PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 12, December 2018, Pages 5451–5461
http://dx.doi.org/10.1090/proc/13997
Article electronically published on September 17, 2018

GRADIENT ESTIMATES FOR THE p-LAPLACIAN LICHNEROWICZ EQUATION ON SMOOTH METRIC MEASURE SPACES

LIANG ZHAO AND DENGYUN YANG

(Communicated by Lei Ni)

ABSTRACT. In this paper, we consider the weighted p-Laplacian Lichnerowicz equation

$$\triangle_{p,f} u + cu^{\sigma} = 0$$

on smooth metric measure spaces, where $c \geq 0, p > 1$, and $\sigma \leq p-1$ are real constants. A local gradient estimate for positive solutions to this equation is derived, and as applications, we give a corresponding Liouville property and Harnack inequality.

1. INTRODUCTION

We mainly consider some existence results for positive solutions to p-Laplacian Lichnerowicz type equations on smooth metric measure space. Smooth metric measure space is a triple $(M, g, d\mu)$, where (M, g) is a complete n-dimensional Riemannian manifold and $d\mu := e^{-f}dv$ with f a fixed smooth real-valued function on M. Denote by $\nabla, \triangle$, and $Hess$ the gradient, Laplace, and Hessian operators, and by dv the Riemannian volume measure. The smooth metric measure space carries a natural analog of the Ricci curvature, the so-called $m - Bakry - \acute{E}mery$ Ricci curvature, which is defined as

$$Ric_f^m := Ric + Hessf - \frac{\nabla f \otimes \nabla f}{m-n} \qquad (n < m \leq \infty).$$

In particular, when $m = \infty, Ric_f^{\infty} := Ric_f := Ric + Hessf$ is the classical $Bakry-\acute{E}mery$ Ricci curvature, which was introduced by Bakry-Émery [2] in the study of diffusion processes and has been extensively investigated in the theory of Ricci flow. The case where $m = n$ is only defined when f is a constant function. There is also an analog of the p-Laplacian, that is, the weighted p-Laplacian, which is defined by

$$\triangle_{p,f} := e^f \mathrm{div}(e^{-f}|\nabla u|^{p-2}\nabla u).$$

It is also understood in the distribution sense.

Gradient estimates are an important tool in geometric analysis and have been used, among other things, to derive Liouville theorems and Harnack inequalities for positive solutions to a variety of nonlinear equations on Riemannian manifolds. Kotschwar and Ni [6] established a local gradient estimate for p-harmonic functions

Received by the editors October 28, 2016, and, in revised form, October 29, 2016, and September 29, 2017.

2010 *Mathematics Subject Classification.* Primary 58J05; Secondary 58J35.

Key words and phrases. p-Laplacian, positive solutions, gradient estimate.

The first author is the corresponding author.

under the assumption that the sectional curvature is bounded from below. However, their computation involves the Hessian of the distance function when the cut-off function is introduced, so the condition about the sectional curvature has to be assumed. Recently, Wang and Zhang [15] studied p-harmonic functions and derived a local gradient estimate and Harnack inequality with constants depending only on the lower bound of the Ricci curvature, the dimension of manifolds, and the radius of the ball. For the weighted p-Laplacian equation on metric measure space, some results about gradient estimates and the Liouville property were given in [12] and [13].

For the p-Laplacian Lichnerowicz equation

$$\triangle_{p,f} u + c u^{\sigma} = 0 \tag{1.1}$$

on noncompact smooth metric measure space, here $c \geq 0, p > 1, \sigma \leq p-1$. This equation can be seen as a simple version of the Lichnerowicz equation which arises from the Hamiltonian constraint equation for the Einstein-scalar field system in general relativity (see [3, 4, 11] and the references therein). When $p = 2$, Ma [7–10] studied the existence and stability of positive solutions to the Lichnerowicz equation. Li and Zhu [5] also studied the simple Lichnerowicz equation and derived corresponding gradient estimates. The first author of this paper proved some gradient estimates for this equation, which can be referred to in [17–19]. However, if $p > 1$, the p-Laplacian Lichnerowicz equation is referred to as a generalized scalar curvature type equation; it is an extension of the equation of prescribed scalar curvature. The problem of positive solutions to the p-Laplacian Lichnerowicz equation was considered in [4] in the case of a compact manifold, and then Benalili and Maliki [1] extended the corresponding results to the complete Riemannian manifolds.

