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K-THEORY OF LINE BUNDLES AND SMOOTH VARIETIES

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Abstract. We give a K-theoretic criterion for a quasi-projective variety to be smooth. If $\mathbb{L}$ is a line bundle corresponding to an ample invertible sheaf on X, it suffices that $K_q(X) \cong K_q(\mathbb{L})$ for all $q \leq \dim(X) + 1$.

Let X be a quasi-projective variety over a field k of characteristic 0. The main result of this paper gives a K-theoretic criterion for X to be smooth. For affine X, such a criterion was given in [CHW08]: it suffices that X be K_{d+1}-regular for $d = \dim(X)$, i.e., that $K_{d+1}(X) \cong K_{d+1}(X \times \mathbb{A}^m)$ for all m. If X is affine, we also showed that K_{d+1}-regularity of X is equivalent to the condition that $K_i(X) \cong K_i(X \times \mathbb{A}^1)$ for all $i \leq d + 1$.

We also showed that K_{d+1}-regularity is insufficient for quasi-projective X; see [CHW08, Thm. 0.2]. In this paper we prove:

Theorem 0.1. *Let X be quasi-projective over a field k of characteristic 0 of dimension d, and let $\mathbb{L} = \mathrm{Spec}(\mathrm{Sym}\, \mathcal{L})$ be the line bundle corresponding to an ample invertible sheaf $\mathcal{L}$ on X.*

If $K_i(\mathbb{L}) \cong K_i(X)$ for all $i \leq n$, then X is regular in codimension $< n$.

If $K_i(\mathbb{L}) \cong K_i(X)$ for all $i \leq d + 1$, then X is regular.

For example, if $K_i(\mathbb{L}) \cong K_i(X)$ for all $i \leq d$, then X has at most isolated singularities.

In the affine case, of course, every line bundle is ample, and when $\mathbb{L} = \mathbb{A}^1_R$ we recover our previous result, proven in [CHW08, 0.1]:

Corollary 0.2. *If R is essentially of finite type over a field of characteristic 0 and $K_i(R) \cong K_i(R[t])$ for all $i \leq n$, then R is regular in codimension $< n$.*

The affine assumption in this corollary is critical. In [CHW08], we gave an example of a curve Y which is K_n-regular for all n, but which is not regular; no affine open U is even reduced. However, $K_1(X) \neq K_1(\mathbb{L})$ for the line bundle associated to an ample $\mathcal{L}$; see Example 4.1 below. In Theorem 4.3 we give a surface X which is K_n-regular for all n, but which is not regular and such that $K_0(X) \neq K_0(\mathbb{L})$ for the line bundle associated to an ample $\mathcal{L}$; it is a cusp bundle over an elliptic curve.

As in our previous papers [CHSW08, CHW08, CHWW10], our technique is to compare K-theory to cyclic homology using *cdh*-descent and the Chern character. The parts of *cdh* descent we need are developed in Section 1 and applied to give a formula for the cyclic homology of line bundles in Section 2. The main theorem is proven in Section 3, and two examples are given in Section 4.

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Notation. If E is a presheaf of spectra, we write $\pi_n E$ for the presheaf of abelian groups $X \mapsto \pi_n E(X)$; we say that a spectrum E is *n-connected* if $\pi_q E = 0$ for all $q \leq n$. For example, $K_n(X)$ is the homotopy group $\pi_n K(X)$ of the spectrum $K(X)$.

Similarly, if E is a cochain complex of presheaves, we may regard it as a presheaf of spectra via Dold-Kan [Wei94, ch. 10]. Thus $\pi_i E(X)$ is another notation for $H^{-i}E(X)$. We will use the cochain shift convention $E[i]^n = E^{i+n}$, so that the spectrum corresponding to $E[1]$ is the suspension of the spectrum of E, and $\pi_n E[1] = \pi_{n-1}E$. Thus if E is n-connected, then $E[1]$ is $(n+1)$-connected.

1. Zariski and *cdh* descent

In this paper, we fix a field of characteristic 0 and work with the category Sch of schemes X of finite type over the field. We will be interested in the Zariski and *cdh* topologies on Sch.

If τ is a Grothendieck topology on Sch, there is an "injective τ-local" model structure on the category $\mathrm{Psh}(\mathbf{Ch}(\mathbf{Ab}))$ of presheaves of cochain complexes of abelian groups on Sch. In this model structure, a map $A \to B$ is a cofibration if $A(X) \to B(X)$ is an injection for all X, and it is a weak equivalence if $H^n A \to H^n B$ induces an isomorphism on the associated τ-sheaves. The fibrant replacement of A in this model structure is written as $A \to \mathbb{H}_\tau(-, A)$. We say that A *satisfies τ-descent* if the canonical map $A(X) \to \mathbb{H}_\tau(X, A)$ is a quasi-isomorphism for all X. There is a parallel notion of τ-descent for presheaves of spectra.

If A is a sheaf, then $A \to \mathbb{H}_\tau(-, A)$ is an injective resolution; it follows that $\mathbb{H}^n_\tau(X, A) = H^n \mathbb{H}_\tau(X, A)$ for all n. For a complex A, the hypercohomology group $\mathbb{H}^n_\tau(X, A)$ equals $H^n \mathbb{H}_\tau(X, A)$. See [CHSW08, 3.3] for these facts.

The inclusion of complexes of sheaves (for a topology τ) into complexes of presheaves induces an injective τ-local model structure on complexes of sheaves, and the inclusion is a Quillen equivalence; see [Jar15, 5.9].

For the Zariski, Nisnevich, and *cdh* topologies, there is a parallel "injective τ-local" model structure on the category $\mathrm{Psh}(\mathbf{Ch}(\mathcal{O}_\tau))$ of presheaves of complexes of $\mathcal{O}_\tau$-modules, and the functor forgetting the module structure is a Quillen adjunction. In particular, if A is a presheaf of complexes of $\mathcal{O}_\tau$-modules, the forgetful functor sends its fibrant $\mathcal{O}_\tau$-module replacement to a presheaf that is objectwise weak equivalent to $\mathbb{H}_\tau(-, A)$.

