# RELATIVELY WEAKLY OPEN CONVEX COMBINATIONS OF SLICES

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ABSTRACT. We show that  $c_0$  and, in fact, C(K) for any scattered compact Hausdorff space K have the property that finite convex combinations of slices of the unit ball are relatively weakly open.

# 1. INTRODUCTION

Let X be a (real or complex) Banach space with unit ball  $B_X$ , unit sphere  $S_X$ , and dual  $X^*$ . Given  $x^* \in S_{X^*}$  and  $\varepsilon > 0$  we define a slice of  $B_X$  by

$$S(x^*,\varepsilon) := \{ x \in B_X : \operatorname{Re} x^*(x) > 1 - \varepsilon \},\$$

where  $\operatorname{Re} x^*(x)$  denotes the real part of  $x^*(x)$ .

Recall the following successively stronger "big-slice concepts", defined in [3]:

**Definition 1.1.** A Banach space X has the

- (i) local diameter 2 property if every slice of  $B_X$  has diameter 2,
- (ii) diameter 2 property if every nonempty relatively weakly open subset of  $B_X$  has diameter 2,
- (iii) strong diameter 2 property if every finite convex combination of slices of  $B_X$  has diameter 2.

By Bourgain's lemma [7, Lemma II.1] every nonempty relatively weakly open subset of  $B_X$  contains a finite convex combination of slices; hence the strong diameter 2 property implies the diameter 2 property. It was shown in [4] that the two properties are not equivalent. Since a slice is relatively weakly open, the diameter 2 property implies the local diameter 2 property. Even though the converse is not true in general, as shown in [5], for some spaces it is. For example, it is known that if a Banach space X satisfies that every  $x \in S_X$  is an extreme point of  $B_{X^{**}}$ , then every nonempty relatively weakly open subset of  $B_X$  contains a slice by Choquet's lemma (cf., e.g., Proposition 1.3 in [1]).

On a particularly sunny day at a conference at the University of Warwick in 2015, Olav Nygaard asked if the converse of Bourgain's lemma is ever true for  $B_X$ . The aim of this short note is to answer this question affirmatively by showing that  $c_0$  and, in fact, C(K) for any scattered compact Hausdorff space K have the much stronger property that finite convex combinations of slices of the unit ball are relatively weakly open. See Theorems 2.3 and 2.4.

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Let us note that in general it is not true that finite convex combinations of slices of the unit ball are relatively weakly open. Indeed, for some spaces there are finite convex combinations of slices of the unit ball that do not even intersect the sphere. The Banach space  $\ell_2$  is one example [7, Remark IV.5]. In their proof (independent of [4]) that the strong diameter 2 property is stronger than the diameter 2 property, Haller, Langemets, and Põldvere [8] show that if Z is an  $\ell_p$ -sum of two Banach spaces,  $Z = X \oplus_p Y$  with  $1 , then for every <math>\lambda \in (0, 1)$  there exist two slices  $S_1$  and  $S_2$  of  $B_Z$  and a  $\beta > 0$  such that  $\lambda S_1 + (1 - \lambda)S_2 \subset (1 - \beta)B_Z$ .

We should also remark that the positive part of the unit sphere of  $L_1[0, 1]$ ,  $F = \{f \in L_1[0, 1] : f \ge 0, ||f|| = 1\}$ , is another example of a closed convex bounded subset of a Banach space that satisfies a converse to Bourgain's lemma in that finite convex combinations of slices of F are relatively weakly open [7, Remark IV.5].

The notation and conventions we use are standard and follow, e.g., [6].

### 2. Main result

We start by recalling the following definition (see, e.g., [6, Definition 14.19]).

**Definition 2.1.** A compact space K is said to be *scattered compact* if every closed subset  $L \subset K$  has an isolated point in L.

Let K be a scattered compact Hausdorff space and consider the Banach space C(K) of all (complex-valued) continuous functions on K with sup-norm. Rudin [11] showed that  $C(K)^* = \ell_1(K)$  in this case. Pełczyński and Semadeni [10] showed that for a compact Hausdorff space K we have  $C(K)^* = \ell_1(K)$  if and only if K is scattered (= dispersed).

To prove the main result, we will need the following geometric lemma for the unit circle in the complex plane.

