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GEOMETRIC STRUCTURES ON LIE ALGEBRAS AND DOUBLE EXTENSIONS

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Honoring the 60th birthday of O. A. Sánchez-Valenzuela

Abstract. Given a finite-dimensional real or complex Lie algebra $\mathfrak{g}$ equipped with a geometric structure (i.e., either an invariant metric, a symplectic or contact structure), the aim of this work is to show that the double extension process introduced by V. Kac allows one to generate Lie algebras equipped with the same type of geometric structure. In particular, for an exact symplectic Lie algebra, through a double extension process it is possible to construct new exact symplectic Lie algebras.

1. Introduction

The *double extension* process appears for the first time as a couple of exercises in the book *Infinite-dimensional Lie algebras* of V. Kac (see exercises 2.10 and 2.11 of [8] for more details). In fact, the first exercise (exercise 2.10 in [8]) describes the process to construct an $(n+2)$-dimensional *quadratic* Lie algebra, i.e., a Lie algebra equipped with an *invariant metric*, from an n-dimensional quadratic Lie algebra; while in the second exercise (labeled as exercise 2.11 in [8]) the reader is asked to prove that every indecomposable quadratic *solvable* Lie algebra is a double extension of a quadratic Lie algebra of codimension 2. A few years later, almost simultaneously, both A. Medina and P. Revoy (see [11]), and G. Favre and J. L. Santharoubane (see [6]) proved that this result holds for any non-simple quadratic Lie algebra $\mathfrak{g}$ having dimension greater than one and such that every ideal of $\mathfrak{g}$ *degenerates*. In 1991, A. Medina and P. Revoy provided an analogous result concerning *nilpotent* symplectic Lie algebras (see [12]). Observe that in both works, conditions are given for a quadratic or symplectic Lie algebra to be a double extension of another quadratic or symplectic Lie algebra, respectively. Clearly, this process allows one to construct families of either quadratic or symplectic Lie algebras and, in some cases, it also allows one to recover the family of Lie algebras in terms of a *minimal* ideal and successive applications of the double extension process. These ideas described above also have been applied in the context of

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quadratic Lie superalgebras (see [2]), Poisson Lie groups (see [10]), or Lie–Kähler groups (see [5]).

More recently, in [1] and [13] similar problems have been addressed for a *contact* Lie algebra. In [13] conditions are given for a double extension of a contact Lie algebra to be a contact Lie algebra again; moreover, it is shown that there are contact Lie algebras that cannot be expressed as a double extension of a contact Lie algebra of codimension 2. Thus, in general, it is not possible to talk about a minimal contact Lie algebra from which, through the double extension process, one can construct all the contact Lie algebras. On the other hand, in [9], Y. Khakimdjanov, M. Goze, and A. Medina point out that every nilpotent contact Lie algebra can be obtained by the "symplectic double extension" and the contactization. Following this idea, in [1] it is proved that every contact nilpotent Lie algebra having dimension greater than 5 can be constructed from the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$, through a successive application of appropriate double extensions.

Hence, given a finite-dimensional Lie algebra $\mathfrak{g}$ equipped either with an invariant metric, a symplectic structure or a contact structure, the aim of this work is to determine whether a double extension of such Lie algebra $\mathfrak{g}$ produces a Lie algebra equipped with the same type of geometric structure. In all cases, we provide a positive answer, although we shall note that the answer concerning contact Lie algebras has been addressed in [13] and for the sake of completeness we include it.

It is important to point out the difference between this work and the works of V. Kac, A. Medina, and P. Revoy: in the construction of a *quadratic* double extension of a Lie algebra $\mathfrak{g}$ presented in both [8] and [11], it is required that the derivation acts in a trivial way in the central element of the central extension of $\mathfrak{g}$, and the same requirement is made for a *symplectic* double extension given in [12]. Clearly, this produces Lie algebras having a *non-trivial center*. On the other hand, in this work it is proved that for the quadratic case, the condition that the derivation acts in a trivial way in the central element of the central extension of $\mathfrak{g}$ *it is a necessary condition*, whereas for the symplectic case *it is not*. Hence, we can produce *exact* symplectic or *Frobenius* Lie algebras that do not appear in the construction given by A. Medina and P. Revoy.

Finally, the work is presented as follows: in section 1 we provide the basic definitions and the first elementary results, in section 2 we focus on determining whether a double extension of a quadratic Lie algebra is a quadratic Lie algebra again, and as a consequence of this, we can recover Theorem 17 from [14]. Subsequently, in section 3 we address the problem for symplectic Lie algebras and, in particular, we show that for an exact symplectic or Frobenius Lie algebra, through a double extension process it is possible to construct new exact symplectic Lie algebras. Finally, following completely [13], in section 4 we study the case regarding contact Lie algebras.

