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GAP AND RIGIDITY THEOREMS OF λ-HYPERSURFACES

QIANG GUANG

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ABSTRACT. We study λ-hypersurfaces that are critical points of a Gaussian weighted area functional $\int_\Sigma e^{-\frac{|x|^2}{4}} dA$ for compact variations that preserve weighted volume. First, we prove various gap and rigidity theorems for complete λ-hypersurfaces in terms of the norm of the second fundamental form $|A|$. Second, we show that in one dimension, the only smooth complete and embedded λ-hypersurfaces in $\mathbb{R}^2$ with $\lambda \geq 0$ are lines and round circles. Moreover, we establish a Bernstein-type theorem for λ-hypersurfaces which states that smooth λ-hypersurfaces that are entire graphs with polynomial volume growth are hyperplanes. All the results can be viewed as generalizations of results for self-shrinkers.

1. INTRODUCTION

We follow the notation of [6] and call a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ a λ-hypersurface if it satisfies

$$H - \frac{\langle x, \mathbf{n} \rangle}{2} = \lambda, \tag{1.1}$$

where λ is any constant, H is the mean curvature, $\mathbf{n}$ is the outward pointing unit normal, and x is the position vector.

λ-hypersurfaces were first studied by McGonagle and Ross in [19], where they investigated the following isoperimetric type problem in a Gaussian weighted Euclidean space:

Let $\mu(\Sigma)$ be the weighted area functional defined by $\mu(\Sigma) = \int_\Sigma e^{-\frac{|x|^2}{4}} dA$ for any hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$. Consider the variational problem of minimizing $\mu(\Sigma)$ among all Σ enclosing a fixed Gaussian weighted volume. Note that the variational problem is not to consider Σ enclosing a specific fixed weighted volume, but to consider variations that preserve the weighted volume.

It turns out that critical points of this variational problem are λ-hypersurfaces and the only smooth stable ones are hyperplanes; see [19].

In [6], Cheng and Wei introduced the notation of λ-hypersurfaces by studying the weighted volume-preserving mean curvature flow. They proved that λ-hypersurfaces are critical points of the weighted area functional for the weighted volume-preserving variations. Moreover, they defined an F-functional of λ-hypersurfaces and studied the F-stability, which extends a result of Colding and Minicozzi [10].

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Example 1.1. We give three examples of λ-hypersurfaces in $\mathbb{R}^3$:

(1) The sphere $\mathbb{S}^2(r)$ with radius $r = \sqrt{\lambda^2+4} - \lambda$.
(2) The cylinder $\mathbb{S}^1(r) \times \mathbb{R}$, where $\mathbb{S}^1(r)$ has radius $\sqrt{\lambda^2+2} - \lambda$.
(3) The hyperplane in $\mathbb{R}^3$.

Note that when $\lambda = 0$, λ-hypersurfaces are just self-shrinkers and they can be viewed as a generalization of self-shrinkers in some sense.

It is well known that self-shrinkers play a key role in the study of mean curvature flow ("MCF"), since they describe the singularity models of the MCF. In one dimension, smooth complete embedded self-shrinking curves are totally understood and they are just lines and round circles by the work of Abresch and Langer [1]. In higher dimensions, self-shrinkers are more complicated and there are only a few examples; see [2], [16], [20], and [21]. There are many classification and rigidity results of self-shrinkers under certain assumptions. Ecker and Huisken [13] proved that if a self-shrinker is an entire graph with polynomial volume growth, then it is a hyperplane. Later, Wang [22] removed the condition of polynomial volume growth. In [10], Colding and Minicozzi proved that the only smooth complete embedded self-shrinkers with polynomial volume growth and $H \geq 0$ in $\mathbb{R}^{n+1}$ are generalized cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$.

In this paper, we study λ-hypersurfaces from three aspects: gap and rigidity results, the one-dimensional case, and the entire graphic case.

First, partially motivated by the work of Chern, do Carmo, and Kobayashi [8] on minimal submanifolds of a sphere with the second fundamental form of constant length, we consider smooth closed embedded λ-hypersurfaces $\Sigma^2 \subset \mathbb{R}^3$ with $|A| =$ *constant* and $\lambda \geq 0$. We prove that they are just round spheres. It can be thought of as a generalization of the result that any smooth self-shrinker in $\mathbb{R}^3$ with $|A| =$ *constant* is a generalized cylinder; see [12] and [14].