**Lemma 2.2.** Let  $\alpha, \beta \in \mathbb{R}$  such that  $e^{i\alpha}$  and  $e^{i\beta}$  are distinct points on the unit circle with distance  $d = |e^{i\alpha} - e^{i\beta}|$ . If  $0 < \mu < \frac{1}{2}$ , then the point  $c = \mu e^{i\alpha} + (1-\mu)e^{i\beta}$  on the line segment between  $e^{i\alpha}$  and  $e^{i\beta}$  satisfies

$$|c| \le 1 - \frac{d^2\mu}{4}.$$

*Proof.* A straightforward calculation shows that  $d^2 = 2 - 2\cos(\alpha - \beta)$  and that  $|c|^2 = \mu^2 + (1 - \mu)^2 + \mu(1 - \mu)2\cos(\alpha - \beta)$ . Hence  $|c|^2 = 1 - d^2\mu(1 - \mu)$ . Since  $\sqrt{1 + x} \le 1 + \frac{x}{2}$  for  $x \ge -1$  and  $\mu(1 - \mu) \ge \frac{\mu}{2}$  for  $\mu \in [0, \frac{1}{2}]$ , we get

$$|c| = \sqrt{1 - d^2 \mu (1 - \mu)} \le 1 - \frac{1}{2} d^2 \mu (1 - \mu) \le 1 - \frac{d^2 \mu}{4},$$

as desired.

**Theorem 2.3.** Let K be a scattered compact Hausdorff space. Then every finite convex combination of slices of the unit ball of C(K) is relatively weakly open.

*Proof.* Let  $\{S(f_j, \varepsilon_j)\}_{j=1}^k$  be slices of  $B_{C(K)}$  with  $f_j \in \ell_1(K)$ ,  $||f_j|| = 1$ , and  $\varepsilon_j > 0$  for  $j = 1, 2, \ldots, k$ . Let  $\lambda_j > 0$  with  $\sum_{j=1}^k \lambda_j = 1$ , and consider the convex combination of these slices

$$C = \sum_{j=1}^{k} \lambda_j S(f_j, \varepsilon_j).$$

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Let  $x = \sum_{j=1}^{k} \lambda_j z_j \in C$  with  $z_j \in S(f_j, \varepsilon_j)$ . Our goal is to find a nonempty relatively weakly open neighborhood of x that is contained in C.

Let  $d = \min\{\operatorname{Re} f_j(z_j) - (1 - \varepsilon_j) : 1 \le j \le k\}$  and let  $\eta > 0$  be such that  $\eta < d/3$ . Let  $E \subset K$  be a finite set such that  $\sum_{t \notin E} |f_j(t)| < \eta$  for  $1 \le j \le k$ .

Define

$$\mathcal{U} = \{ y \in B_{C(K)} : |y(t) - x(t)| < \delta, t \in E \},\$$

where  $\delta > 0$ . Next we specify how  $\delta$  is chosen.

Let  $L = \max\{\frac{1}{\lambda_j} : j = 1, 2, ..., k\}$ . Let

$$E_I = \{t \in E : \text{there exists } 1 \le j_0 \le k \text{ such that } |z_{j_0}(t)| < 1\}$$

Define

$$\delta_I = (1+3L)^{-1} \min \left\{ 1 - |z_{j_0}(t)| : t \in E_I, |z_{j_0}(t)| < 1 \right\}$$

if  $E_I$  is nonempty and  $\delta_I = 1$  otherwise. Let

$$E_{III} = \{t \in E \setminus E_I : \text{there exists } j \neq m \text{ such that } z_j(t) \neq z_m(t)\}$$

and define

(1) 
$$D = \min_{t \in E_{III}} \min_{z_j(t) \neq z_m(t)} \{ |z_j(t) - z_m(t)|^2 \}.$$

Choose  $0 < \rho < \min\{D/8, \eta/4L\}$ . Define  $\delta_{III} = D\rho(4(1+3L))^{-1}$  if  $E_{III}$  is nonempty and  $\delta_{III} = 1$  otherwise. Finally we choose  $\delta < \min\{\eta/6L, \delta_I, \delta_{III}\}$ .

Let  $y \in \mathcal{U}$ . We will define  $y_j \in S(f_j, \varepsilon_j), j = 1, 2, ..., k$ , and show that y can be written  $y = \sum_{j=1}^k \lambda_j y_j \in C$ .