2. Preliminaries

For the sake of completeness, in this section we collect some well-known facts about cohomology of Lie algebras and we also introduce the definitions needed to develop the main results of this work. Here and subsequently, $\mathbb{F}$ will denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

2.1. **Cohomology of Lie algebras.** For this the best general reference is [4]. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Given a linear representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, $C^k(\mathfrak{g}, V)$ denotes the space of k-linear functions from $\mathfrak{g}$ into V. As usual, $C^0(\mathfrak{g}, V) := V$ and $C^1(\mathfrak{g}, V) = \mathrm{Hom}(\mathfrak{g}, V)$. An element $\phi \in C^k(\mathfrak{g}, V)$ is called a *k-form.* The exterior differential $d : C^k(\mathfrak{g}, V) \to C^{k+1}(\mathfrak{g}, V)$ is defined as follows: for $\phi \in C^0(\mathfrak{g}, V) := V$, $(d\phi)(x) := \rho(x)(\phi)$, whereas for $\phi \in C^k(\mathfrak{g}, V)$ with $k \geq 1$, then

$$
\begin{aligned}
(d\phi)(x_1, \dots, x_{k+1}) = \frac{1}{k+1} \sum_{i=1}^{k+1} (-1)^{i+1} \rho(x_i) \phi(x_1, \dots, \widehat{x_i}, \dots, x_{k+1}) \\
+ \frac{1}{k+1} \sum_{i<j} (-1)^{i+j} \phi([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}),
\end{aligned}
$$

where $\widehat{\ }$ indicates that the i-th element is removed. If $\dim V = 1$ one can consider the trivial representation $\rho(x) \equiv 0$ for all $x \in \mathfrak{g}$ and in this case, $C^k(\mathfrak{g}, V)$ is denoted by $(\Lambda^k \mathfrak{g})^*$.

Given a Lie algebra $\mathfrak{g}$ and a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we define

$$
\begin{aligned}
Z^k(\mathfrak{g}, V) &= \mathrm{Ker}(d : C^k(\mathfrak{g}, V) \to C^{k+1}(\mathfrak{g}, V)), \\
B^k(\mathfrak{g}, V) &= \mathrm{Im}(d : C^{k-1}(\mathfrak{g}, V) \to C^k(\mathfrak{g}, V)).
\end{aligned}
$$

A k-form $\omega \in Z^k(\mathfrak{g}, V)$ is called *closed* or a *k-cocycle*, whereas a k-form $\eta \in B^k(\mathfrak{g}, V)$ is called *exact*.

The *k-th group of cohomology with coefficients in the representation* ρ is defined by:

$$
\begin{aligned}
H^k(\mathfrak{g}, V) &= \frac{Z^k(\mathfrak{g}, V)}{B^k(\mathfrak{g}, V)}, \quad k \geq 1, \\
H^0(\mathfrak{g}, V) &= Z^0(\mathfrak{g}, V).
\end{aligned}
$$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}$. We say that $(\mathfrak{g}, B)$ is a *quadratic* or *metric* Lie algebra if B satisfies that $B([x, y], z) = B(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$. In this case we will say that B is an *invariant scalar product* or an *invariant metric* on $\mathfrak{g}$.

Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra and let $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ be a non-degenerate skew-symmetric bilinear form on $\mathfrak{g}$. We say that $(\mathfrak{g}, \omega)$ is a *symplectic* Lie algebra if ω satisfies that $d\omega = 0$, that is, if ω is a non-degenerate 2-cocycle for the scalar cohomology of $\mathfrak{g}$. In this case we will say that ω is a *symplectic structure* on $\mathfrak{g}$. Furthermore, we say that $(\mathfrak{g}, \omega)$ is a *symplectic exact* or *Frobenius* Lie algebra if in addition there exists $\varphi \in \mathfrak{g}^*$ such that $\omega = d\varphi$. In this case ω will be called a *symplectic exact structure* on $\mathfrak{g}$. Let $\mathfrak{g}$ be a $(2n + 1)$-dimensional Lie algebra and let $\alpha : \mathfrak{g} \to \mathbb{F}$ be a 1-form. We say that $(\mathfrak{g}, \alpha)$ is a *contact* Lie algebra if α satisfies that $\alpha \wedge (d\alpha)^n \neq 0$. In this case we will say that α is a *contact structure* on $\mathfrak{g}$.

Here and subsequently, by a geometric structure defined on a Lie algebra $\mathfrak{g}$, we will understand that the Lie algebra $\mathfrak{g}$ is equipped with either an invariant metric, a symplectic structure, or a contact structure.

2.2. **Double extensions.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $\theta \in (\Lambda^2 \mathfrak{g})^*$ be a 2-*cocycle*. Let $\langle e \rangle := \mathrm{Span}_{\mathbb{F}}\{e\}$ be a 1-dimensional trivial Lie algebra.

Then, letting

$$[x,y]_\theta = [x,y]_{\mathfrak{g}} + \theta(x,y)e, \quad x,y \in \mathfrak{g},$$
$$[x,e]_\theta = 0, \quad x \in \mathfrak{g},$$

the vector space $\mathfrak{g}_\theta(e) := \mathfrak{g} \oplus \langle e \rangle$ is clearly a Lie algebra that it is called the *central extension* of $\mathfrak{g}$ by the 2-cocycle θ.

