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ON THE NUMBER OF DOMINATING FOURIER COEFFICIENTS OF TWO NEWFORMS

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Abstract. Let $f = \sum_{n\geq 1} \lambda_f(n) n^{(k_1-1)/2} q^n$ and $g = \sum_{n\geq 1} \lambda_g(n) n^{(k_2-1)/2} q^n$ be two newforms with real Fourier coeffcients. If f and g do not have complex multiplication and are not related by a character twist, we prove that

$$\#\{n \leq x \mid \lambda_f(n) > \lambda_g(n)\} \gg x.$$

1. Introduction

For an even positive integer k and a positive integer N, denote by $S_k(N)^{\text{new}}$ the set of newforms of weight k, level N, and trivial nebentypus. Every $f \in S_k(N)^{\text{new}}$ has a Fourier expansion at infinity

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{(k-1)/2} q^n \ (q = e^{2\pi i z}),$$

in the upper half-plane $\Im(z) > 0$. Lau and Wu [7] have shown that a positive proportion of the coefficients $\lambda_f(n)$ are positive, and a positive proportion are negative. In this note we consider two newforms without complex multiplication that are not related by a character twist and establish a similar result for the difference $\lambda_f(n) - \lambda_g(n)$. More precisely, our main result is the following.

Theorem 1.1. *Let $k_1, k_2 \geq 2$ be even integers, and let $N_1, N_2 \geq 1$ be integers. Let $f \in S_{k_1}(N_1)^{new}$ and $g \in S_{k_2}(N_2)^{new}$ be two newforms without complex multiplication. Assume that $f \neq g \otimes \chi$, for any Dirichlet character χ. Then there exist two positive constants C and x_0 (both dependening only on f and g) such that for all $x \geq x_0$ the following inequality holds:*

$$\#\{n \leq x \mid \lambda_f(n) > \lambda_g(n)\} \geq Cx.$$

Our approach is based on the method of $\mathcal{B}$-free numbers (as outlined by Lau and Wu in [7]), combined with a result of Kowalski, Robert and Wu [6] about the distribution of the vanishing Fourier coefficients at prime powers. Another important ingredient is a version of a result of Harris [4], in the form used by Murty and Pujahari [9], about the joint Sato–Tate distribution for two newforms (cf. Proposition 2.1), which we need to deduce the existence of two primes with certain properties.

If, instead of considering the set of dominating coefficients over all positive integers, one restricts the analysis just to those indexed by prime numbers, then the joint Sato–Tate distribution readily implies that the corresponding set of primes has

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density 1/2. Without such a powerful tool, the author [2] obtained a lower bound of 1/16 for the analytic density of that set of primes using the holomorphy and the nonvanishing only of the first few symmetric power L-functions (see Section 3 for further discussion).

2. Results

We shall denote by $\mathcal{F}$ the set

$$\mathcal{F} := \{n \mid \lambda_f(n) > \lambda_g(n)\}.$$

A key point in our proof is the existence of two primes, p' and p'', for which

$$\lambda_f(p') > 0 > \lambda_g(p') \text{ and } \lambda_f(p'') < \lambda_g(p'') < 0. \tag{2.1}$$

While we only need a pair (p', p'') as above, we note that there are infinitely many such primes that we can choose from. In fact, as we shall later see in Proposition 2.1 (whose proof we postpone to the end of this section), the above inequalities hold for a positive proportion of primes.

Next, we consider a set $\mathcal{B}$ consisting of the following primes:

$$\mathcal{B} := \{p \mid \lambda_f(p^\nu)\cdot\lambda_g(p^\nu) = 0 \text{ for some } \nu \geq 1\} \cup \{p', p''\} \cup \{p \mid p \text{ divides } N_1N_2\}. \tag{2.2}$$

Let $\mathcal{A}$ be the set of $\mathcal{B}$-free numbers, by which we mean the set of positive integers that are not divisible by any of the elements of $\mathcal{B}$. The multiplicative property of the coefficients imply that, for every $\mathcal{B}$-free number $n = \prod_{i=1}^r p_i^{\nu_i} \in \mathcal{A}$, we have

$$\lambda_f(n) = \prod_{i=1}^{r} \lambda_f(p_i^{\nu_i}) \neq 0,$$

and similarly, $\lambda_g(n) \neq 0$. This means that we can partition the set $\mathcal{A}$ into the following three subsets:

$$\begin{aligned} S &:= \{n \in \mathcal{A} \mid \lambda_f(n) > \lambda_g(n)\} = \mathcal{A} \cap \mathcal{F}, \\ S' &:= \{n \in \mathcal{A} \mid \lambda_f(n) = \lambda_g(n) > 0 \text{ or } \lambda_g(n) > \lambda_f(n) > 0\}, \\ S'' &:= \{n \in \mathcal{A} \mid \lambda_f(n) = \lambda_g(n) < 0 \text{ or } \lambda_f(n) < \lambda_g(n) < 0 \text{ or } \lambda_f(n) < 0 < \lambda_g(n)\}. \end{aligned}$$

Using (2.1) we note that if $n \in S'$, then $p' \cdot n \in \mathcal{F}$, whereas if $n \in S''$, then $p'' \cdot n \in \mathcal{F}$. Therefore,

$$\mathcal{F} \supseteq S \cup p'S' \cup p''S'',$$

which implies that for x large enough

$$\#\{n \leq x \mid n \in \mathcal{F}\} \geq \#\{n \leq x/p'p'' \mid n \in \mathcal{A}\}. \tag{2.3}$$