Let  $\{\mathcal{V}_t\}_{t\in E}$  be a collection of pairwise disjoint neighborhoods for the points in E chosen such that for each  $t \in E$  we have  $|z_j(t) - z_j(s)| < \delta$ ,  $1 \leq j \leq k$ ,  $|x(t) - x(s)| < \delta$ , and  $|y(t) - y(s)| < \delta$  for all  $s \in \mathcal{V}_t$ . If  $t \in E$  is an isolated point, we let  $\mathcal{V}_t = \{t\}$ . Note that, in particular, we get  $|x(s) - y(s)| < 3\delta$  for all  $s \in \mathcal{V}_t$ .

**Definition of**  $y_j$  **outside**  $\bigcup_{t \in E} \mathcal{V}_t$ . For  $s \in K \setminus \bigcup_{t \in E} \mathcal{V}_t$  we define  $y_j(s) = y(s)$  for all  $1 \leq j \leq k$ .

**Definition of**  $y_j$  on  $\bigcup_{t \in E} \mathcal{V}_t$ . For each  $t \in E$  the way we define  $y_j$  on  $\mathcal{V}_t$  depends on whether  $t \in E_I$ ,  $t \in E_{III}$ , or neither, so we have to consider three cases. Let  $t \in E$ . Choose by Urysohn's lemma a real-valued nonnegative continuous function  $n_t \in S_{C(K)}$  with  $n_t(t) = 1$  such that  $n_t(s) = 0$  off  $\mathcal{V}_t$ . Define w(t) = y(t) - x(t) for all  $t \in K$ .

Case I. Assume  $t \in E_I$ . Then by definition of  $E_I$  there exists  $1 \leq j_0 \leq k$  with  $|z_{j_0}(t)| < 1$ . Now, for  $s \in \mathcal{V}_t$  let

$$y_{j_0}(s) = n_t(s)[z_{j_0}(s) + \lambda_{j_0}^{-1}w(s)] + [1 - n_t(s)]y(s)$$

and for  $j \neq j_0$  we let

$$y_j(s) = n_t(s)z_j(s) + [1 - n_t(s)]y(s).$$

It is straightforward to see that  $\sum_{j=1}^{k} \lambda_j y_j(s) = y(s)$  and that by the choice of  $\delta$ 

$$|z_{j_0}(s) + \lambda_{j_0}^{-1}w(s)| \le |z_{j_0}(t)| + |z_{j_0}(s) - z_{j_0}(t)| + L|y(s) - x(s)|$$
  
$$\le |z_{j_0}(t)| + \delta + 3L\delta < 1$$

for all  $s \in \mathcal{V}_t$ . Thus we have  $|y_j(s)| \leq 1$  for every  $1 \leq j \leq k$ .

We will need that  $|y_{j_0}(t) - z_{j_0}(t)| \le \lambda_{j_0}^{-1} |y(t) - x(t)| < L\delta < \eta$  and  $|y_j(t) - z_j(t)| = 0$  for  $j \ne j_0$ .

Case II. If for all  $1 \leq j, m \leq k$  we have  $z_j(t) = z_m(t)$  with  $|z_j(t)| = 1$ , then  $x(t) = z_j(t)$  and we can just let  $y_j(s) = y(s)$  for all  $1 \leq j \leq k$  and  $s \in \mathcal{V}_t$ . We will need that  $|y_j(t) - z_j(t)| = |y(t) - x(t)| < \delta < \eta$ .

Case III. The remaining case is that  $t \in E_{III}$ ; that is,  $|z_j(t)| = 1$  for all  $1 \leq j \leq k$ , but not all  $z_j(t)$  are equal. Order the set  $\{\arg z_j(t) : 1 \leq j \leq k\}$  as an increasing sequence  $\{\theta_1 < \theta_2 < \cdots < \theta_q\}$  and define  $\theta_0 = \theta_q$ . We put  $A_p = \{j : \arg z_j(t) = \theta_p\}$ and  $\Lambda_p = \sum_{j \in A_p} \lambda_j$ .