Consider now a central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by a 2-cocycle $\theta \in (\Lambda^2\mathfrak{g})^*$, and let $D \in \mathrm{Der}_{\mathbb{F}}(\mathfrak{g}_\theta(e))$ be a derivation of $\mathfrak{g}_\theta(e)$. The *double extension* of $\mathfrak{g}$ by the pair (D,θ) is defined as the semidirect product $\mathfrak{g}(D,\theta) := \langle D \rangle \ltimes \mathfrak{g}_\theta(e)$ of the abelian Lie algebra $\langle D \rangle$ with $\mathfrak{g}_\theta(e)$, where the bracket is given by

$$[x,y]_{D,\theta} = [x,y]_\theta, \quad x,y \in \mathfrak{g}_\theta(e),$$
$$[D,x]_{D,\theta} = D(x), \quad x \in \mathfrak{g}_\theta(e).$$

A direct calculation shows the following result:

Lemma 2.1. *Let $\mathfrak{g}_\theta(e) = \mathfrak{g} \oplus \langle e \rangle$ be a central extension of a real Lie algebra $\mathfrak{g}$ by a closed 2-form θ. Then the center of $\mathfrak{g}_\theta(e)$ is given by*

$$Z(\mathfrak{g}_\theta(e)) = (Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)) \oplus \langle e \rangle,$$

where $\mathrm{Rad}(\theta) = \{z \in \mathfrak{g} \mid \theta(z,x) = 0 \ \forall x \in \mathfrak{g}\}$.

Given a central extension $\mathfrak{g}_\theta(e)$ of a Lie algebra $\mathfrak{g}$ by a closed 2-form θ, the next result characterizes the derivations $D \in \mathrm{Der}_{\mathbb{F}}(\mathfrak{g}_\theta(e))$.

Lemma 2.2. *A linear transformation $D \in \mathfrak{gl}_{\mathbb{F}}(\mathfrak{g}_\theta(e))$ consists of a 4-tuple $(A,f,v,b) \in \mathrm{End}_{\mathbb{F}}(\mathfrak{g}) \times \mathfrak{g}^* \times \mathfrak{g} \times \mathbb{F}$ satisfying*

$$\begin{aligned} D(x) &= A(x) + f(x)e \text{ for } x \in \mathfrak{g},\\ D(e) &= v + be. \end{aligned} \tag{2.1}$$

Moreover, such $D \in \mathfrak{gl}_{\mathbb{F}}(\mathfrak{g}_\theta(e))$ is a derivation if and only if the following conditions hold for all $x,y \in \mathfrak{g}$:

(1) $dA(x,y) = \theta(x,y)v$ (*in this case d is the exterior differential for the adjoint representation*),
(2) $-df(x,y) = \theta(A(x),y) + \theta(x,A(y)) - b\,\theta(x,y)$ (*in this case d is the exterior differential for the trivial representation*),
(3) $v \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$.

Remark 2.3. In the previous lemma suppose that $\theta \not\equiv 0$; then $v = 0$ if and only if $A \in \mathrm{Der}(\mathfrak{g})$. In particular, this holds when either $Z(\mathfrak{g}) = \{0\}$ or $\mathrm{Rad}(\theta) = \{0\}$.

Corollary 2.4. *Let $D \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$ be a derivation as in Lemma 2.2 above. It follows that*

(1) *$D(e) \in Z(\mathfrak{g}_\theta(e))$.*
(2) *If $x \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$, then $A(x) \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$.*
(3) *If $v = 0$, then D is a nilpotent derivation if and only if $A \in \mathrm{Der}_{\mathbb{F}}(\mathfrak{g})$ is a nilpotent derivation and clearly, $b = 0$.*

Lemma 2.5. *Let $\mathfrak{g}(D,\theta)$ be a double extension of $\mathfrak{g}$ by the pair (D,θ) and suppose that $D(e) \neq 0$. Then $w \in Z(\mathfrak{g}(D,\theta))$ if and only if there exist $w_0 \in \mathfrak{g}, p, q \in \mathbb{R}$ such that*

(1) *$w = w_0 + pe + qD$,*
(2) *$w_0 \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$,*

(3) $q = 0$,
(4) $D(w_0 + pe) = 0$.

3. Quadratic Lie algebras

The goal of this section is to determine whether a double extension of a quadratic Lie algebra is a quadratic Lie algebra. For doing this, let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a fixed basis for $\mathfrak{g}$. For any linear or bilinear transformation T defined on $\mathfrak{g}$, we will denote by $[T]$ its associated matrix in the chosen basis. A straightforward calculation shows the following result:

Lemma 3.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a fixed basis for $\mathfrak{g}$ and consider a bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$. Then the following statements are equivalent:*

(1) *B is an invariant form on $\mathfrak{g}$,*
(2) *$[\mathrm{ad}(x)]^T[B] + [B][\mathrm{ad}(x)] = 0$ for all $x \in \mathfrak{g}$,*
(3) *$[\mathrm{ad}(e_i)]^T[B] + [B][\mathrm{ad}(e_i)] = 0$ for all $1 \leq i \leq n$.*