Theorem 1.2. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a smooth closed and embedded λ-hypersurface with $\lambda \geq 0$. If the second fundamental form of Σ^2 is of constant length, i.e., $|A| = constant$, then Σ^2 is a round sphere.*

The proof of Theorem 1.2 has two key ingredients. The first ingredient is to consider the point where the norm of the position vector $|x|$ achieves its minimum. This will give that the genus is 0. The second ingredient is an interesting result from [15] that any smooth closed special W-surface of genus 0 is a round sphere.

The second main result is the following gap theorem for λ-hypersurfaces in terms of the norm of the second fundamental form $|A|$.

Theorem 1.3. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth complete embedded λ-hypersurface satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$ with polynomial volume growth, which satisfies*

$$|A| \leq \frac{\sqrt{\lambda^2+2} - |\lambda|}{2}, \tag{1.2}$$

then Σ is one of the following:

(1) *a round sphere $\mathbb{S}^n$,*
(2) *a cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for $1 \leq k \leq n-1$,*
(3) *a hyperplane in $\mathbb{R}^{n+1}$.*

Remark 1.4. Note that when $\lambda = 0$, then Σ is a self-shrinker satisfying $|A|^2 \leq 1/2$. So this theorem implies the gap theorem of Cao and Li [3] in the codimension one case. Cheng, Ogata, and Wei [5] obtained a gap theorem for λ-hypersurfaces in terms of $|A|$ and H, which also generalizes Cao and Li's result.

We also give the following Bernstein-type theorem for λ-hypersurfaces, which generalizes Ecker and Huisken's result [13].

Theorem 1.5. *If a λ-hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is an entire graph with polynomial volume growth satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$, then Σ is a hyperplane.*

In the last part, we turn to the one-dimensional case. Following an argument in [18], we show that just as self-shrinkers in $\mathbb{R}^2$, the only smooth complete and embedded λ-hypersurfaces (λ-curves) in $\mathbb{R}^2$ with $\lambda \geq 0$ are lines and round circles.

Theorem 1.6. *Any smooth complete embedded λ-hypersurface (λ-curve) γ in $\mathbb{R}^2$ satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$ with $\lambda \geq 0$ must either be a line or a round circle.*

In contrast to embedded self-shrinking curves, the dynamical pictures suggest that there exist some embedded λ-curves with $\lambda < 0$ which are not round circles. There also exist Abresch–Langer-type curves for immersed λ-curves; see [4] for more details.

Remark 1.7. λ-hypersurfaces with $\lambda \geq 0$ are special cases of hypersurfaces with non-negative rescaled mean curvature, i.e., $H - \frac{1}{2}\langle x, \mathbf{n}\rangle \geq 0$. Such hypersurfaces behave nicely under the rescaled mean curvature flow. In particular, if Σ_0 is a closed hypersurface with nonnegative rescaled mean curvature, then the nonnegative rescaled mean curvature is preserved under the rescaled mean curvature flow. Moreover, if $H - \frac{1}{2}\langle x, \mathbf{n}\rangle > 0$ holds at least at one point of Σ_0, then the rescaled mean curvature flow will develop a singularity in finite time; see [9] for more details.

2. Background and preliminaries

In this section, we recall some background and collect several useful formulas for λ-hypersurfaces. Throughout this paper, we always assume hypersurfaces to be smooth complete embedded, without boundary, and with polynomial volume growth.

2.1. **Notion and conventions.** Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface. Then ∇_Σ, div, and Δ are the gradient, divergence, and Laplacian, respectively, on Σ. $\mathbf{n}$ is the outward unit normal, $H = \mathrm{div}_\Sigma \mathbf{n}$ is the mean curvature, A is the second fundamental form, and x is the position vector. With this convection, the mean curvature H is n/r on the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ of radius r. If e_i is an orthonormal frame for Σ, then the coefficients of the second fundamental form are defined to be $a_{ij} = \langle \nabla_{e_i} e_j, \mathbf{n}\rangle$.