Refining a classical result of Serre's, Kowalski, Robert, and Wu [6, Lemma 2.3] have proved that

$$\#\{p \leq x \mid \lambda_f(p^\nu) = 0 \text{ for some } \nu \geq 1\} \ll_{f,\varepsilon} \frac{x}{(\log x)^{1+\varepsilon}},$$

for $x \geq 2$ and any $\varepsilon < \frac{1}{2}$. This estimate ensures that the set $\mathcal{B}$ defined in (2.2) is not too big, in the sense that

$$\sum_{b \in \mathcal{B}} \frac{1}{b} < \infty.$$

Hence, the infinite product

$$\prod_{b \in \mathcal{B}} \left(1 - \frac{1}{b}\right)$$

converges to some constant $\delta > 0$. However, this infinite product also represents the proportion of the positive integers that are not divisble by any of the elements of $\mathcal{B}$, so by the definition of $\mathcal{A}$,

$$\#\{n \le x \mid n \in \mathcal{A}\} \sim \delta x \text{ as } x \to \infty. \tag{2.4}$$

In other words, the set $\mathcal{A}$ has positive asymptotic density $\delta > 0$. Then, combining (2.3) and (2.4) we get that

$$\#\{n \le x \mid n \in \mathcal{F}\} \gg x \text{ as } x \to \infty,$$

which proves Theorem 1.1.

The only thing left to show is the existence of the pair (p', p'') from (2.1) that we used in our construction of the set $\mathcal{B}$. Below we give a more general result in this direction.

Proposition 2.1. *Let f and g be two newforms without complex multiplication. Assume that $f \neq g \otimes \chi$, for any Dirichlet character χ. Then:*

(i) *The set $\{p \mid \lambda_f(p) > 0 > \lambda_g(p)\}$ has density $1/4$.*
(ii) *The set $\{p \mid \lambda_f(p) < \lambda_g(p) < 0\}$ has density $1/8$.*
(iii) *The set $\{p \mid \lambda_f(p) > \lambda_g(p)\}$ has density $1/2$.*
(iv) *The set $\{p \mid sign(\lambda_f(p)) = sign(\lambda_g(p))\}$ has density $1/2$.*

Proof. The Deligne bound $|\lambda_f(p)| \le 2$ implies that there is a unique Frobenius angle $\theta_f(p) \in [0, \pi]$ such that

$$\lambda_f(p) = 2\cos\theta_f(p).$$

The Sato–Tate conjecture for non-CM elliptic modular newforms (proved by Barnet-Lamb, Geraghty, Harris, and Taylor [1]) says that the Frobenius angles $\theta_f(p)$ are equidistributed in $[0, \pi]$ with respect to the probability measure

$$\frac{2}{\pi}\sin^2(\theta)d\theta.$$

What we need for our purposes is a natural generalization of this result for the joint distribution of two newforms. This was done by Harris [4, Theorem 5.4];[1] see also [3, Theorem 2.4] for two nonisogenous elliptic curves. In [9, Section 4], Murty and Pujahari have extended the argument for two Hecke eigenforms, provided that they are not twists of each other. As a result, it follows that the pairs

$$(\theta_f(p), \theta_g(p)) \in [0, \pi] \times [0, \pi]$$

are uniformly distributed with respect to the product measure

$$\frac{4}{\pi^2}\sin^2(\theta_1)\sin^2(\theta_2)d\theta_1 d\theta_2.$$

Therefore, part (i) is clear once we observe that

$$\frac{4}{\pi^2}\int_0^{\pi/2}\int_{\pi/2}^{\pi}\sin^2(\theta_1)\sin^2(\theta_2)d\theta_2 d\theta_1 = \frac{1}{4}.$$

The remaining parts are obtained similarly. This finishes the proof of the proposition and of the main theorem. □

[1] The note added in proof on page 2 of [4] clarifies that the Expected Theorems in that paper have now been established.

3. Concluding remarks

As was already mentioned in the introduction, without appealing to the joint Sato–Tate distribution, one can still give a lower bound for the density of the set of primes from part (iii) of Proposition 2.1. This was done by the author in [2], where it is proved that if $f \neq g$, then $\{p \mid \lambda_f(p) > \lambda_g(p)\}$ has analytic density at least $1/16$. Moreover, it is also shown that if f and g do not have complex multiplication, and neither is a quadratic twist of the other, then the same lower bound holds for the set $\{p \mid \lambda_f^2(p) > \lambda_g^2(p)\}$.

Similarly, the set from part (iv) was considered by Kowalski, Lau, Soundararajan, and Wu in [5], where the analysis was also carried out without the use of the joint Sato-Tate. Theorem 5 of [5] states that if $\lambda_f(p)$ and $\lambda_g(p)$ have the same sign for every prime p, except those in a set of analytic density $\leq 1/32$, then $f = g$ (assuming that neither f nor g has complex multiplication). This estimate was further improved to $6/25$ by Matomäki ([8, Theorem 2]), who employed the "common" version (as opposed to the joint one) of the Sato–Tate conjecture for non-CM elliptic modular newforms proved in [1].

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