Now, consider a double extension $\mathfrak{g}(D,\theta)$ of a finite-dimensional Lie algebra $\mathfrak{g}$ by a pair (D,θ). In what follows $\mathrm{ad}_{D,\theta} : \mathfrak{g}(D,\theta) \to \mathfrak{gl}(\mathfrak{g}(D,\theta))$ stands for the adjoint representation of the double extension $\mathfrak{g}(D,\theta)$ of $\mathfrak{g}$ by the pair (D,θ), whereas for $x \in \mathfrak{g}$, θ_x denotes the linear function $\theta_x \in \mathfrak{g}^*$ such that $\theta_x(y) := \theta(x,y)$ for all $y \in \mathfrak{g}$. Observe that fixing a basis $\widehat{\mathcal{B}} = \mathcal{B} \cup \{e, D\}$ of $\mathfrak{g}(D,\theta)$, according to Lemma 2.2 one can explicitly compute the associated matrix to any derivation of $\mathfrak{g}(D,\theta)$. Hence, relative to basis $\widehat{\mathcal{B}}$ one gets that the associated matrix of an inner derivation of $\mathfrak{g}(D,\theta)$ is given by one of the following expressions:

$$[\mathrm{ad}_{D,\theta}(x)] = \begin{pmatrix} [\mathrm{ad}(x)] & 0 & -[Ax] \\ [\theta_x] & 0 & f(x) \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } x \in \mathfrak{g},$$

$$[\mathrm{ad}_{D,\theta}(e)] = \begin{pmatrix} 0 & 0 & -v \\ 0 & 0 & -b \\ 0 & 0 & 0 \end{pmatrix}, \qquad [\mathrm{ad}_{D,\theta}(D)] = \begin{pmatrix} [A] & v & 0 \\ [f]^T & b & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose now that $\mathfrak{g}$ is a quadratic Lie algebra, that is, $\mathfrak{g}$ is equipped with an invariant metric $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$. On the double extension $\mathfrak{g}(D,\theta)$ of $\mathfrak{g}$ by the pair (D,θ) it can be defined as a non-degenerate symmetric bilinear form $B_{D,\theta} : \mathfrak{g}(D,\theta) \times \mathfrak{g}(D,\theta) \to \mathbb{F}$ as follows:

$$\begin{aligned} B_{D,\theta}(x,y) &= B(x,y) \quad \text{for } x, y \in \mathfrak{g}, \\ B_{D,\theta}(x,e) &= B_{D,\theta}(x,D) = 0 \quad \text{for } x \in \mathfrak{g}, \\ B_{D,\theta}(e,e) &= B_{D,\theta}(D,D) = 0, \\ B_{D,\theta}(e,D) &= B_{D,\theta}(D,e) = \lambda \neq 0. \end{aligned} \tag{I}$$

Clearly, the associated matrix of $B_{D,\theta}$ with respect to the basis $\widehat{\mathcal{B}}$ is given by

$$[B_{D,\theta}] = \begin{pmatrix} [B] & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix}.$$

Then we can state the main theorem of this section.

Theorem 3.2. *Let $\mathfrak{g}(D,\theta)$ be a double extension of a finite-dimensional Lie algebra $\mathfrak{g}$ by the pair (D,θ). Let $B_{D,\theta}:\mathfrak{g}(D,\theta)\times\mathfrak{g}(D,\theta)\to\mathbb{F}$ be the symmetric bilinear form defined by the equations given by* (I). *Then $(\mathfrak{g}(D,\theta),B_{D,\theta})$ is a quadratic Lie algebra if and only if $f(x)=0$, $D(e)=0$, and $\theta(x,y)=\frac{1}{\lambda}B(Ax,y)$ for all $x,y\in\mathfrak{g}$.*

Proof. Let $\widehat{\mathcal{B}}=\{e_1,\dots,e_n,e,D\}$ be a basis of $\mathfrak{g}(D,\theta)$. From Lemma 3.1 it follows that $(\mathfrak{g}(D,\theta),B_{D,\theta})$ is a quadratic Lie algebra if and only if the next conditions are satisfied:

(1) $\lambda[\theta_x]=[x]^T[A]^T[B]$,
(2) $f(x)=0$,
(3) $v=0$,
(4) $b=0$,
(5) $[A]^T[B]+[B][A]=0$.

Let $\mathfrak{g}_\theta(e)$ be the central extension of $\mathfrak{g}$ by the 2-cocycle θ. Observe that conditions (3) and (4) are equivalent to the fact that the derivation $D\in\operatorname{Der}(\mathfrak{g}_\theta(e))$ acts trivially on the center $Z(\mathfrak{g}_\theta(e))$ of the central extension $\mathfrak{g}_\theta(e)$, that is, $D(e)=0$. Moreover, since $v=0$, from Lemma 2.2 follows that $A\in\operatorname{Der}(\mathfrak{g})$. On the other hand, equation (1) is equivalent to $\lambda\theta(x,y)=B(Ax,y)$ for all $x,y\in\mathfrak{g}$ and since $\theta\in(\Lambda^2\mathfrak{g})^*$ is a 2-cocyle, A is skew-symmetric with respect to B, but this is precisely equation (5). □

Remark 3.3. Given a quadratic Lie algebra $(\mathfrak{g},B)$, it is well known that there exists a one-to-one correspondence between 2-cocycles $\theta\in(\Lambda^2\mathfrak{g})^*$ and skew-symmetric derivations $D\in\operatorname{Der}(\mathfrak{g})$ with respect to the bilinear form B. Since the invariant metric $B:\mathfrak{g}\times\mathfrak{g}\to\mathbb{F}$ is non-degenerate, the correspondence is obtained considering $\theta(x,y)=B(Dx,y)$ for all $x,y\in\mathfrak{g}$. Moreover, θ is non-degenerate if and only if D is invertible.

