambda_i(\Omega)$ for the successive minima of Ω . Theorem 1.2 of [20] implies that there exists $\mathbf{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$ such that

$$|\mathbf{y}| < \frac{\det \Lambda_K(\mathcal{M})}{\lambda_1(\Omega)^{sd-1}} \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \lambda_1(\Omega).$$

Our first goal is to make this bound more explicit in terms of the parameters of \mathcal{M} . First notice that by Minkowski's Successive Minima Theorem,

$$\lambda_1(\Omega) \leq \left(\prod_{i=1}^{sd} \lambda_i(\Omega) \right)^{1/sd} \leq (\det \Omega)^{1/sd} \leq \mathcal{D}^{1/sd}.$$

We also need a lower bound on $\lambda_1(\Omega)$. Observe that $\lambda_1(\Omega) \geq \lambda_1$, while $\lambda_1 \geq \frac{1}{\sqrt{2}}h(\alpha)^{-1}$ for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$, by (9) above. Putting these estimates together, we see that

$$(25) \quad |\mathbf{y}| < \left(\sqrt{2}h(\alpha) \right)^{sd-1} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{1/sd}$$

for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$.

Since $\mathbf{y} \in \Lambda_K(\mathcal{M})$ and $|\Lambda_K(\mathcal{M}) : \Gamma_i| = \mathcal{D}_i / \det \Lambda_K(\mathcal{M})$ for each $1 \leq i \leq m$, it follows that

$$(g|\Lambda_K(\mathcal{M}) : \Gamma_i|) \mathbf{y} = \frac{g\mathcal{D}_i}{\det \Lambda_K(\mathcal{M})} \mathbf{y} \in \Gamma_i$$

for every $g \in \mathbb{Z}$, and hence

$$\left(\frac{g\mathcal{D}_1 \cdots \mathcal{D}_m}{(\det \Lambda_K(\mathcal{M}))^m} \right) \mathbf{y} = \left(\frac{g\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m} \right) \mathbf{y} \in \Omega$$

for every $g \in \mathbb{Z}$. Therefore, it must be true that

$$\left(\frac{g\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m} + 1 \right) \mathbf{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$$

for every $g \in \mathbb{Z}$. For brevity, let us write $\mathcal{D}' = \frac{\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m}$.

From here on, the argument is largely similar to the proof of Theorem 1.1 above, but with some notable changes. For each $1 \leq i \leq t$, let θ_i be as in (19) for our choice of $\mathbf{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$ satisfying (25) as above; then

$$L_i((g\mathcal{D}' + 1)\mathbf{y}) = (g\mathcal{D}' + 1)\theta_i \quad \forall 1 \leq i \leq t.$$

Using (20) with (25) instead of (18), we obtain that

$$\begin{aligned} & \max\{1, |\theta_i|_v\} \\ & \leq (wd)^{\frac{3}{2}} \left(\left(\sqrt{2}h(\alpha) \right)^{sd} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}} \sqrt{2}h(\alpha) \right) \max\{1, |L_i|_v\} \end{aligned}$$

for all archimedean $v \in M(E)$, while for the nonarchimedean $v \in M(E)$,

$$\max\{1, |\theta_i|_v\} \leq \max\{1, |\alpha^{-1}|_v\} \max\{1, |L_i|_v\},$$

as in (22). Taking the product over all places of E , we have for every $1 \leq i \leq t$:

$$(26) \quad \begin{aligned} h(\theta_i) & \leq (wd)^{\frac{3}{2}} \sqrt{2}h(\alpha) h(\alpha^{-1}) h(B) \\ & \times \left(\left(\sqrt{2}h(\alpha) \right)^{sd-1} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}} \right), \end{aligned}$$

and $1, \theta_1, \dots, \theta_t$ (and hence $1, \mathcal{D}'\theta_1, \dots, \mathcal{D}'\theta_t$) are \mathbb{Q} -linearly independent by the same reasoning as in the proof of Theorem 1.1.

Now let $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and let $\varepsilon > 0$, as in the statement of our theorem. Notice that for each $1 \leq i \leq t$,

$$|(g\mathcal{D}' + 1)\theta_i - a_i - p_i| = |g(\mathcal{D}'\theta_i) + (\theta_i - a_i) - p_i|$$

for any integers p_1, \dots, p_t . Then, applying Theorem 3.1 to approximate the vector $(\theta_1 - a_1, \dots, \theta_t - a_t)$ by the fractional parts of the integer multiples of the vector $(\mathcal{D}'\theta_1, \dots, \mathcal{D}'\theta_t)$, we conclude that there exists $g \in \mathbb{Z}$ and $\mathbf{p} \in \mathbb{Z}^t$ such that

$$(27) \quad |g| \leq 2^{\ell t(\ell-1)}(t+1)^{3\ell-1}(t!)^{2\ell} \times \left((wd)^{\frac{3}{2}} \sqrt{2} h(\alpha) h(\alpha^{-1}) h(B) \mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m) \right)^{\ell^2(t+1)-\ell} \varepsilon^{-\ell+1},$$

where $\mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m)$ is as in (6), and

$$|g(\mathcal{D}'\theta_i) + (\theta_i - a_i) - p_i| < \varepsilon \quad \forall \quad 1 \leq i \leq t.$$

Letting $\mathbf{x} = (g\mathcal{D}' + 1)\mathbf{y}$, we see that $(g\mathcal{D}' + 1)\theta_i = L_i(\mathbf{x})$ for each $1 \leq i \leq t$ and $|\mathbf{x}| = |g\mathcal{D}' + 1||\mathbf{y}|$. Combining these observations with (25), (27), and (8) finishes the proof of the theorem.

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