2.2. **Simons-type identity.** Now we will derive a Simons-type identity for λ-hypersurfaces which plays a key role in our proof of Theorem 1.3. First, recall the operators $\mathcal{L}$ and L from [10] defined by

$$\mathcal{L} = \Delta - \frac{1}{2}\langle x, \nabla \cdot\rangle,$$
$$L = \Delta - \frac{1}{2}\langle x, \nabla \cdot\rangle + |A|^2 + \frac{1}{2}.$$

Lemma 2.1. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a λ-hypersurface satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$, then*

$$LA = A - \lambda A^2, \tag{2.1}$$

$$LH = H + \lambda |A|^2, \tag{2.2}$$

$$\mathcal{L}|A|^2 = 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 - 2\lambda\langle A^2, A\rangle + 2|\nabla A|^2. \tag{2.3}$$

Remark 2.2. More general results of the above formulas were already obtained by Colding and Minicozzi; see Proposition 1.2 in [11]. For completeness we also include a proof here. Note that when $\lambda = 0$, these formulas are just Simons' equations for self-shrinkers in [10].

Proof of Lemma 2.1. Recall that for a general hypersurface, the second fundamental form A satisfies

$$\Delta A = -|A|^2 A - HA^2 - Hess_H. \tag{2.4}$$

Now we fix a point $p \in \Sigma$ and choose a local orthonormal frame e_i for Σ such that its tangential covariant derivatives vanish. So at this point, we have $\nabla_{e_i} e_j = a_{ij}\mathbf{n}$. Thus,

$$\begin{aligned} 2Hess_H(e_i, e_j) &= \nabla_{e_j}\nabla_{e_i}\langle x, \mathbf{n}\rangle = \langle x, -a_{ik}e_k\rangle_j \\ &= -a_{ikj}\langle x, e_k\rangle - a_{ij} - a_{ik}a_{jk}\langle x, \mathbf{n}\rangle \\ &= -(\nabla_{x^T} A)(e_i, e_j) - A(e_i, e_j) - \langle x, \mathbf{n}\rangle A^2(e_i, e_j). \end{aligned} \tag{2.5}$$

Combining (2.4) with (2.5) gives

$$LA = \Delta A - \frac{1}{2}\nabla_{x^T} A + \Big(\frac{1}{2} + |A|^2\Big)A = A - \Big(H - \frac{\langle x, \mathbf{n}\rangle}{2}\Big)A^2 = A - \lambda A^2.$$

This gives (2.1) and taking the trace gives (2.2). For (2.3), we have that

$$\begin{aligned} \mathcal{L}|A|^2 &= 2\langle \mathcal{L}A, A\rangle + 2|\nabla A|^2 \\ &= 2|A|^2 - 2\lambda\langle A^2, A\rangle - 2\Big(\frac{1}{2} + |A|^2\Big)|A|^2 + 2|\nabla A|^2 \\ &= 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 - 2\lambda\langle A^2, A\rangle + 2|\nabla A|^2. \end{aligned}$$

This completes the proof. □

We also need the following lemma.

Lemma 2.3. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth λ-hypersurface, then*

$$\mathcal{L}|x|^2 = 2n - |x|^2 - 2\lambda\langle x, \mathbf{n}\rangle.$$

Proof. Recall that for any hypersurface, we have $\Delta x = -H\mathbf{n}$. Therefore,

$$\begin{aligned} \mathcal{L}|x|^2 = \Delta|x|^2 - \frac{1}{2}\langle x, \nabla|x|^2\rangle &= 2\langle \Delta x, x\rangle + 2|\nabla x|^2 - |x^T|^2 \\ &= -2H\langle x, \mathbf{n}\rangle + 2n - |x^T|^2 \\ &= 2n - |x|^2 - 2\lambda\langle x, \mathbf{n}\rangle. \end{aligned}$$

□

2.3. **Weighted integral estimates for $|A|$.** In this subsection, we prove a result which will justify the integration when hypersurfaces are noncompact and with bounded $|A|$.

Proposition 2.4. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a complete λ-hypersurface with polynomial volume growth satisfying $|A| \leq C_0$, then*

$$\int_\Sigma |\nabla A|^2 e^{-\frac{|x|^2}{4}} < \infty.$$

The proof of Proposition 2.4 relies on the following two lemmas from [10] which show that the linear operator $\mathcal{L}$ is self-adjoint in a weighted L^2 space.