The following result is a direct consequence of Theorem 3.2 above.

Corollary 3.4. *If $(\mathfrak{g}(D,\theta),B_{D,\theta})$ is a quadratic Lie algebra, then*

(1) $\operatorname{Rad}(\theta)=\operatorname{Ker}A$,
(2) *$w\in Z(\mathfrak{g}(D,\theta))$ if and only if there exist $w_0\in\mathfrak{g}$, $p,q\in\mathbb{R}$ such that $w=w_0+pe+qD$ and*
   (a) $w_0\in\operatorname{Rad}(\theta)$,
   (b) $\operatorname{ad}_{\mathfrak{g}}(w_0)=-qA$.
(3) *$\langle e\rangle\subset Z(\mathfrak{g}(D,\theta))$. Moreover, $\langle e\rangle$ is an isotropic ideal.*

On the the other hand, the proof of the next result follows immediately from Exercise 2.10 of [8].

Theorem 3.5. *Let $n\geq 5$ and $(\mathfrak{h},\Phi)$ be an n-dimensional quadratic Lie algebra having a 1-dimensional central isotropic ideal. Then, there exists an $(n-2)$-dimensional quadratic Lie algebra $(\mathfrak{g},B)$, a 2-cocycle $\theta\in(\Lambda^2\mathfrak{g})^*$ and a derivation $D\in\operatorname{Der}(\mathfrak{g}_\theta(e))$ of the central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by θ such that $(\mathfrak{h},\Phi)$ is isometrically isomorphic to $(\mathfrak{g}(D,\theta),B_{D,\theta})$, where $\mathfrak{g}(D,\theta)$ is the double extension of $\mathfrak{g}$ by a pair (D,θ) and $B_{D,\theta}:\mathfrak{g}(D,\theta)\times\mathfrak{g}(D,\theta)\to\mathbb{F}$ is the invariant metric defined on $\mathfrak{g}(D,\theta)$ by equations* (I).

Remark 3.6. From Theorem 3.5 above it follows that for $n\geq 1$, a $(2n+3)$-dimensional quadratic Lie algebra having a 1-dimensional central isotropic ideal is isometrically isomorphic to a double extension of a $(2n+1)$-dimensional quadratic Lie algebra. Observe that if one is restricted to the class of *nilpotent* Lie algebras endowed with either a symplectic or a contact structure, analogous results can be obtained since, in both cases, there exists a 1-dimensional central isotropic ideal. However, this is not necessarily true for the general case.

3.1. **Example.** Consider the quadratic Lie algebra $(\mathbb{R}^{2n}, B)$ where $\mathbb{R}^{2n}$ is the $2n$-dimensional abelian Lie algebra and $B : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is the usual invariant metric on $\mathbb{R}^{2n}$. On the other hand, consider also $\theta : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ as the usual symplectic form on $\mathbb{R}^{2n}$. Clearly, θ is a 2-cocycle for the scalar cohomology of $\mathbb{R}^{2n}$ and hence, letting $\langle z\rangle$ be a 1-dimensional trivial Lie subalgebra, one can take the central extension $\mathbb{R}^{2n}_{\theta}(z)$ of $\mathbb{R}^{2n}$ by θ. Observe that $\mathbb{R}^{2n}_{\theta}(z)$ is just the $(2n+1)$-dimensional Heisenberg Lie algebra, that is, $\mathfrak{h}_{2n+1} = \mathbb{R}^{2n}_{\theta}(z)$. Choose now a derivation $D \in \mathrm{Der}(\mathbb{R}^{2n}_{\theta}(z))$ of the Heisenberg Lie algebra $\mathbb{R}^{2n}_{\theta}(z)$ in order to construct the double extension $\mathbb{R}^{2n}(D,\theta)$ of $\mathbb{R}^{2n}$ by the pair (D,θ). Hence, as a consequence of Theorem 3.2 we get the next result:

Theorem 3.7 (See [14], Theorem 17)**.** $\mathbb{R}^{2n}(D,\theta)$ *is a quadratic Lie algebra if and only if* $\mathrm{Ker}(D) = \langle z\rangle$*, where* $Z(\mathbb{R}^{2n}_{\theta}(z)) = \langle z\rangle$*.*

Proof. From Remark 3.3 it follows that the restriction $D|_{\mathbb{R}^{2n}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is an invertible transformation $D|_{\mathbb{R}^{2n}}$ and therefore, $\mathrm{Ker}(D) = \langle z\rangle$. Moreover, $D|_{\mathbb{R}^{2n}} \in \mathfrak{sp}(2n,\mathbb{R})$. □

4. Symplectic Lie algebras

As we did in section 2, we shall determine whether a double extension of a symplectic Lie algebra is a symplectic Lie algebra again. In particular, we shall show that for an exact symplectic Lie algebra, through a double extension process it is possible to construct new exact symplectic Lie algebras.