In this paper, we will establish the local gradient estimate for positive solutions to equation (1.1).

Theorem 1.1. *Let $(M, g, d\mu)$ be a smooth metric measure space with $Ric_{f,m} \geq -(m-1)K$, where K is a positive constant. Suppose that u is a positive solution to* (1.1) *on the ball $B_o(R) \subset M$ under the condition $\sigma \leq p-1, p > 1$. Then there exists a constant $C_{p,m}$ such that*

$$\frac{|\nabla u|}{u} \leq C_{p,m} \frac{(1+\sqrt{K}R)^{\frac{3}{4}}}{R}.$$

As applications of Theorem 1.1, we can obtain the following two corollaries.

Corollary 1.2. *If the positive solution u is defined globally on manifolds, then we get only constant solutions to equation* (1.1).

Corollary 1.3. *Under the same conditions as in Theorem* 1.1, *given any $x, y \in B_o(\frac{R}{2})$ and any minimal geodesic $\gamma(s) : [0,1] \to B_o(\frac{R}{2})$ with $\gamma(0) = x, \gamma(1) = y$, the following Harnack inequality holds:*

$$u(x) \leq u(y)^{\rho(x,y) C_{p,m} \frac{1+\sqrt{K}R}{R}},$$

where $\rho = \rho(x, y)$ denotes the geodesic distance x and y.

2. Proof of Theorem 1.1

Assume that u is a positive solution to (1.1). The linearized operator of the weighted p-Laplacian at point $u \in C^2(M)$ is given by

$$\mathcal{L}_f(\psi) = e^f \operatorname{div}(e^{-f} |\nabla u|^{p-2} A(\nabla \psi)),$$

where

$$A_{ij} = g_{ij} + (p-2)\frac{\nabla u \otimes \nabla u}{|\nabla u|^2}.$$

We should mention that since equation (2.1) can be either degenerate or singular at the points such that $|\nabla u| = 0$, we usually use an ϵ-regularization technique by replacing the linearized operator $\mathcal{L}_f$ with its approximate operator, i.e.,

$$\mathcal{L}_{f,\epsilon}(\psi) = e^f \operatorname{div}(e^{-f} w_\epsilon^{\frac{p}{2}-1} A_\epsilon(\nabla\psi)),$$

where $\epsilon > 0, w_\epsilon = |\nabla u_\epsilon|^2 + \epsilon, A_\epsilon = g_{ij} + (p-2)\frac{\nabla u_\epsilon \otimes \nabla u_\epsilon}{|\nabla u|^2}$, and u_ϵ is a solution to the approximate equation

$$e^f \operatorname{div}(e^{-f} w_\epsilon^{\frac{p}{2}-1} \nabla u_\epsilon) + c u_\epsilon^\sigma = 0.$$

In order to avoid tedious presentation, we omit the details. The interested reader can refer to [7] for details.

Lemma 2.1 (See [14])**.** *Let $(M, g, d\mu)$ be a smooth metric measure space. Given a C^3 function u, if $|\nabla u| \neq 0$, then*

$$\mathcal{L}_f(|\nabla u|^p) \quad = \quad p|\nabla u|^{2p-4}(|Hess\ u|_A^2 + Ric_f(\nabla u, \nabla u)) + p|\nabla u|^{p-2}\langle \nabla u, \nabla \triangle_{p,f} u\rangle,$$

where $|Hess\ u|_A^2 = A^{ik}A^{jl}u_{ij}u_{kl}$.

Set $v = (p-1)\log u$. From Lemma 2.1, we can obtain

$$\mathcal{L}_f(|\nabla v|^p) = p|\nabla v|^{2p-4}(|Hess\ v|_A^2 + Ric_f(\nabla v, \nabla v)) + |\nabla v|^{p-2}\langle \nabla v, \nabla \triangle_{p,f} v\rangle.$$

Moreover, in terms of v, equation (1.1) has the following form:

$$\begin{aligned} \triangle_{p,f} u + c u^\sigma &= e^f \operatorname{div}(e^{-f}|\nabla e^{\frac{v}{p-1}}|^{p-2}\nabla e^{\frac{v}{p-1}}) + c e^{\frac{v\sigma}{p-1}} \\ &= (p-1)^{1-p} e^v (|\nabla v|^p + \triangle_{p,f} v) + c e^{\frac{v\sigma}{p-1}} \\ &= 0. \end{aligned}$$