Example 1.1. Let k be a subfield of our fixed base field. (For example, k could be $\mathbb{Q}$.) The Hochschild complex HH/k satisfies Zariski descent by [WG91, 0.4]. By definition, the cochain complex $HH(X/k)$ is concentrated in negative cohomological degrees and has the Zariski sheaf $\mathcal{O}_X^{\otimes_k n+1}$ in cohomological degree $-n$; its cohomology sheaves are quasi-coherent. When k is understood, we drop the '$/k$' from the notation. We sometimes regard HH as a sheaf of spectra, using Dold-Kan, and use the notation $HH_q(X) = \pi_q HH(X)$ for $\mathbb{H}^{-q}_{\mathrm{zar}}(X, HH)$. Recall from [WG91, 4.6] that if X is noetherian, then $HH_q(X) = 0$ for $q < -\dim(X)$.

If E is a complex of Zariski sheaves of $\mathcal{O}$-modules on Sch/X, we may assume that $\mathbb{H}_{\mathrm{zar}}(-, E)$ is a complex of Zariski sheaves of $\mathcal{O}$-modules, and similarly for $\mathbb{H}_{\mathrm{cdh}}(-, E)$. (See [Jar15, 8.6].) Thus it makes sense to form the sheaf tensor product $\mathbb{H}_\tau(-, E) \otimes_{\mathrm{zar}} \mathcal{L}$ with a Zariski sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules.

If E is a Zariski sheaf of $\mathcal{O}_X$-modules on X, then there is a Zariski sheaf E' of $\mathcal{O}$-modules on Sch/X, unique up to unique isomorphism, such that for every $f : Y \to X$ in Sch/X the restriction of E' to the small Zariski site of Y is naturally isomorphic to the sheaf f^*E. In this paper we will always work with this sheaf on the big site, so for example "an invertible sheaf $\mathcal{L}$ on X" will indicate the sheaf on the big site associated in this way to an invertible sheaf on X.

Lemma 1.2. *If $\mathcal{L}$ is an invertible sheaf on X, $\otimes_{\mathrm{zar}}\mathcal{L}$ is an auto-equivalence of the category* $\mathrm{Sh}(\mathbf{Ch}(\mathcal{O}_{\mathrm{zar}}))/X$ *of sheaves of complexes of $\mathcal{O}_{\mathrm{zar}}$-modules on* Sch/X *which preserves cofibrations, fibrations, and weak equivalences.*

Proof. The functor $\otimes_{\mathrm{zar}}\mathcal{L}^{-1}$ is a quasi-inverse to $\otimes_{\mathrm{zar}}\mathcal{L}$. Since $\mathcal{L}$ is flat, $\otimes_{\mathrm{zar}}\mathcal{L}$ preserves injections. Since $\mathcal{L}$ is locally trivial on X (and hence on any X-scheme), and $A \otimes_{\mathrm{zar}} \mathcal{O}_X \cong A$, $\otimes_{\mathrm{zar}}\mathcal{L}$ preserves weak equivalences. Now suppose that $C \to D$ is a Zariski-local fibration; we want to see that $C \otimes_{\mathrm{zar}} \mathcal{L} \to D \otimes_{\mathrm{zar}} \mathcal{L}$ is a Zariski-local fibration. By invertibility, it suffices to observe that if $A \to B$ is a trivial cofibration of $\mathcal{O}_{\mathrm{zar}}$ modules, then so is $A \otimes_{\mathrm{zar}} \mathcal{L}^{-1} \to B \otimes_{\mathrm{zar}} \mathcal{L}^{-1}$, a fact we have just verified. □

Corollary 1.3. *If $\mathcal{L}$ is an invertible sheaf on X and A is a complex of Zariski sheaves of $\mathcal{O}$-modules, then there is a quasi-isomorphism on* Sch/X:

$$\mathbb{H}_{\mathrm{zar}}(-, A) \otimes_{\mathrm{zar}} \mathcal{L} \xrightarrow{\simeq} \mathbb{H}_{\mathrm{zar}}(-, A \otimes_{\mathrm{zar}} \mathcal{L}).$$

Proof. This follows immediately from Lemma 1.2. □

We write (a^*, a_*) for the usual adjunction between Zariski and *cdh* sheaves associated to the change-of-topology morphism $a : (\mathrm{Sch}/k)_{\mathrm{cdh}} \to (\mathrm{Sch}/k)_{\mathrm{zar}}$ and its restrictions $a : (\mathrm{Sch}/X)_{\mathrm{cdh}} \to (\mathrm{Sch}/X)_{\mathrm{zar}}$. Thus if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathrm{cdh}}$-modules on $(\mathrm{Sch}/X)_{\mathrm{cdh}}$, $a_*\mathcal{F}$ is the underlying sheaf of $\mathcal{O}_{\mathrm{zar}}$-modules, and for any Zariski sheaf E of $\mathcal{O}_X$-modules on X, we may form the Zariski sheaf $a_*\mathcal{F} \otimes_{\mathcal{O}_X} E$ on Sch/X.

Recall from [EGA, $0_I(5.4.1)$] that a Zariski sheaf E of $\mathcal{O}_X$-modules is *locally free* if each point of X has an open neighborhood U such that $E|_U$ is a free $\mathcal{O}_U$-module, possibly of infinite rank.

Lemma 1.4. *If E is a locally free sheaf on X and $\mathcal{F}$ is a cdh sheaf of $\mathcal{O}_{\mathrm{cdh}}$-modules, then $a_*\mathcal{F} \otimes_{\mathcal{O}_X} E$ is a cdh sheaf on* (Sch/X).

Proof. Since the question is local on X, we may replace X by an open subscheme to assume that E is free. Because the *cdh*-topology on Sch/X is noetherian, and therefore arbitrary direct sums of sheaves are sheaves, we are reduced to the trivial case $E = \mathcal{O}_X$ when $a_*\mathcal{F} \otimes_{\mathcal{O}_X} E = a_*\mathcal{F}$. □

Definition 1.5. If $\mathcal{F}$ is a *cdh* sheaf of $\mathcal{O}_{\mathrm{cdh}}$-modules, we will write $\mathcal{F} \otimes_{\mathrm{zar}} E$ for the *cdh* sheaf $a_*\mathcal{F} \otimes_{\mathcal{O}_X} E$.