Proposition 4.1. *Let $\mathfrak{g}$ be a finite-dimensional symplectic Lie algebra with a symplectic structure $\omega \in (\Lambda^2\mathfrak{g})^*$, and let $\theta \in (\Lambda^2\mathfrak{g})^*$. Then for almost all $\lambda \in \mathbb{F}$ with the exception of a finite number of values, $\omega + \lambda\theta$ is a symplectic structure on $\mathfrak{g}$, i.e.,*

$$(\omega + \lambda\theta)^n \neq 0.$$

Proof. Let $\{e_1, \dots, e_{2n}\}$ be a basis for $\mathfrak{g}$ and let $\{e^1, \dots, e^{2n}\}$ be its dual basis. Since $\omega \in (\Lambda^2\mathfrak{g})^*$ is a symplectic structure on $\mathfrak{g}$, it follows that $\omega^n = a_0\, e^1 \wedge \cdots \wedge e^{2n}$ for some $a_0 \neq 0$. On the other hand, for each $k \in \{1, \dots, n\}$, $\omega^{n-k} \wedge \theta^k = a_k\, e^1 \wedge \cdots \wedge e^{2n}$ with $a_k \in \mathbb{F}$. Hence,

$$\begin{aligned}(\omega + \lambda\theta)^n &= \sum_{k=0}^{n} \binom{n}{k} \lambda^k\, \omega^{n-k} \wedge \theta^k \\ &= \Big(a_0 + \sum_{k=1}^{n} \binom{n}{k} a_k\, \lambda^k\Big)(e^1 \wedge \cdots \wedge e^{2n}).\end{aligned}$$

Let $p(x) = a_0 + \sum_{k=1}^{n} \binom{n}{k} a_k x^k \in \mathbb{F}[x]$ be a non-zero polynomial of degree n. Since $p(x)$ has at most n roots, there exists $\lambda \in \mathbb{F}$ such that $p(\lambda) \neq 0$. Therefore, for such $\lambda \in \mathbb{F}$, $(\omega + \lambda\theta)^n \neq 0$. □

Theorem 4.2. *Let $\mathfrak{g}$ be a finite-dimensional Frobenius Lie algebra with an exact symplectic structure given by $\omega = d\alpha$, $\alpha \in \mathfrak{g}^*$. Given a 2-cocycle $\theta \in (\Lambda^2\mathfrak{g})^*$, consider a central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by θ and let $\beta \in (\mathfrak{g}_\theta(e))^*$ be defined by $\beta = \alpha + \lambda e^*$, $\lambda \in \mathbb{F}$. If there exists a derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ such that $\beta(D(e)) \neq 0$, then the double extension $\mathfrak{g}(D,\theta)$ of $\mathfrak{g}$ by the pair (D,θ) is a Frobenius Lie algebra with exact symplectic form $\omega_\theta = d_\theta\beta$ for some $\lambda \in \mathbb{F}$.*

Proof. Let $[\theta] \in H^2(\mathfrak{g},\mathbb{F})$. Since the elements of $H^2(\mathfrak{g},\mathbb{F})$ are in a one-to-one correspondence with the isomorphism classes of central extensions of $\mathfrak{g}$ (see [4] for more details), if $[\theta] = 0$ (for example if $\theta = -d\alpha$), then the central extension of $\mathfrak{g}$ by θ is isomorphic to a direct sum of Lie algebras:

$$\mathfrak{g}_\theta(e) \simeq \mathfrak{g} \oplus \langle e\rangle.$$

Hence, defining $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ by $D|_{\mathfrak{g}} \equiv 0$ and $D(e) = e$, it follows that

$$\mathfrak{g}(D,\theta) \simeq \mathfrak{g} \oplus \langle e, D\rangle,$$

where $[D,e] = D(e) = e$. Since $\langle e, D\rangle$ is an exact symplectic Lie algebra, it follows that $\mathfrak{g}(D,\theta)$ is a Frobenius Lie algebra with exact symplectic form $\omega_\theta = d\beta$, $\beta = \alpha + e^*$ and clearly, $\lambda = 1$.

Suppose now that $[\theta] \neq 0$ and let $\beta = \alpha + \lambda\ e^* \in (\mathfrak{g}(D,\theta))^*$. We shall use the Maurer-Cartan equations of $\mathfrak{g}(D,\theta)$ in order to prove that $\omega_\theta = d\beta$ is an exact symplectic structure for some $\lambda \in \mathbb{F}$. For this we shall use the following notation: d_θ stands for the Chevalley-Eilenberg differential operator associated with the trivial representation on $\mathfrak{g}(D,\theta)$, and d stands for the Chevalley-Eilenberg differential operator associated with the trivial representation on $\mathfrak{g}$.