Lemma 2.5 ([10])**.** *If $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface, u is a C^1 function with compact support, and v is a C^2 function, then*

$$\int_\Sigma u(\mathcal{L}v)e^{-\frac{|x|^2}{4}} = -\int_\Sigma \langle \nabla u, \nabla v\rangle e^{-\frac{|x|^2}{4}}.$$

Lemma 2.6 ([10])**.** *Suppose that $\Sigma \subset \mathbb{R}^{n+1}$ is a complete hypersurface without boundary. If u, v are C^2 functions with*

$$\int_\Sigma \big(|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|\big)e^{-\frac{|x|^2}{4}} < \infty,$$

then we get

$$\int_\Sigma u(\mathcal{L}v)e^{-\frac{|x|^2}{4}} = -\int_\Sigma \langle \nabla u, \nabla v\rangle e^{-\frac{|x|^2}{4}}.$$

Proof of Proposition 2.4. By Lemma 2.1 and $|A| \leq C_0$, we have

$$\begin{aligned} \mathcal{L}|A|^2 &= 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 - 2\lambda\langle A^2, A\rangle + 2|\nabla A|^2 \\ &\geq 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 - 2|\lambda||A|^3 + 2|\nabla A|^2 \geq 2|\nabla A|^2 - C, \end{aligned} \tag{2.6}$$

where C is a positive constant depending only on λ and C_0. We allow C to change from line to line. For any smooth function ϕ with compact support, we integrate (2.6) against $\frac{1}{2}\phi^2$.

By Lemma 2.5, we obtain that

$$-2\int_\Sigma \phi|A|\langle \nabla\phi, \nabla|A|\rangle e^{-\frac{|x|^2}{4}} \geq \int_\Sigma \phi^2(|\nabla A|^2 - C)e^{-\frac{|x|^2}{4}}.$$

Using the absorbing inequality $\epsilon a^2 + \frac{b^2}{\epsilon} \geq 2ab$ gives

$$\int_\Sigma (\epsilon\phi^2|\nabla|A||^2 + \frac{1}{\epsilon}|A|^2|\nabla\phi|^2)e^{-\frac{|x|^2}{4}} \geq \int_\Sigma \phi^2(|\nabla A|^2 - C)e^{-\frac{|x|^2}{4}}. \tag{2.7}$$

Now we choose $|\phi| \leq 1$, $|\nabla\phi| \leq 1$, and $\epsilon = 1/2$. Combining this with $|\nabla A| \geq |\nabla|A||$, we see that (2.7) gives

$$\int_\Sigma (4|A|^2 + C)e^{-\frac{|x|^2}{4}} \geq \int_\Sigma \phi^2|\nabla A|^2 e^{-\frac{|x|^2}{4}}.$$

The conclusion follows from the monotone convergence theorem and the fact that Σ has polynomial volume growth. $\square$

A direct consequence of Proposition 2.4 and Lemma 2.6 is the following corollary.

Corollary 2.7. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a complete λ-hypersurface with polynomial volume growth satisfying $|A| \leq C_0$, then*

$$\int_\Sigma \mathcal{L}|A|^2 e^{-\frac{|x|^2}{4}} = 0.$$

3. Closed λ-hypersurfaces with the second fundamental form of constant length

This section is devoted to proving Theorem 1.2. Recall that if $\Sigma^2 \subset \mathbb{R}^3$ is a smooth complete embedded self-shrinker with $|A| = \textit{constant}$, then one can show that Σ is a generalized cylinder $\mathbb{S}^k \times \mathbb{R}^{2-k}$ for some $k \leq 2$; see [12] and [14]. One way to prove this is to consider the point where the norm of position vector $|x|$ achieves its minimum. For λ-hypersurfaces, we will also use this idea to prove Theorem 1.2. In addition, we need some important results from [15].

First, we recall the following ingredients from [15].

Definition 3.1. A surface in $\mathbb{R}^3$ is called a special Weingarten surface (special W-surface) if its Gauss curvature and mean curvature,[1] K and H, are connected by an identity

$$F(K, H) = 0$$

in which F satisfies the following condition:

- The function $F(K, H)$ is defined and of class C^2 on the portion $4K \leq H^2$ of the (K, H)-plane and satisfies

$$F_H + HF_K \neq 0 \quad \text{when} \quad 4K = H^2.$$

In [15], Hartman and Wintner proved the following theorem for special W-surfaces.

Theorem 3.2 ([15])**.** *Let S be a (small piece of a) special W-surface of class C^2. Then, unless S is part of a plane or a sphere, the umbilical points (if any) are isolated and their indices are negative.*

A direct consequence is the following result which serves as a key ingredient in the proof of Theorem 1.2.

Theorem 3.3 ([15])**.** *If a closed orientable surface S of genus 0 is a special W-surface of class C^2, then S is a round sphere.*

One may easily check that a closed surface with $|A| = \textit{constant}$ is a special W-surface. Hence, by Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a smooth closed embedded surface of genus 0. If $|A| = \textit{constant}$, then Σ is a round sphere.*

3.1. **Proof of Theorem 1.2.** By Corollary 3.4, in order to prove Theorem 1.2, all we need to show is that any closed λ-hypersurface with constant $|A|$ has genus 0. In the proof of Theorem 1.2, we also need the following gap result for closed λ-hypersurfaces. The proof will be given in Section 4.2.