Note that $\mathbb{H}^*_{\mathrm{zar}}(X, \mathcal{F} \otimes_{\mathrm{zar}} E) \neq \mathbb{H}^*_{\mathrm{zar}}(X, \mathcal{F}) \otimes E(X)$. For example, $E(X) = 0$ does not imply that $(\mathcal{F} \otimes_{\mathrm{zar}} E)(X) = 0$.

Lemma 1.6. *If E is locally free on X, then $\otimes_{\mathrm{zar}}E$ preserves weak equivalences and cofibrations for complexes of cdh sheaves of $\mathcal{O}_{\mathrm{cdh}}$-modules on* Sch/X.

Proof. As in the proof of Lemma 1.4, we may replace X by an open subscheme and assume that E is a sheaf of free modules. Since $A \otimes_{\mathrm{zar}} E$ is a sum of copies of A, it follows that $A \mapsto A \otimes_{\mathrm{zar}} E$ preserves weak equivalences and cofibrations. □

Definition 1.7. Given a cochain complex A of presheaves of abelian groups on Sch and a scheme X, we write $F_A(X)$ for the homotopy fiber (the shifted mapping cone) of the canonical map $A(X) \to \mathbb{H}_{\mathrm{cdh}}(X, A)$, so for each X there is a long exact sequence

$$\cdots \mathbb{H}^{n-1}_{\mathrm{cdh}}(X, A) \to H^n F_A(X) \to H^n A(X) \to \mathbb{H}^n_{\mathrm{cdh}}(X, A) \to H^{n+1} F_A(X) \cdots .$$

If A is a complex of sheaves (in some topology) of $\mathcal{O}$-modules, then $\mathbb{H}_{\mathrm{cdh}}(-, A)$ can be represented by a complex of sheaves of $\mathcal{O}$-modules as well (see [Jar15, 8.1]), and hence so can F_A. We also write $F_K(X)$ for the homotopy fiber of $K(X) \to KH(X)$. (Recall that KH is equivalent to the *cdh*-fibrant replacement of K by [Hae04] or [Cis13], so this notation is consistent.)

It is well known that HH, HC, and K-theory satisfy Zariski descent. It follows that F_{HH}, F_{HC}, and F_K also satisfy Zariski descent.

Proposition 1.8. *If $\mathcal{L}$ is an invertible sheaf on X and A is a complex of Zariski sheaves of $\mathcal{O}$-modules on* Sch $/X$*, then*

$$\mathbb{H}_{\mathrm{cdh}}(-, A) \otimes_{\mathrm{zar}} \mathcal{L} \xrightarrow{\simeq} \mathbb{H}_{\mathrm{cdh}}(-, A \otimes_{\mathrm{zar}} \mathcal{L}).$$

Consequently, $F_A \otimes_{\mathrm{zar}} \mathcal{L} \xrightarrow{\simeq} F_{A\otimes\mathcal{L}}$.

Proof. Arguing as in the proof of Lemma 1.2, Lemma 1.4 shows that $\otimes_{\mathrm{zar}}\mathcal{L}$ preserves *cdh*-local fibrations (in addition to cofibrations and weak equivalences). The first statement follows immediately from this. Because $\otimes_{\mathrm{zar}}\mathcal{L}$ is exact, the second statement follows from the triangles

$$F_A \to A \to \mathbb{H}_{\mathrm{cdh}}(-, A) \to \quad \text{and} \quad F_{A\otimes\mathcal{L}} \to A \otimes \mathcal{L} \to \mathbb{H}_{\mathrm{cdh}}(-, A \otimes \mathcal{L}) \to . \qquad \square$$

Lemma 1.9. *Let A_i be cochain complexes of presheaves on* Sch $/X$*. Then for every X-scheme Y, the canonical maps*

$$\bigoplus_i \mathbb{H}_{\mathrm{zar}}(Y, A_i) \to \mathbb{H}_{\mathrm{zar}}(Y, \bigoplus_i A_i)$$

and

$$\bigoplus_i \mathbb{H}_{\mathrm{cdh}}(Y, A_i) \to \mathbb{H}_{\mathrm{cdh}}(Y, \bigoplus_i A_i)$$

are quasi-isomorphisms.

Proof. These sites are noetherian, and thus cohomology in them commutes with direct limits. $\square$

2. Homology of line bundles

Suppose that R is a (commutative) noetherian algebra over a field k of characteristic 0. In [CHWW10, 3.2, 4.1], we showed that $NK(R) = K(R[t])/K(R)$ is weakly equivalent to $NF_{HC/\mathbb{Q}}(R)[1]$ as well as $F_{HH/\mathbb{Q}}(R)[1] \otimes_R tR[t]$. In this section, we replace $R[t]$ by the symmetric algebra $R[L] = \mathrm{Sym}_R(L)$ of a rank 1 projective R-module and the ideal $tR[t]$ by $LR[L]$. More generally, if $\mathcal{L}$ is an invertible sheaf on a scheme X, we replace $X \times \mathbb{A}^1$ by the line bundle $\mathbb{L} = \mathrm{Spec}(\mathrm{Sym}_X \mathcal{L})$.

Lemma 2.1. *Let L be a rank* 1 *projective R-module. Then the symmetric algebra $R[L] = \mathrm{Sym}_R(L)$ satisfies:*

$$HH(R[L]) \simeq HH(R) \otimes_R R[L] \;\oplus\; HH(R)[1] \otimes_R LR[L],$$
$$HC(R[L]) \simeq HC(R) \;\oplus\; HH(R) \otimes_R LR[L].$$

Similarly, if X is a scheme over R and $X[L]$ denotes $X \times_R \operatorname{Spec}(R[L])$, then

$$\begin{aligned} HH(X[L]) &\simeq HH(X) \otimes_R R[L] \;\oplus\; HH(X)[1] \otimes_R LR[L]), \\ HC(X[L]) &\simeq HC(X) \;\oplus\; HH(X) \otimes_R LR[L]. \end{aligned}$$

Note that, as an R-module, $LR[L] = R[L] \otimes_R L$ is just $\bigoplus_{j=1}^{\infty} L^{\otimes j}$.