Let $\{e_1,\dots,e_{2n},e,D\}$ be a basis for $\mathfrak{g}(D,\theta)$ and $\{e^1,\dots,e^{2n},e^*,D^*\}$ be its dual basis. Then $\alpha = \sum_{i=1}^{2n}\alpha_i e^i$ for $\alpha_i \in \mathbb{R}$ and, in the chosen basis, $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ has a matrix representation given by

$$[D]_{\mathfrak{g}_\theta(e)} = \begin{pmatrix} [D|_{\mathfrak{g}}] & v \\ u^T & b \end{pmatrix},$$

where $v, u \in \mathbb{F}^n$, $b \in \mathbb{F}$ and $[D|_{\mathfrak{g}}]$ is a matrix representation for $D|_{\mathfrak{g}} \in \mathrm{End}_{\mathbb{F}}(\mathfrak{g})$, with $([D|_{\mathfrak{g}}])_{ij} = D_{ij} \in \mathbb{F}$.

Take $\beta = \alpha + \lambda\ e^* \in (\mathfrak{g}(D,\theta))^*$ and suppose that $\beta(D(e)) \neq 0$. Clearly, $d_\theta\beta = d_\theta\alpha + \lambda d_\theta e^*$ where

$$d_\theta\alpha = d\alpha + \frac{1}{2}\sum_{i=1}^{2n}\left(\sum_{k=1}^{2n}\alpha_k D_{ki}\right)e^i \wedge D^* + \frac{1}{2}\left(\sum_{i=1}^{2n}\alpha_i v_i\right)e^* \wedge D^*,$$

and

$$d_\theta e^* = -\frac{\lambda}{2}\left(\sum_{i<j=1}^{2n}\theta(e_i,e_j)e^i \wedge e^j - \sum_{i=1}^{2n}u_i e^i \wedge D^* - be^* \wedge D^*\right).$$

Now, let X be the 2-form in $(\Lambda^2\mathfrak{g})^*$ given by

$$\begin{aligned} X &= d\alpha - \frac{\lambda}{2}\sum_{i<j=1}^{2n}\theta(e_i,e_j)e^i \wedge e^j \\ &= d\alpha - \frac{\lambda}{2}\ \theta, \end{aligned}$$

and let Y be the 2-form in $(\Lambda^2\mathfrak{g}(D,\theta))^*$ given by

$$Y = \frac{1}{2}\left(\sum_{i=1}^{2n} \widehat{u}_i(\lambda)\ e^i \wedge D^* + \widehat{b}(\lambda)\ e^* \wedge D^*\right),$$

where

$$\widehat{u}_i(\lambda) = \lambda u_i + \sum_{k=1}^{2n} \alpha_k D_{ki},$$

$$\widehat{b}(\lambda) = \lambda b + \sum_{i=1}^{2n} \alpha_i v_i = \beta(D(e)) \neq 0.$$

Clearly, $X^k = 0$ for $k \geq n+1$, whereas $Y^k = 0$ for $k \geq 2$. Then $d_\theta\beta$ can be written as $X + Y$, and it follows that

$$\begin{aligned}(d_\theta\beta)^{n+1} &= (X+Y)^{n+1}\\ &= (n+1)X^n \wedge Y\\ &= \frac{n+1}{2}\sum_{i=1}^{2n} \widehat{u}_i(\lambda)(d\alpha - \frac{\lambda}{2}\theta)^n \wedge e^i \wedge D^*\\ &\qquad + (n+1)\widehat{b}(\lambda)(d\alpha - \frac{\lambda}{2}\theta)^n \wedge e^* \wedge D^*\\ &= (n+1)\widehat{b}(\lambda)(d\alpha - \frac{\lambda}{2}\theta)^n \wedge e^* \wedge D^* \neq 0\end{aligned}$$

because $(d\alpha - \frac{\lambda}{2}\theta)^n \wedge e^i$ is a $2n+1$-form on $\mathfrak{g}$ and $\dim \mathfrak{g} = 2n$. Hence, $(d_\theta\beta)^{n+1} \neq 0$. □

Remark 4.3. It follows from the proof of Theorem 4.2 above that if the central extension $\mathfrak{g}_\theta(e)$ is trivial, i.e., $[\theta] = 0$, then the Frobenius Lie algebra $\mathfrak{g}(D,\theta)$ obtained from the double extension process is a direct product of the original Frobenius Lie algebra $\mathfrak{g}$ and the 2-dimensional affine Lie algebra spanned by $\langle D, e\rangle$ with $[D,e] = e$, that is, $\mathfrak{g}(D,\theta) = \mathfrak{g} \oplus \langle D, e\rangle$.

Corollary 4.4. *If $\mathfrak{g}(D,\theta)$ is a Frobenius Lie algebra, then $\mathfrak{g}(D,\theta)$ is non-nilpotent.*

Proof. It follows either from Lemma 1 from [3] or from Proposition 6 given in [7]. □

Corollary 4.5. *If $\mathfrak{g}(D,\theta)$ is a Frobenius Lie algebra, then $N(\mathfrak{g}(D,\theta)) = 0$, where N denotes the number of functionally independent invariants for the coadjoint representation.*

Proof. See [3]. □

Following the same ideas it is straightforward to prove the following result:

Theorem 4.6. *Let $\mathfrak{g}$ be a finite-dimensional symplectic Lie algebra with symplectic structure given by ω. Given a closed 2-form $\theta \in (\Lambda^2\mathfrak{g})^*$, consider a central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by θ and let $\beta \in (\Lambda^2\mathfrak{g})^*$ be defined by $\beta = \omega + \lambda d_\theta e^*$, $\lambda \in \mathbb{F}$. If there exists a derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ such that $D(e) = v + be$ with $b \neq 0, v \in \mathfrak{g}$, then the double extension $\mathfrak{g}(D,\theta)$ of $\mathfrak{g}$ by the pair (D,θ) is a symplectic Lie algebra with symplectic form β for some $\lambda \in \mathbb{F}$.*

Remark 4.7. Both Theorem 4.2 and Theorem 4.6 above give conditions in order to construct symplectic Lie algebras and Frobenius Lie algebras with trivial center.