With  $\rho$  as above we define for  $1 \le p \le q$ 

$$c_p = \rho(e^{i\theta_{p-1}} - e^{i\theta_p}).$$

Let  $s \in \mathcal{V}_t$  and define (for  $j \in A_p$ )

$$y_j(s) = n_t(s) \left[ z_j(s) + \frac{c_p}{\Lambda_p} + \frac{w(s)}{q\Lambda_p} \right] + (1 - n_t(s))y(s).$$

We have

$$\sum_{j=1}^{k} \lambda_j y_j(s) = \sum_{p=1}^{q} \sum_{j \in A_p} \lambda_j y_j(s)$$
  
=  $\sum_{p=1}^{q} n_t(s) \sum_{j \in A_p} \lambda_j z_j(s) + \sum_{p=1}^{q} n_t(s) c_p + \sum_{p=1}^{q} n_t(s) \frac{w(s)}{q} + (1 - n_t(s))y(s)$   
=  $n_t(s) \sum_{j=1}^{k} \lambda_j z_j(s) + n_t(s)0 + n_t(s)w(s) + (1 - n_t(s))y(s)$   
=  $n_t(s)x(s) + n_t(s)(y(s) - x(s)) + y(s) - n_t(s)y(s) = y(s)$ 

With  $\mu = \rho / \Lambda_p$ 

$$z_j(t) + \frac{c_p}{\Lambda_p} = e^{i\theta_p} + \mu(e^{i\theta_{p-1}} - e^{i\theta_p}) = \mu e^{i\theta_{p-1}} + (1-\mu)e^{i\theta_p}.$$

So, by Lemma 2.2 and (1)

$$\left| z_j(t) + \frac{c_p}{\Lambda_p} \right| \le 1 - \frac{|e^{i\theta_{p-1}} - e^{i\theta_p}|^2 \rho}{4\Lambda_p} \le 1 - \frac{D\rho}{4\Lambda_p} < 1 - \frac{D\rho}{4} < 1 - (1+3L)\delta.$$

Hence

$$\left| z_j(s) + \frac{c_p}{\Lambda_p} + \frac{w(s)}{q\Lambda_p} \right| \le \left| z_j(t) + \frac{c_p}{\Lambda_p} \right| + \left| z_j(s) - z_j(t) \right| + \left| \frac{w(s)}{q\Lambda_p} \right|$$
  
$$< 1 - (1 + 3L)\delta + \delta + 3L\delta = 1.$$

Thus we have  $|y_j(s)| \leq 1$ . We will also need that

$$|y_j(t) - z_j(t)| = \left|\frac{c_p}{\Lambda_p} + \frac{w(t)}{q\Lambda_p}\right| \le \rho |e^{i\theta_{p-1}} - e^{i\theta_p}|L + 3\delta L \le 2L\rho + 3L\delta \le \eta.$$

Conclusion. So far we have defined  $y_j \in B_{C(K)}$  and shown that  $y = \sum_{j=1}^k \lambda_j y_j$ . Note that for each  $1 \leq j \leq k$  the function  $y_j$  is continuous on K since  $y_j$  is a combination of the continuous functions  $z_j$ , y, x, and  $n_t$ . Also  $n_t$  is zero off  $\mathcal{V}_t$ ; hence  $y_j = y$  on  $K \setminus \bigcup_{t \in E} \mathcal{V}_t$ .

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It only remains to show that  $y_j \in S(f_j, \varepsilon_j)$ . We have

$$\sum_{t \notin E} |f_j(t)(y_j(t) - z_j(t))| < \eta ||y_j - z_j|| \le 2\eta$$

and

$$\sum_{t \in E} |f_j(t)(y_j(t) - z_j(t))| < ||f_j|| \eta < \eta.$$

Hence  $|f_j(y_j - z_j)| < 3\eta$  so that

$$\operatorname{Re} f_j(y_j) \ge \operatorname{Re} f_j(z_j) - 3\eta > \operatorname{Re} f_j(z_j) - d > 1 - \varepsilon_j,$$

and we are done.

The above theorem applies to  $C[0, \alpha]$  for any infinite ordinal  $\alpha$  and in particular to  $c = C[0, \omega]$ . It should be clear that the proof also works for real scalars and that it proves the following result.

**Theorem 2.4.** Every finite convex combination of slices of the unit ball of  $c_0$  is relatively weakly open.

## 3. Questions and remarks

We will end with some questions and remarks.

- (i) Which Banach spaces satisfy that finite convex combinations of slices of the unit ball are relatively weakly open?
- (ii) Which Banach spaces satisfy that finite convex combinations of slices of the unit ball contain a nonempty relatively weakly open neighborhood of some point in the combination?
- (iii) Which Banach spaces satisfy that finite convex combinations of slices of the unit ball always have nonempty intersection with the sphere?
- (iv) If finite convex combinations of slices of both  $B_X$  and  $B_Y$  are relatively weakly open, is the same true for the unit ball of  $X \oplus_{\infty} Y$  and/or  $X \oplus_1 Y$ ?