Proof. The cochain complex $HH(R[L])$ ends: $\to R[L] \otimes R[L] \xrightarrow{0} R[L] \to 0$. Therefore there are natural maps from $R[L]$ and $R[L] \otimes L[1]$ to $HH(R[L])$. Using the shuffle product, we get a natural map $\mu(R)$ from the direct sum of $HH(R) \otimes_R R[L]$ and $HH(R) \otimes_R (R[L] \otimes L)[1]$ to $HH(R[L])$. For each prime ideal $\wp$ of R, we have $R_\wp[L] \cong R_\wp[t]$ and $\mu(R_\wp)$ is a quasi-isomorphism by the Künneth formula [Wei94, 9.4.1]. It follows that $\mu(R)$ is a quasi-isomorphism. The formula for $HC(R[L])$ follows by induction on the SBI sequence using [Wei94, 9.9.1], just as it does for $HC(R[t])$ (see [CHWW10, sec. 1]).

If X is a scheme over R, the same argument applies to $\pi_* HH(\mathcal{O}_X[L])$, the direct image along $X[L] \xrightarrow{\pi} X$ of the cochain complex $HH(\mathcal{O}_X[L])$ on $X[L]$ of sheaves with quasi-coherent cohomology described in Example 1.1. Because π is affine, we have a quasi-isomorphism

$$\mathbb{H}_{\mathrm{zar}}(X[L], HH(\mathcal{O}_X[L])) \cong \mathbb{H}_{\mathrm{zar}}(X, \pi_* HH(\mathcal{O}_X[L])).$$

Now the assertions about $X[L]$ follow from Corollary 1.3 and Lemma 1.9. □

Corollary 2.2. $F_{HC}(R[L]) \cong F_{HC}(R) \;\oplus\; \bigoplus_{j=1}^{\infty} \left(F_{HH} \otimes_R L^{\otimes j}\right)(R)$.

Proof. Suppose that X is a scheme over R. Then

$$\mathbb{H}_{\mathrm{cdh}}(X[L], HH) \cong \mathbb{H}_{\mathrm{cdh}}(X, HH(-[L]))$$

and similarly for HC. Indeed, when X is smooth as a scheme over k, this follows from the analogous statement in the Zariski topology (observed in the proof of Lemma 2.1) and the result, proved in [CHW08, Theorem 2.4], that Hochschild and cyclic homology satisfy *cdh* descent on smooth schemes. The equivalence for general X follows using resolution of singularities, induction on the dimension of X, and the fact that $X[L] \to X$ is a smooth morphism.

The assertion of the corollary now follows from Lemma 2.1, Proposition 1.8, and Lemma 1.9. □

Now suppose that X is a scheme of finite type over a field of characteristic 0, containing k, and write HH, HC, etc., for HH/k, HC/k, etc.

Lemma 2.3. *Let $\mathbb{L}$ be a line bundle over X, and write $\mathcal{F}_{HH}$ for the cochain complex of Zariski sheaves on X associated to the complex of presheaves $U \mapsto F_{HH}(\mathbb{L}|_U)$. Then $F_{HH}(\mathbb{L}) \xrightarrow{\simeq} \mathbb{H}_{\mathrm{zar}}(X, \mathcal{F}_{HH})$.*

Proof. As observed after Definition 1.7, the presheaf of complexes F_{HH} satisfies Zariski descent: $F_{HH}(\mathbb{L}) \simeq \mathbb{H}_{\mathrm{zar}}(\mathbb{L}, F_{HH})$. By [Tho85, 1.56],

$$\mathbb{H}_{\mathrm{zar}}(\mathbb{L}, F_{HH}) \xrightarrow{\simeq} \mathbb{H}_{\mathrm{zar}}(X, \mathcal{F}_{HH}).$$

□

In what follows, we write $\otimes$ for the tensor product of $\mathcal{O}_X$-modules.

Proposition 2.4. *Let $\mathbb{L}$ be the line bundle* $\mathrm{Spec}(\mathrm{Sym}\,\mathcal{L})$ *on X associated to an invertible sheaf $\mathcal{L}$, and let $p : \mathbb{L} \to X$ be the projection. Write $K(\mathbb{L}, X)$ for the relative K-theory spectrum $K(\mathbb{L})/K(X)$ and similarly for HC. Then we have quasi-isomorphisms:*

$$\begin{aligned} HC(\mathbb{L}) &\simeq HC(X) \oplus \mathbb{H}_{\mathrm{zar}}(X, HH \otimes \mathrm{Sym}(\mathcal{L}) \otimes \mathcal{L}),\\ \mathbb{H}_{\mathrm{cdh}}(X, p_*HC) &\simeq \mathbb{H}_{\mathrm{cdh}}(X, HC) \oplus \mathbb{H}_{\mathrm{cdh}}(X, HH \otimes \mathrm{Sym}(\mathcal{L}) \otimes \mathcal{L}),\\ F_{HC}(\mathbb{L}) &\simeq F_{HC}(X) \ \oplus \ \bigoplus\nolimits_{j=1}^{\infty} \left(F_{HH} \otimes \mathcal{L}^{\otimes j}\right)(X),\\ K(\mathbb{L}, X) &\simeq F_{HC/\mathbb{Q}}(\mathbb{L}, X)[1]. \end{aligned}$$

Proof. Using Zariski descent, we may assume that $X = \mathrm{Spec}(R)$ for some R. The first two quasi-isomorphisms are immediate from Lemma 2.1, while the third is immediate from Corollary 2.2. By Theorem 1.6 of [CHW08],

$$K(\mathbb{L})/K(X) \cong F_K(\mathbb{L})/F_K(X) \simeq F_{HC/\mathbb{Q}}(\mathbb{L})[1]/F_{HC/\mathbb{Q}}(X)[1].$$

This shows the final weak equivalence. $\square$

Now suppose that R is a commutative $\mathbb{Q}$-algebra. Then $K_n(R[L], R)$ is a $\mathbb{Q}$-module [Wei87], and the Adams operations give an R-module decomposition

$$K_n(R[L], R) \cong \bigoplus_{i=0}^{\infty} K_n^{(i)}(R[L], R)$$

with $K_n^{(0)}(R[L], R) = 0$ for all n. The relative terms $F_K(R) \cong F_{HC}(R)[1]$ have a similar decomposition, and $F_K^{(i)}(R[L], R) \simeq F_{HC}^{(i-1)}(R[L], R)[1]$.