[1] In [15], they use the average rather than the sum of the principal curvatures.

Theorem 3.5. *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a smooth closed λ-hypersurface with $\lambda \geq 0$. If Σ satisfies*

$$|A|^2 \leq \frac{1}{2} + \frac{\lambda(\lambda + \sqrt{\lambda^2 + 2n})}{2n}, \tag{3.1}$$

then Σ is a round sphere with radius $\sqrt{\lambda^2+2n} - \lambda$.

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. First, by the Gauss–Bonnet Formula, the Minkowski Integral Formulas and the Stokes' theorem, we have

$$\int_\Sigma H^2 = \int_\Sigma |A|^2 + 8\pi(1-g), \tag{3.2}$$

$$\int_\Sigma H\langle x, \mathbf{n}\rangle = 2\text{Area}(\Sigma),$$

$$\int_\Sigma \langle x, \mathbf{n}\rangle = 3\text{Volume}(\Omega),$$

where g is the genus of Σ and Ω is the region enclosed by Σ.

Combining above identities, we deduce that

$$\int_\Sigma H^2 \geq (\lambda^2+1)\int_\Sigma = (\lambda^2+1)\text{Area}(\Sigma). \tag{3.3}$$

Next, we consider the point $p \in \Sigma$ where $|x|$ achieves its minimum. By Lemma 2.3, at point p, we have

$$H^2(p) \leq \frac{2 + \lambda^2 + \lambda\sqrt{\lambda^2+4}}{2}. \tag{3.4}$$

At point p, we can choose a local orthonormal frame $\{e_1, e_2\}$ such that the second fundamental form $a_{ij} = \lambda_i\delta_{ij}$ for $i, j = 1, 2$. Thus, we have

$$|\nabla H|^2 = (a_{111} + a_{221})^2 + (a_{112} + a_{222})^2. \tag{3.5}$$

Since $|A|^2 = \textit{constant}$, we see that

$$a_{11}a_{111} + a_{22}a_{221} = a_{11}a_{112} + a_{22}a_{222} = 0. \tag{3.6}$$

Note that at point p, $|\nabla H| = 0$. This implies

$$a_{111} + a_{221} = a_{112} + a_{222} = 0.$$

Combining this with (3.5) and (3.6), we get

$$a_{111}(a_{11} - a_{22}) = a_{222}(a_{11} - a_{22}) = 0.$$

If $a_{11} = a_{22}$, then by (3.4), we have

$$|A|^2 = \frac{H^2}{2} \leq \frac{2 + \lambda^2 + \lambda\sqrt{\lambda^2+4}}{4}.$$

By Theorem 3.5, this implies that Σ is a round sphere.

If $a_{111} = a_{222} = 0$, then $|\nabla A|^2 = 0$. Hence,

$$\Big(\frac{1}{2} - |A|^2\Big)|A|^2 = \lambda\langle A^2, A\rangle.$$

Thus, we have

$$\Big(|A|^2 - \frac{1}{2}\Big)|A|^2 = -\lambda\langle A^2, A\rangle \leq \lambda|A|^3.$$

Therefore,

$$|A|^2 \leq \frac{1+\lambda^2+\lambda\sqrt{\lambda^2+2}}{2}.$$

Combining this with (3.2) and (3.3) gives

$$(\lambda^2+1)\mathrm{Area}(\Sigma) \leq \int_\Sigma H^2 \leq \frac{1+\lambda^2+\lambda\sqrt{\lambda^2+2}}{2}\mathrm{Area}(\Sigma) + 8\pi(1-g).$$

Observe that

$$\lambda^2+1 > \frac{1+\lambda^2+\lambda\sqrt{\lambda^2+2}}{2};$$

then we get that the genus $g=0$. By Corollary 3.4, we conclude that Σ is a round sphere. This completes the proof. □

Remark 3.6. Note that our method does not apply to higher dimensions. It is desirable that one may remove the conditions of closedness and $\lambda \geq 0$ to prove that any λ-hypersurface $\Sigma^2 \subset \mathbb{R}^3$ with $|A| = \mathit{constant}$ is a generalized cylinder. We also conjecture that in higher dimensions, all λ-hypersurfaces with $|A| = \mathit{constant}$ must be generalized cylinders.