It is not clear that there is a connection between having relatively weakly open convex combinations of slices and the diameter 2 properties. But we have the following observation.

Remark 3.1. Let X be an (infinite-dimensional) Banach space such that there exists a slice  $S_1 = S(x^*, \varepsilon)$  of  $B_X$  with diam  $S_1 < 1$ . Then with  $S_2 = S(-x^*, \varepsilon)$  and  $C = \frac{1}{2}S_1 + \frac{1}{2}S_2$  it is easy to see that  $C \cap S_X = \emptyset$ ; hence C is a convex combination of slices which is not relatively weakly open.

Regarding question (iii) we have the following examples of spaces where finite convex combinations of slices intersect the sphere.

**Example 3.2.** Finite convex combinations of slices of the unit ball of  $L_1[0, 1]$  always intersect the sphere. Here slices are given by functions  $g_j \in S_{L_{\infty}[0,1]}$ . We may assume that the  $g_j$ 's are simple functions and find sets  $B_j \subset [0,1]$  with  $B_j \cap B_k = \emptyset$  for  $j \neq k$  and  $\|\chi_{B_j}g_j\|_{\infty}$  almost 1. The functions  $f_j = m(B_j)^{-1}\chi_{B_j}$  do the job (m is Lebesgue measure).

**Example 3.3.** Let X be a Banach space such that whenever  $S_j = S(x_j^*, \varepsilon_j)$  with  $x_j^* \in S_{X^*}$  and  $\varepsilon_j > 0$  for  $1 \le j \le k$  are slices of  $B_X$ , then there exists  $x_j \in S_j \cap S_X$  and  $y \in S_X$  such that  $||x_j \pm y|| = 1$  and  $x_j + y \in S_j$ .

Spaces that satisfy this condition include  $\ell_{\infty}^{c}(\Gamma)$  for  $\Gamma$  uncountable since this space is almost square with  $\varepsilon = 0$  [2, Remark 2.11]. It also includes  $\ell_{\infty}$  and C[0, 1] since the slices there are defined by measures of bounded variation.

If X is a space with this property, then finite convex combinations of slices of  $B_X$  always intersect the sphere. Indeed, let  $\lambda_j > 0$  with  $\sum_{j=1}^k \lambda_j = 1$  and let  $S_j = S(x_j^*, \varepsilon_j)$  be slices of  $B_X$  with  $x_j^* \in S_{X^*}$  and  $\varepsilon_j > 0$  for  $1 \le j \le k$ .

By assumption, there exists  $x_j \in S_j \cap S_X$  and  $y \in S_X$  such that  $||x_j \pm y|| = 1$ and  $x_j + y \in S_j$ .

Choose  $y^* \in S_{X^*}$  such that  $y^*(y) = 1$ . Then

$$1 = ||x_j \pm y|| \ge y^*(y) \pm y^*(x_j) = 1 \pm y^*(x_j);$$

hence  $y^*(x_j) = 0$ . Now  $\sum_{j=1}^k \lambda_j (x_j + y) \in \sum_{j=1}^k \lambda_j S_j$  and  $\left\| \sum_{j=1}^k \lambda_j (x_j + y) \right\| \ge \sum_{j=1}^k \lambda_j y^*(y) = 1.$ 

**Example 3.4.** If X has the Daugavet property, then finite convex combinations of weak\*-slices of  $B_{X*}$  intersect the sphere  $S_{X*}$ . To see this let  $x_j \in S_X$ , let  $\varepsilon_j > 0$ , and let  $S(x_j, \varepsilon_j)$  be slices of  $B_{X*}$  for  $1 \le j \le k$ . Consider  $\sum_{j=1}^k \lambda_j S(x_j, \varepsilon_j)$  where  $\lambda_j > 0$  and  $\sum_{j=1}^k \lambda_j = 1$ .

By using [9, Lemma 2.12] and an induction argument we can, for  $1 \le j \le k$ , find  $x_j^* \in S(x_j, \varepsilon_j) \cap S_{X^*}$  such that  $\|\sum_{j=1}^k \lambda_j x_j^*\| = \sum_{j=1}^k \lambda_j = 1$ .

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