As in [CHWW10, 5.1], we define the *typical piece* $TK_n(R)$ to be $H^{1-n}(F_{HH}(R))$ and set $TK_n^{(i)}(R) = H^{1-n}(F_{HH}^{(i-1)}(R))$. Since these groups were detemined in [CHWW10], we may rephrase the last part of Proposition 2.4 as follows:

Corollary 2.5. *If R is a commutative $\mathbb{Q}$-algebra, $K_n(R[L], R) \cong TK_n(R) \otimes_R LR[L]$ and*

$$K_n^{(i)}(R[L]) \cong K_n^{(i)}(R) \oplus TK_n^{(i)}(R) \otimes_R LR[L].$$

Moreover,

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R), & \text{if } \ i < n, \\ H_{\mathrm{cdh}}^{i-n-1}(R, \Omega^{i-1}), & \text{if } \ i \geq n+2. \end{cases}$$

(The formulas for $TK_n^{(n)}$ and $TK_n^{(n+1)}$ are more complicated; see [CHWW10].) The following special case $n = 0$ of Corollary 2.5, which is an analogue of [CHWW10, (0.5)], shows that we cannot twist out the example in [CHW08, Thm. 0.2].

Corollary 2.6. *Let L be a rank 1 projective R-module, where R is a d-dimensional commutative $\mathbb{Q}$-algebra, with seminormalization R^+, and let $R[L]$ be the twisted polynomial ring. Then*

$$K_0(R[L], R) \cong \left((R^+/R) \oplus \bigoplus_{p=1}^{d-1} \mathbb{H}_{\mathrm{cdh}}^p(R, \Omega^p) \right) \otimes_R LR[L].$$

In particular, $K_0(R) = K_0(R[t])$ if and only if $K_0(R) = K_0(R[L])$.

Proof. This follows from the fact that $\mathbb{H}_{\mathrm{cdh}}(X, HH^{(i)}) \cong Ra_*a^*\Omega^i[-i]$, so that when $i > 1$ we have $K_0^{(i)}(R[L], R) \cong \mathbb{H}_{\mathrm{cdh}}^{i-1}(R, \Omega^{i-1}) \otimes_R LR[L]$; see [CHW08, 2.2]. $\square$

Remark 2.7. Corollary 2.5 shows that $K_*(R[L],R)$ is a graded $R[L]$-module. As in [CHWW10], this reflects the fact that locally $R[L]$ is a polynomial ring and $K_*(R[t],R)$ has a continuous module structure over the ring of big Witt vectors $W(R)$, compatible with the operations V_n and F_n. When $\mathbb{Q}\subset R$, such modules are graded $R[t]$-modules. Since $H^0(\operatorname{Spec} R,\widetilde{W})=W(R)$, patching the structures via Zariski descent proves that $K_*(R[L],R)$ is a graded $R[L]$-module.

When X is no longer affine, this Zariski descent argument shows that

$$K_n(\mathbb{L},X)=\oplus H^{1-n}(X,F_{HH}\otimes_{\mathrm{zar}}\mathcal{L}^{\otimes i})$$

is a graded module over $S=\oplus H^0(X,\mathcal{L}^{\otimes i})$. This is clear from Proposition 2.4. Previously, using [Wei87], it was only known that the $K_n(\mathbb{L},X)$ are continuous modules over $H^0(X,\widetilde{W})=W(k)=\prod_1^\infty k$.

3. Proof of Theorem 0.1

In order to use Proposition 2.4, we need to analyze $\mathbb{H}^n_{\mathrm{zar}}\left(X,F_{HH/\mathbb{Q}}\otimes\mathcal{L}^j\right)$. For this, we use the hypercohomology spectral sequence (see [Wei94, 5.7.10])

$$E_2^{p,q}=H^p_{\mathrm{zar}}(X,H^qE)\Rightarrow\mathbb{H}^{p+q}_{\mathrm{zar}}(X,E). \tag{3.1}$$

Here E is a cochain complex which need not be bounded below, and (by abuse of notation) the E_2 term denotes cohomology with coefficients in the Zariski sheaf associated to H^qE. The spectral sequence converges if X is noetherian and finite dimensional. When $E=F_{HH}\otimes\mathcal{L}^j$, we have $H^qE=H^q(F_{HH})\otimes\mathcal{L}^j$, because $\mathcal{L}^j$ is flat.

In this section, we write k for our (fixed) base field of characteristic zero. When discussing Hochschild homology (or cyclic homology, or differentials, etc.) relative to $\mathbb{Q}$, we will suppress the base from the notation. For example, if X is a k-scheme, then $HH_n(X)$ and Ω^n_X will mean $HH_n(X/\mathbb{Q})$ and $\Omega^n_{X/\mathbb{Q}}$.

Lemma 3.2. *If X is noetherian and finite dimensional, and E is a complex of Zariski sheaves such that $H^p_{\mathrm{zar}}(X,H^qE)=0$ for $1\le p\le\dim(X)$ and $p+q=s,s+1$, then $\mathbb{H}^s_{\mathrm{zar}}(X,E)\cong H^0_{\mathrm{zar}}(X,H^sE)$.*

Proof. This is immediate from the hypercohomology spectral sequence (3.1). □

In the remainder of this section, we will write $H^p(X,-)$ for $H^p_{\mathrm{zar}}(X,-)$. By a "quasi-coherent" (or "coherent") sheaf on Sch/k we mean a Zariski sheaf whose restriction to every small Zariski site is quasi-coherent (or coherent).

Recall that when $\mathbb{Q}\subseteq k$, the Hochschild homology complex relative to k decomposes into a direct sum of weight pieces $HH^{(j)}(-/k)$; this induces decompositions on $\mathbb{H}_{\mathrm{cdh}}(-,HH(/k))$, the fiber $F_{HH(/k)}$, and on their cohomology sheaves and hypercohomology groups as well. As in [CHW08], we use versions of a spectral sequence introduced by Kassel and Sletsjøe in [KS92] to obtain information about $F_{HH(/k)}$ from information about F_{HH}.