4. Gap theorems for λ-hypersurfaces

In this section, we prove the gap theorems for λ-hypersurfaces.

4.1. **Proof of Theorem 1.3.** Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemma 2.1, we have

$$\begin{aligned}\frac{1}{2}\mathcal{L}|A|^2 &= \Big(\frac{1}{2}-|A|^2\Big)|A|^2 - \lambda\langle A^2, A\rangle + |\nabla A|^2\\ &\geq \Big(\frac{1}{2}-|A|^2\Big)|A|^2 - |\lambda||A|^3 + |\nabla A|^2.\end{aligned}$$

Then Proposition 2.4 and Corollary 2.7 give

$$0 = \int_\Sigma \mathcal{L}|A|^2 e^{-\frac{|x|^2}{4}} \geq \int_\Sigma \Big(\frac{1}{2}-|A|^2-|\lambda||A|\Big)|A|^2 e^{-\frac{|x|^2}{4}} + \int_\Sigma |\nabla A|^2 e^{-\frac{|x|^2}{4}}. \tag{4.1}$$

Note that when

$$|A| \leq \frac{\sqrt{\lambda^2+2}-|\lambda|}{2},$$

we have

$$\frac{1}{2}-|A|^2-|\lambda||A| \geq 0.$$

This implies that the first term of (4.1) on the right-hand side is nonnegative. Therefore, (4.1) implies that all inequalities are equalities. Moreover, we have

$$|\nabla A| = \Big(\frac{1}{2}-|A|^2-|\lambda||A|\Big)|A|^2 = 0.$$

By Theorem 4 of Laswon [17] that every smooth hypersurface with $\nabla A = 0$ splits isometrically as a product of a sphere and a linear space, we finish the proof. □

By the proof of Theorem 1.3, we have the following gap result.

Corollary 4.1. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth complete embedded λ-hypersurface satisfying $H - \frac{\langle x,\mathbf{n}\rangle}{2} = \lambda$ with polynomial volume growth, which satisfies*

$$|A| < \frac{\sqrt{\lambda^2+2} - |\lambda|}{2},$$

then Σ is a hyperplane in $\mathbb{R}^{n+1}$.

4.2. **Gap theorems for closed λ-hypersurfaces.** In Theorem 1.3, when Σ^n is a round sphere, this forces $\lambda = 0$. We address this issue by providing the gap theorem for closed λ-hypersurfaces with $\lambda \geq 0$, i.e., Theorem 3.5, which is used in the proof of Theorem 1.2. Now we give the proof of Theorem 3.5.

Proof of Theorem 3.5. Since Σ is closed, we consider the point p where $|x|$ achieves its maximum. At point p, x and $\mathbf{n}$ are in the same direction. This implies $2H(p) = 2\lambda + |x|(p)$.

By (3.1), we have

$$\Big(\lambda + \frac{|x|(p)}{2}\Big)^2 = H^2(p) \leq n|A|^2 \leq n\Big(\frac{1}{2} + \frac{\lambda(\lambda + \sqrt{\lambda^2+2n})}{2n}\Big).$$

This gives

$$\max_{\Sigma} |x| \leq |x|(p) \leq \sqrt{\lambda^2 + 2n} - \lambda. \tag{4.2}$$

By Lemma 2.3, we have

$$\mathcal{L}|x|^2 = 2n - |x|^2 - 2\lambda\langle x, \mathbf{n}\rangle.$$

Combining this with (4.2), the maximum principle gives that Σ is a round sphere. $\square$

4.3. **A Bernstein-type theorem for λ-hypersurfaces.** The aim of this subsection is to prove Theorem 1.5 which generalizes Ecker and Huisken's result [13]. The key ingredient is that for a λ-hypersurface Σ, the function $\langle v, \mathbf{n}\rangle$ is an eigenfunction of the operator L with eigenvalue $1/2$, where $v \in \mathbb{R}^{n+1}$ is any constant vector. Note that the result is also true for self-shrinkers. This eigenvalue result was also obtained by McGonagle and Ross [19].