That is to say,

$$\triangle_{p,f} v = -c(p-1)^{p-1} e^{(\frac{\sigma}{p-1}-1)v} - |\nabla v|^p. \tag{2.1}$$

Let $w = |\nabla v|^2$. Note that in terms of w,

$$\begin{aligned} \triangle_{p,f} w &= e^f \operatorname{div}(e^{-f} w^{\frac{p-2}{2}} \nabla v) \\ &= w^{\frac{p-2}{2}} \triangle_f v + \frac{p-2}{2}\langle \nabla w, \nabla v\rangle. \end{aligned}$$

Therefore, equation (2.1) has the equivalent form

$$w^{\frac{p-2}{2}} \triangle_f v + \frac{p-2}{2}\langle \nabla w, \nabla v\rangle = -c(p-1)^{p-1} e^{(\frac{\sigma}{p-1}-1)v} - w^{\frac{p}{2}}. \tag{2.2}$$

Assume that $Q = |\nabla v|^p$. Then

$$\mathcal{L}_f(Q) = p w^{p-2}[|Hess\ v|_A^2 + Ric_f(\nabla v, \nabla v)] - p w^{\frac{p}{2}-1}\langle \nabla v, \nabla Q\rangle - cph(\frac{\sigma}{p-1} - 1)Q,$$

where $h = (p-1)^{p-1} e^{(\frac{\sigma}{p-1}-1)v}$.

Now we are in position to estimate $|\nabla\nabla v|_A^2$. We only need to estimate it over the points where $w > 0$. Choose a local orthonormal frame $\{e_i\}_{i=1}^n$ near any such given point so that $\nabla v = |\nabla v| e_1$. Then $w = v_1^2, w_1 = 2v_{i1}v_i = 2v_{11}v_1$, and for $j \geq 2, w_j = 2v_{j1}v_1$. Hence $2v_{j1} = \frac{w_j}{w^{\frac{1}{2}}}$.

From (2.2), we immediately deduce that

$$\begin{aligned}\sum_{j=2}^n v_{jj} &= -chw^{1-\frac{p}{2}} - (\frac{p}{2}-1)\frac{w_1 v_1}{w} - v_{11} + f_1 v_1 - w\\ &= -chw^{1-\frac{p}{2}} - (p-1)v_{11} + f_1 v_1 - w.\end{aligned}$$

It is easy to see that

$$\begin{aligned}|Hess\ v|_A^2 &= |Hess\ v|^2 + \frac{(p-2)^2}{4w^2}\langle\nabla v, \nabla w\rangle^2 + \frac{p-2}{2w}|\nabla w|^2\\ &= \sum_{i,j=1}^n v_{ij}^2 + (p-2)v_{11}^2 + 2(p-2)\sum_{j=1}^n v_{ij}^2\\ &= (p-1)^2 v_{11}^2 + 2(p-1)\sum_{j=2}^n v_{1j}^2 + \sum_{i,j=2}^n v_{ij}^2\\ &\geq (p-1)^2 v_{11}^2 + 2(p-1)\sum_{j=2}^n v_{1j}^2 + \frac{1}{n-1}\big(\sum_{j=2}^n v_{jj}\big)^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|Hess\ v|_A^2 &\geq (p-1)^2 v_{11}^2 + 2(p-1)\sum_{j=2}^n v_{1j}^2\\ &\quad+\frac{1}{n-1}\big(-chw^{1-\frac{p}{2}} - (p-1)v_{11} + f_1 v_1 - w\big)^2\\ &\geq \alpha\sum_{j=1}^n v_{j1}^2 + \frac{1}{m-1}(chw^{1-\frac{p}{2}} + w)^2\\ &\quad+\frac{2(p-1)v_{11}}{m-1}(chw^{1-\frac{p}{2}} + w) - \frac{(f_1 v_1)^2}{m-n},\end{aligned}$$

where $\alpha = \min\{2(p-1), \frac{m(p-1)^2}{m-1}\}$, and we applied the inequality $(a-b)^2 \geq \frac{a^2}{1+\delta} - \frac{b^2}{\delta}$ with $\delta = \frac{m-n}{n-1} > 0$. Substituting the identities,