Lemma 3.3 (Kassel-Sletsjøe)**.** *Let $\mathbb{Q}\subseteq k$ and $p\ge1$ be fixed, and let X be a scheme over k. Then there are bounded cohomological spectral sequences of quasi-coherent sheaves on* Sch/k *($p>s\ge0$):*

$$E_1^{s,t}=\Omega^s_k\otimes_k H^{2s+t-p}HH^{(p-s)}(-/k)\Rightarrow H^{s+t-p}HH^{(p)}(-/\mathbb{Q})$$

(for $s+t \le 0$) and

$$E_1^{s,t} = \Omega_k^s \otimes_k H^{s+t}(Ra_*\Omega^{(p-s)}_{(-/k),\mathrm{cdh}}) \Rightarrow H^{s+t}(Ra_*\Omega^p_{\mathrm{cdh}})$$

and a morphism of spectral sequences between them. If k has finite transcendence degree, then both spectral sequences are spectral sequences of coherent sheaves.

We remark that the second spectral sequence is just the Zariski sheafification of the spectral sequence in [CHW08, 4.2].

Proof. If $X = \mathrm{Spec}(R)$, the homological spectral sequence in [KS92, 4.3a] is

$${}_pE^1_{-i,i+j} = \Omega^i_k \otimes_k HH^{(p-i)}_{p-i+j}(R/k) \Rightarrow HH^{(p)}_{p+j}(R)$$

($0 \le i < p$, $j \ge 0$); see [CHW08, 4.1].

We claim that this is a spectral sequence of R-modules, compatible with localization of R. Indeed, following the construction in [KS92, Thm. 3.2], the exact couple underlying the spectral sequence is constructed by choosing $\mathbb{Q}$-cofibrant simplicial resolutions $P_\bullet \to k$ and $Q_\bullet \to R$ and then filtering the differential modules $\Omega^p_{Q_\bullet/\mathbb{Q}}$ by certain $Q_\bullet$-submodules, leading to a filtration of $\Omega^p_{Q_\bullet/\mathbb{Q}} \otimes_{Q_\bullet} R$ by R-modules. (Although the filtration steps are defined as certain $P_\bullet$-submodules in [KS92, sec. 3], they are in fact $Q_\bullet$-submodules.) The identification of the associated graded via [KS92, Lem. 3.1] is easily checked to be a R-module isomorphism. The whole construction commutes with localization because forming differential modules does. Thus (using a functorial choice of $\mathbb{Q}$-cofibrant simplicial resolutions) we may sheafify this construction for the Zariski topology, resulting in a filtered cochain complex of sheaves of $\mathcal{O}_X$-modules on X with quasi-coherent cohomology sheaves. Setting $\ell = i + j$, the spectral sequence in the affine case is

$${}_pE^1_{-i,\ell} = \Omega^i_k \otimes_k HH^{(p-i)}_{p+\ell-2i}(R/k) \Rightarrow HH^{(p)}_{p+\ell-i}(R), \quad \ell \ge i.$$

As sheafification is exact, we can identify the spectral sequence associated to the filtered cochain complex of sheaves constructed above by sheafifying this spectral sequence. Reindexing cohomologically, with $s = i$ and $t = -\ell$, we have

$${}^pE_1^{s,t} = \Omega^s_k \otimes_k H^{2s+t-p}(HH^{(p-s)})(-/k) \Rightarrow H^{s+t-p}(HH^{(p)}).$$

This yields the first spectral sequence. To obtain the second, we apply the functor Ra_*a^* to the filtered cochain complex of sheaves used to produce the first one. The result is a tower of cochain complexes of sheaves, and it follows from [CHW08, Lem. 2.8] that these cochain complexes have quasi-coherent cohomology sheaves. That the spectral sequence associated to this tower of cochain complexes has the indicated form follows from the quasi-isomorphisms $a^*HH^{(p)} \cong \Omega^p_{\mathrm{cdh}}[p]$. The morphism between the spectral sequences is induced by the unit of the adjunction (a^*, Ra_*).

Finally, if k has finite transcendence degree, then the E_1-terms of both spectral sequences are coherent (apply [CHW08, Lem. 2.8] again for the second one) and hence so are the abutments. $\square$

Corollary 3.4. *There is a bounded spectral sequence of quasi-coherent sheaves*

$$E_1^{s,t} = \Omega^s_k \otimes_k H^{2s+t-p}(F^{(p-s)}_{HH/k}) \Rightarrow H^{s+t-p}(F^{(p)}_{HH}).$$

If k has finite transcendence degree, this is a spectral sequence of coherent sheaves.

Proof. The morphism of spectral sequences in Lemma 3.3 comes from a morphism $HH^{(p)} \to HH^{(p)}_{\mathrm{cdh}}$ of filtered complexes of quasi-coherent sheaves on Sch/k. By a lemma of Eilenberg–Moore [Wei94, Ex. 5.4.4], there is a filtration on the [shifted] mapping cone $F^{(p)}_{HH}$ of $HH^{(p)} \to HH^{(p)}_{\mathrm{cdh}}$, yielding a spectral sequence converging to $H^*(F_{HH})$. This is the displayed spectral sequence. □

Proposition 3.5. *Assume that k has finite transcendence degree. If $\mathcal{L}$ is an ample line bundle on X, then for every n and $p \geq 0$ there is an $N_0 = N_0(n,p)$ such that for all $N > N_0$ the Zariski sheaf $H^n F^{(p)}_{HH} \otimes \mathcal{L}^{\otimes N}$ is generated by its global sections, and $H^q(X, H^n F^{(p)}_{HH} \otimes \mathcal{L}^{\otimes N}) = 0$ for all $q > 0$.*

Proof. The complex $F^{(0)}_{HH}$ is quasi-isomorphic to the cone of the map from the structure sheaf $\mathcal{O}$ to $Ra_*a^*\mathcal{O}$ and thus has coherent cohomology by [CHSW08, Lem. 6.5]. If $p > 0$, then by Corollary 3.4 the cohomology sheaves in question are coherent as well. Now apply Serre's Theorem B. □

Let $\mathcal{L}$ be an ample sheaf on X and let $\mathbb{L}$ be the line bundle $\mathrm{Spec}(\mathrm{Sym}\,\mathcal{L})$. Recall that for any Y, $F_{HC}(Y)$ is n-connected if and only if $F_{HH}(Y)$ is n-connected; see [CHW08, 1.7]. If $\mathbb{L}$ is a line bundle over X, we define $F_{HH/k}(\mathbb{L}, X)$ to be the cokernel of the canonical split injection $F_{HH/k}(X) \to F_{HH/k}(\mathbb{L})$, and similarly for cyclic homology.