Lemma 4.2. *If $\Sigma \subset \mathbb{R}^{n+1}$ is a λ-hypersurface, then for any constant vector $v \in \mathbb{R}^{n+1}$, we have*

$$L\langle v, \mathbf{n}\rangle = \frac{1}{2}\langle v, \mathbf{n}\rangle.$$

Proof. Set $f = \langle v, \mathbf{n}\rangle$. Working at a fixed point p and choosing e_i to be a local orthonormal frame, we have

$$\nabla_{e_i} f = \langle v, \nabla_{e_i}\mathbf{n}\rangle = -a_{ij}\langle v, e_j\rangle.$$

Differentiating again and using the Codazzi equation gives that

$$\nabla_{e_k}\nabla_{e_i} = -a_{ijk}\langle v, e_j\rangle - a_{ij}a_{jk}\langle v, \mathbf{n}\rangle.$$

Therefore,

$$\Delta f = \langle v, \nabla H\rangle - |A|^2 f. \tag{4.3}$$

Using the equation of λ-hypersurfaces, we have

$$\langle v, \nabla H\rangle = \langle v, -\frac{1}{2}a_{ij}\langle x, e_j\rangle e_i\rangle = \frac{1}{2}\langle x, \nabla f\rangle. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain that

$$Lf = \Delta f - \frac{1}{2}\langle x, \nabla f\rangle + \Big(\frac{1}{2} + |A|^2\Big) f = \frac{1}{2} f.$$

□

We now give the proof of Theorem 1.5.

Proof of Theorem 1.5. Since Σ is an entire graph, we can find a constant vector v such that $f = \langle v, \mathbf{n}\rangle > 0$. Let $u = 1/f$. Then we have

$$\nabla u = -\frac{\nabla f}{f^2} \text{ and } \Delta u = -\frac{\Delta f}{f^2} + \frac{2|\nabla f|^2}{f^3}.$$

By Lemma 4.2, we can easily get

$$\mathcal{L} u = |A|^2 u + \frac{2|\nabla u|^2}{u}.$$

Since Σ has polynomial volume growth, we get

$$\int_\Sigma \Big(|A|^2 u + \frac{2|\nabla u|^2}{u}\Big) e^{-\frac{|x|^2}{4}} = 0.$$

Therefore, $|A| = 0$ and Σ is a hyperplane in $\mathbb{R}^{n+1}$. □

Remark 4.3. A similar result is also obtained later by Cheng and Wei [7] under the assumption of properness instead of polynomial volume growth. Note that they proved properness of λ-hypersurfaces implies polynomial volume growth; see Theorem 9.1 in [6].

5. Embedded λ-hypersurfaces in $\mathbb{R}^2$

In this section, we will follow the argument in [18] to show that any λ-hypersurface (λ-curve) in $\mathbb{R}^2$ with $\lambda \geq 0$ must either be a line or a round circle, i.e., Theorem 1.6.

Proof of Theorem 1.6. Suppose s is an arclength parameter of γ; then the curvature is $H = -\langle \nabla_{\gamma'}\gamma', \mathbf{n}\rangle$. Note that $\nabla_{\gamma'}\mathbf{n} = H\gamma'$, so we have

$$2H' = \nabla_{\gamma'}\langle x, \mathbf{n}\rangle = H\langle x, \gamma'\rangle.$$

If at some point $H = 0$, then $H' = 0$. By the uniqueness theorem of ODE, we conclude that $H \equiv 0$, and, thus, γ is just a line. Therefore, we may assume that H is always nonzero and possibly reversing the orientation of the curve to make $H > 0$, i.e., γ is strictly convex.

Differentiating $|x|^2$ gives

$$(|x|^2)' = 2\langle x, \gamma'\rangle = 4\frac{H'}{H}.$$

Thus $H = Ce^{\frac{|x|^2}{4}}$ for some constant $C > 0$.

Since the curve is strictly convex, we introduce a new variable θ by $\theta = \arccos\langle \mathbf{E}_1, n\rangle$.

Differentiating with respect to the arclength parameter gives

$$\partial_s \theta = -H,$$

$$H_\theta = -\frac{H'}{H} = -\frac{\langle x, \gamma'\rangle}{2},$$

and

$$H_{\theta\theta} = \frac{\partial_s H_\theta}{-H} = \frac{1 - 2H(H-\lambda)}{2H} = \frac{1}{2H} - H + \lambda. \tag{5.1}$$

Multiplying the above equation by $2H_\theta$, we get

$$\partial_\theta(H_\theta^2 + H^2 - \log H - 2\lambda H) = 0.$$

Therefore, the quantity

$$E = H_\theta^2 + H^2 - \log H - 2\lambda H$$

is a constant.