5. Contact Lie algebras

Analogously to sections 2 and 3, the goal of this section is to determine whether a double extension of a contact Lie algebra is a contact Lie algebra again. It is important to point out that Proposition 5.1 and Theorem 5.2 below are studied and proved in [13], and we shall present them for the sake of completeness of this work.

Proposition 5.1. *Let $\mathfrak{g}$ be a finite-dimensional contact Lie algebra with a contact structure $\alpha \in \mathfrak{g}^*$, and let $\theta \in (\Lambda^2\mathfrak{g})^*$. Then for almost all $\lambda \in \mathbb{F}$ with the exception of a finite number of values it follows that*

$$\alpha \wedge (d\alpha + \lambda\theta)^n \neq 0.$$

Hence, we can state the main result of this section:

Theorem 5.2. *Let $\mathfrak{g}$ be a finite-dimensional contact Lie algebra with a contact structure $\alpha \in \mathfrak{g}^*$. Given a 2-cocycle $\theta \in (\Lambda^2\mathfrak{g})^*$, consider a central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by θ and let $\beta \in (\mathfrak{g}_\theta(e))^*$ be defined by $\beta = \alpha + \lambda e^*$, $\lambda \in \mathbb{F}$. If there exists a derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ such that $\beta(D(e)) \neq 0$, then the double extension $\mathfrak{g}(D,\theta)$ of $\mathfrak{g}$ by the pair (D,θ) is a contact Lie algebra with contact form β for some $\lambda \in \mathbb{F}$.*

Remark 5.3. From the condition $\beta(D(e)) \neq 0$ it follows that e can be chosen in such a way that $D(e)$ is the Reeb vector of the contact form β.

References

[1] M. A. Alvarez, M. C. Rodríguez-Vallarte, and G. Salgado, *Contact nilpotent Lie algebras*, Proc. Amer. Math. Soc. **145** (2017), no. 4, 1467–1474. MR3601539

[2] Hedi Benamor and Saïd Benayadi, *Double extension of quadratic Lie superalgebras*, Comm. Algebra **27** (1999), no. 1, 67–88. MR1668212

[3] Rutwig Campoamor-Stursberg, *A characterization of exact symplectic Lie algebras $\mathfrak{g}$ in terms of the generalized Casimir invariants*, New developments in mathematical physics research, Nova Sci. Publ., Hauppauge, NY, 2004, pp. 55–84. MR2076274

[4] Claude Chevalley and Samuel Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124. MR0024908

[5] Jean-Michel Dardié and Alberto Médina, *Algèbres de Lie kählériennes et double extension* (French, with English summary), J. Algebra **185** (1996), no. 3, 774–795. MR1419723

[6] G. Favre and L. J. Santharoubane, *Symmetric, invariant, nondegenerate bilinear form on a Lie algebra*, J. Algebra **105** (1987), no. 2, 451–464. MR873679

[7] Michel Goze and Elisabeth Remm, *Contact and Frobeniusian forms on Lie groups*, Differential Geom. Appl. **35** (2014), 74–94. MR3231748

[8] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219

[9] Yu. Khakimdjanov, M. Goze, and A. Medina, *Symplectic or contact structures on Lie groups*, Differential Geom. Appl. **21** (2004), no. 1, 41–54. MR2067457

[10] Alberto Medina and Philippe Revoy, *Groupes de Lie Poisson et double extension* (French, with English summary), Séminaire Gaston Darboux de Géométrie et Topologie Différentielle, 1990–1991 (Montpellier, 1990), Univ. Montpellier II, Montpellier, 1992, pp. iv, 87–105. MR1186222

[11] Alberto Medina and Philippe Revoy, *Algèbres de Lie et produit scalaire invariant* (French), Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 553–561. MR826103

[12] Alberto Medina and Philippe Revoy, *Groupes de Lie à structure symplectique invariante* (French), Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 20, Springer, New York, 1991, pp. 247–266. MR1104932

[13] M. C. Rodríguez-Vallarte and G. Salgado, *5-dimensional indecomposable contact Lie algebras as double extensions*, J. Geom. Phys. **100** (2016), 20–32. MR3435759

[14] M. C. Rodríguez-Vallarte, G. Salgado, and O. A. Sánchez-Valenzuela, *Heisenberg Lie superalgebras and their invariant superorthogonal and supersymplectic forms*, J. Algebra **332** (2011), 71–86. MR2774679

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