Theorem 3.6. *If $F_{HC}(\mathbb{L}, X)$ is n-connected for some ample line bundle $\mathcal{L}$ on X, then $F_{HH}(\mathbb{L}, X)$ is n-connected and:*

(1) *The Zariski sheaf F_{HH} is n-connected on X.*
(2) *X is regular in codimension $\leq n$.*
(3) *If $F_{HC}(\mathbb{L}, X)$ is d-connected for $d = \dim(X)$, then X is regular.*

Proof. There is a finitely generated subfield k_0 of k, a k_0-scheme X_0, and an ample line bundle $\mathcal{L}_0$ such that $X = X_0 \otimes_{k_0} k$ and $\mathcal{L} = \mathcal{L}_0 \otimes_{k_0} k$. The Künneth formula for Hochschild homology implies (see [CHWW10, Thm. 6.4]) that $F_{HH}(\mathbb{L}, X) = F_{HH}(\mathbb{L}_0, X_0) \otimes \Omega^*_{k/k_0}$, whence $F_{HH}(\mathbb{L}, X)$ is n-connected if and only if $F_{HH}(\mathbb{L}_0, X_0)$ is. Thus we may assume that k has finite transcendence degree.

(1) Recall [CHW08, 2.1] that $F_{HH}(\mathbb{L}, X) = \prod F^{(p)}_{HH}(\mathbb{L}, X)$. Thus it suffices to fix p and show that $F^{(p)}_{HH}$ is n-connected. Set $\mathcal{G}_N = \mathcal{L}^N \otimes F^{(p)}_{HH}$, and note that $H^q\mathcal{G}_N = \mathcal{L}^N \otimes H^q F^{(p)}_{HH}$. By Proposition 3.5 and Lemma 3.2, $H^s(X, \mathcal{G}_N) \cong H^0(X, H^s\mathcal{G}_N)$ for $s \geq -n$ and N large enough (how large may depend on s).

By assumption and Zariski descent on X, the groups

$$\pi_s F^{(p)}_{HH}(\mathbb{L}, X) = \mathbb{H}^{-s}_{\mathrm{zar}}(X, F^{(p)}_{HH}(\mathbb{L}|-, -)) = \mathbb{H}^{-s}_{\mathrm{zar}}(X, F^{(p)}_{HH}(\mathbb{L}|-)/F^{(p)}_{HH})$$

vanish for $s \leq n$. By Lemma 2.4, this implies that for all $N > 0$:

$$H^0(X, H^{-s}\mathcal{G}_N) \cong H^{-s}(X, \mathcal{G}_N) = H^{-s}(X, \mathcal{L}^N \otimes F^{(p)}_{HH}) = 0, s \leq n.$$

Since $\mathcal{L}$ is ample, the sheaves $H^s\mathcal{G}_N = \mathcal{L}^N \otimes H^s F^{(p)}_{HH}$ are generated by their global sections $H^0(X, H^s\mathcal{G}_N)$ for large N and $s \geq -n$. This implies that the sheaves $\mathcal{L}^N \otimes H^s F^{(p)}_{HH}$ vanish and hence that the sheaves $H^s F^{(p)}_{HH}$ vanish for $s \geq -n$. This proves (1).

Given (1), the stalks $F_{HH}(\mathcal{O}_{X,x})$ are n-connected. We proved in [CHW08, 4.8] that this implies that each $F_{HH/k}(\mathcal{O}_{X,x})$ is n-connected. If $\dim(\mathcal{O}_{X,x}) \geq n$, we proved in [CHW08, 3.1] that $\mathcal{O}_{X,x}$ is smooth over k and hence regular. □

Variant 3.7. *Let X, $\mathcal{L}$, and $\mathbb{L}$ be as in Proposition* 3.6*. Suppose that $F_{HC/k}(\mathbb{L}, X)$ is n-connected. Then the proof of Theorem* 3.6 *goes through to show that:*

1. *The sheaf $\mathcal{F}_{HH/k}$ is n-connected.*
2. *X is regular in codimension $\leq n$.*
3. *If $F_{HH/k}(\mathbb{L}, X)$ is d-connected for $d = \dim(X)$, then X is regular.*

Proof of Theorem 0.1. Suppose that $K_i(\mathbb{L}) \cong K_i(X)$ for all $i \leq n$. By Proposition 2.4, $F_{HC/\mathbb{Q}}(\mathbb{L}, X)$ is $(n-1)$-connected. By Theorem 3.6, $F_{HH/\mathbb{Q}}(\mathbb{L}, X)$ is $(n-1)$-connected and X is regular in codimension $< n$. $\square$

4. Two examples

We conclude with two quick examples. Let E be an elliptic curve over $\mathbb{Q}$ with basepoint Q, and let P be a point such that $P - Q$ does not have finite order in $\mathrm{Pic}(E)$.

Example 4.1. Consider the non-reduced scheme $Y = \mathrm{Spec}(\mathcal{O}_E \oplus J)$, where J is the invertible sheaf $\mathcal{O}(P - Q)$. We showed in [CHW08, 0.2] that Y is K_n-regular for all n, because $K_n(Y \times \mathbb{A}^1) \cong K_n(Y) \cong K_n(E)$ for all n.

Let $\mathcal{L}$ be the sheaf $\mathcal{O}(Q)$ and set $\mathbb{L} = \mathrm{Spec}_Y(\mathrm{Sym}\,\mathcal{L})$. Then

$$K_1(\mathbb{L}) \cong K_1(Y) \oplus \mathbb{Q}[x, y].$$

Indeed,

$$K_1(\mathbb{L})/K_1(Y) \cong H^0(\mathbb{L}, \mathcal{O}^*)/H^0(Y, \mathcal{O}^*) \cong \bigoplus_{n\geq 1} H^0(E, J \otimes \mathcal{L}^n) \cong \mathbb{Q}[x, y],$$

since $\dim H^0(E, J \otimes \mathcal{L}^n) = n$ by Riemann–Roch.