Consider the function $f(t) = t^2 - \log t - 2\lambda t$, $t > 0$. It is easy to verify that

$$f(t) \geq f\Big(\frac{\lambda + \sqrt{\lambda^2+2}}{2}\Big).$$

Hence,

$$E \geq f\Big(\frac{\lambda + \sqrt{\lambda^2+2}}{2}\Big).$$

If $E = f(\frac{\lambda+\sqrt{\lambda^2+2}}{2})$, then H is constant and γ must be a round circle.

Now we assume that $E > f(\frac{\lambda+\sqrt{\lambda^2+2}}{2})$. Note that $H = Ce^{\frac{|x|^2}{4}}$ and $H \leq |x/2| + |\lambda|$. Then H has an upper bound and $|x|$ is bounded. By the embeddedness and completeness of γ, we conclude that γ must be closed, simple, and strictly convex.

If γ is not a round circle, then we consider the critical points of the curvature H. By our assumption that $E > f(\frac{\lambda+\sqrt{\lambda^2+2}}{2})$, when $H_\theta = 0$, we have $H_{\theta\theta} = \frac{1}{2H} - H + \lambda \neq 0$. So the critical points are not degenerate. By the compactness of the curve, they are finite and isolated.

Without loss of generality, we may assume $H(0) = H_{max}$ and $H(\bar{\theta})$ is the first subsequent critical point of H for $\bar{\theta} > 0$. Combining the fact that the curvature is strictly decreasing in the interval $[0, \bar{\theta}]$ with the second-order ODE of the function H is symmetric with respect to $\theta = 0$ and $\theta = \bar{\theta}$, we conclude that $H(\bar{\theta})$ must be the minimum of the curvature.

By the four-vertex theorem, we know that γ has at least four pieces like the one described above. Since our curve is closed and embedded, the curvature H is periodic with period $T < \pi$ and $\frac{T}{2} = \bar{\theta}$.

Next, we will evaluate an integral to produce a contradiction.

Since $H_{\theta\theta} = \frac{1}{2H} - H + \lambda$, we have

$$(H^2)_{\theta\theta\theta} + 4(H^2)_\theta = \frac{2H_\theta}{H} + 6\lambda H_\theta.$$

Now we consider the following integral:

$$2\int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta = \int_0^{\frac{T}{2}} \sin 2\theta \Big[(H^2)_{\theta\theta\theta} + 4(H^2)_\theta - 6\lambda H_\theta\Big] d\theta.$$

Integration by parts gives

$$\begin{aligned} 2\int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta = & \sin 2\theta (H^2)_{\theta\theta}\Big|_0^{\frac{T}{2}} - 2\int_0^{\frac{T}{2}} \cos 2\theta (H^2)_{\theta\theta} d\theta + 4\int_0^{\frac{T}{2}} \sin 2\theta (H^2)_\theta d\theta \\ & - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta. \end{aligned}$$

Hence,

$$
\begin{aligned}
2\int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta &= 2\sin T\Big[H_\theta^2(\frac{T}{2}) + H(\frac{T}{2})H_{\theta\theta}(\frac{T}{2})\Big] - 2\cos 2\theta (H^2)_\theta\Big|_0^{\frac{T}{2}} \\
&\quad - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta \\
&= 2\sin T H(\frac{T}{2})H_{\theta\theta}(\frac{T}{2}) - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta.
\end{aligned}
$$

By (5.1) and $H_\theta(0) = H_\theta(\frac{T}{2}) = 0$, we get

$$
2\int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta = 2\sin T\Big[\frac{1}{2} - H^2(\frac{T}{2}) + \lambda H(\frac{T}{2})\Big] - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta. \tag{5.2}
$$

Since H is decreasing from 0 to $\frac{T}{2}$ and $\sin 2\theta$ is nonnegative, the left-hand side of (5.2) is nonpositive. For the right-hand side, the first term is nonnegative since $H(\frac{T}{2})$ is a minimum, and $\lambda \geq 0$ implies the second term is nonpositive. So the right-hand side of (5.2) is nonnegative, and this gives a contradiction. Therefore, we conclude that γ is a round circle. □

Remark 5.1. For the noncompact case, we do not need the condition $\lambda \geq 0$ to prove it is a line, and we do need $\lambda \geq 0$ for the closed case. When $\lambda < 0$, there exist some embedded λ-curves which are not round circles; see [4].

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Department of Mathematics, University of California Santa Barbara, Santa Barbara, California 93106
Email address: guang@math.ucsb.edu