For our second example, recall that if R is a regular $\mathbb{Q}$-algebra and J is a rank 1 projective R-module and A is the subring $R[J^2, J^3]$ of $R[J] = \mathrm{Sym}_R(J)$, then $\mathrm{Spec}(A)$ is an affine cusp bundle over $\mathrm{Spec}(R)$. Set $V_0 = V_1 = 0$, and for $n \geq 2$, set

$$V_n(R) = \begin{cases} J^{6(i-1)} \oplus (J^{6(i-2)} \otimes \Omega^2_R) \oplus \cdots \oplus (R \otimes \Omega^{n-2}_R), & n = 2i \geq 2; \\ J^{6(i-1)} \otimes \Omega^1_R \oplus (J^{6(i-2)} \otimes \Omega^3_R) \oplus \cdots \oplus (R \otimes \Omega^{n-2}_R), & n = 2i+1 \geq 3. \end{cases}$$

In particular, $V_2(R) = R$ and $V_3(R) = \Omega^1_R$. Let us write $\widetilde{K}_n(A)$ for $K_n(A)/K_n(R)$.

Proposition 4.2. *If $A = R[J^2, J^3]$ and R is a regular $\mathbb{Q}$-algebra, then*

$$\widetilde{K}_n(A) \cong (J^5 \oplus J^6) \otimes V_n(R) \oplus (J \otimes \Omega^n_R).$$

In particular, $\widetilde{K}_0(A) \cong J$, $\widetilde{K}_1(A) \cong J \otimes \Omega^1_R$, and

$$\widetilde{K}_2(A) \cong (J^5 \oplus J^6) \oplus (J \otimes \Omega^2_R).$$

Proof. For $J = R$, this is Theorem 9.2 of [GRW89], which holds for any regular $\mathbb{Q}$-algebra R (not just for any field). In order to pass to $R[J^2, J^3]$, we need more detail. Using the classical Mayer-Vietoris sequence for $A \subset R[J]$, it is easy to see that $K_0(A)/K_0(R) \cong J$ and $K_1(A)/K_1(R) \cong J \otimes \Omega^1_R$.

For $n \geq 2$ the factors in $K_n(A)$ come from $HH_{n-1}(A)$ via the maps $HH_*(A) \to HC_*(A)$ and $\widetilde{K}_n(A) \to \widetilde{HC}_{n-1}(A)$. The summand $J \otimes \Omega^n_R$ of $K_n(A)$ comes from the $J \otimes \Omega^1_R$ in $K_1(A)$ (or $HH_0(A, R[J], J)$) by multiplication by $HH_{n-1}(R) \cong \Omega^{n-1}_R$.

The V_n factors come from the explicit description of the corresponding cyclic homology cycles (coming from cycles in Hochschild homology $HH_{n-1}(A)$) in Lemma

4.3, Remark 4.7, and Example 5.8 of [GRW89]. Locally, J is generated by an element t; we set $x = t^2 \in J^2$, $y = t^3 \in J^3$ so that $y^2 = x^3$. The summands J^5 and J^6 of $K_2(A)$ are locally generated by the cycles $z = 2x[y]+3y[x]$ and $tz = 2y[y]+3x^2[x]$ in $HH_1(A)$. Multiplication by Ω_R^{n-2} gives the summands $(J^5 \oplus J^6) \otimes \Omega_R^{n-2}$ in $K_n(A)$.

Now consider the summand J^6 in the degree 2 part $A^{\otimes 3}$ of the Hochschild complex for A, locally generated by the element $w = [y|y] - x[x|x] - [x^2|x]$. The product zw^{i-1} is a cycle in $HH_{2i-1}(A)$ and locally generates a summand $J^{5+6(i-1)}$ of $HH_{2i-1}(A)$, corresponding to the factor $J^{5+6(i-1)}$ of the summand $J^5 \otimes V_{2i}(R)$ of $K_{2i}(A)$. As above, multiplication by Ω_R^* gives the rest of the summands. □

Remark 4.2.1. In the spirit of Corollary 2.5, we note that $NK_n(A) \cong TK_n(A) \otimes_R LR[L]$, where

$$TK_n(A) = \widetilde{K}_n(A) \oplus \widetilde{K}_n(A).$$

Theorem 4.3. *Let J be the invertible sheaf $\mathcal{O}(P-Q)$ on the ellipic curve E and let X denote the affine cusp bundle $\operatorname{Spec}_E(\mathcal{O}_E[J^2, J^3])$ over E. (X has a codimension 1 singular locus.) If J does not have finite order in $\operatorname{Pic}(E)$, then X is K_n-regular for all integers n: for all $m \geq 0$ we have*

$$K_n(X) \cong K_n(X \times \mathbb{A}^m) \cong K_n(E).$$

On the other hand, if $\mathbb{L} = \operatorname{Sym}_E(\mathcal{O}(Q)) \times_E X$, then $K_{-1}(\mathbb{L}) = K_{-1}(X)$, but $K_0(\mathbb{L}) \neq K_0(X)$.

Proof. Since $\Omega_E \cong \mathcal{O}_E$, $V_n(\mathcal{O}_E)$ is a sum of terms J^i for $i > 0$; the same is true for the pushforward of the sheaf $V_n(\mathcal{O}_E[t_1, \dots, t_m])$ to E. Recall that $H^p(E, J^r) = 0$ for all $r \neq 0$, because J does not have finite order in the Picard group. From the Zariski descent spectral sequence

$$E_2^{p,q} = H^p(E, K_{-q}(\mathcal{O}_E[J^2, J^3][t_1, \dots, t_m])/K_{-q}(\mathcal{O}_E)) \Rightarrow K_{-p-q}(X \times \mathbb{A}^m)/K_{-p-q}(E)$$

we see that $K_n(X \times \mathbb{A}^m) \cong K_n(E)$ for all n.

On the other hand, Proposition 4.2 yields $\widetilde{K}_{-1}(\mathbb{L}) \cong \bigoplus_{j \geq 1} H^1(E, J \otimes \mathcal{L}^j)$, where $\mathcal{L} = \mathcal{O}(Q)$ and

$$\widetilde{K}_0(\mathbb{L}) \cong \bigoplus_{j \geq 1} H^0(E, J \otimes \mathcal{L}^j) \oplus \widetilde{K}_{-1}(\mathbb{L}).$$

The first group is zero; the second is non-zero because $\mathcal{L}$ is ample. □

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