$$2wv_{11} = \langle\nabla v, \nabla w\rangle, \sum_{j=1}^n v_{j1}^2 = \frac{1}{4}\frac{|\nabla w|^2}{w},$$

we can obtain

$$\begin{aligned}|Hess\ v|_A^2 &\geq \frac{\alpha}{4}\frac{|\nabla w|^2}{w} + \frac{w^2}{m-1}(chw^{-\frac{p}{2}} + 1)^2\\ &\quad+\frac{p-1}{m-1}(1 + chw^{\frac{-p}{2}})\langle\nabla v, \nabla w\rangle - \frac{(f_1 v_1)^2}{m-n}.\end{aligned}$$

By the assumption that $Ric_f^m \geq -(m-1)K$, we have

$$\begin{aligned}
\mathcal{L}_f(Q) &= \mathcal{L}_f(w^{\frac{p}{2}}) \\
&\geq pw^{p-2}[\frac{\alpha}{4}\frac{|\nabla w|^2}{w} + \frac{1}{m-1}(w + ch(chw^{1-\frac{p}{2}}))^2 \\
&\quad + \frac{p-1}{m-1}(1 + chw^{\frac{-p}{2}})\langle \nabla v, \nabla w\rangle] \\
&\quad - pw^{p-2} Ric_f^m(\nabla v, \nabla v) - pw^{\frac{p}{2}-1}\langle \nabla v, \nabla w^{\frac{p}{2}}\rangle - cph(\frac{\sigma}{p-1} - 1)w^{\frac{p}{2}} \\
&= \frac{\alpha p}{4} w^{p-3}|\nabla w|^2 + \frac{p}{m-1} w^p (1 + chw^{\frac{-p}{2}})^2 \\
&\quad + \frac{p(p-1)}{m-1} w^{p-2}(1 + chw^{\frac{-p}{2}})\langle \nabla v, \nabla w\rangle \\
&\quad - p(m-1)Kw^{p-1} - pw^{\frac{p}{2}-1}\langle \nabla v, \nabla w^{\frac{p}{2}}\rangle - cph(\frac{\sigma}{p-1} - 1)w^{\frac{p}{2}} \\
&\geq \frac{\alpha p}{4} w^{p-3}|\nabla w|^2 + \frac{p}{m-1} w^p (1 + chw^{\frac{-p}{2}})^2 \\
&\quad + \frac{p(p-1)}{m-1} w^{p-2}(1 + chw^{\frac{-p}{2}})\langle \nabla v, \nabla w\rangle \\
&\quad - p(m-1)Kw^{p-1} - \frac{p^2}{2} w^{p-2}\langle \nabla v, \nabla w\rangle \\
&= \frac{\alpha p}{4} w^{p-3}|\nabla w|^2 + \frac{p}{m-1} w^p (1 + chw^{\frac{-p}{2}})^2 \\
&\quad + [\frac{p(p-1)}{m-1}(1 + chw^{\frac{-p}{2}}) - \frac{p^2}{2}] w^{p-2}\langle \nabla v, \nabla w\rangle - p(m-1)Kw^{p-1}.
\end{aligned}$$

The above inequality holds wherever w is strictly positive. Let $K = \{x \in \Omega : w(x) = 0\}$; here $\Omega \subset M$ is an open set. Then for any nonnegative function ψ with compact support in $\Omega \setminus K$, we have

$$\begin{aligned}
&- \int_\Omega \langle \frac{1}{2} w^{p-2}\nabla w + \frac{1}{2}(p-2)w^{p-3}\langle \nabla v, \nabla w\rangle \nabla v, \nabla \psi\rangle \\
&\geq \int_\Omega \Big(\frac{\alpha p}{4} w^{p-3}|\nabla w|^2 + \frac{p}{m-1} w^p(1 + chw^{\frac{-p}{2}})^2 \\
&\quad + [\frac{p(p-1)}{m-1}(1 + chw^{\frac{-p}{2}}) - \frac{p^2}{2}] w^{p-2}\langle \nabla v, \nabla w\rangle - p(m-1)Kw^{p-1}\Big)\psi.
\end{aligned}$$

In particular, let $\epsilon > 0$ and $\psi = w_\epsilon^b \eta^2$, where $w_\epsilon = (w-\epsilon)^+, \eta \in C_0^\infty(B_R)$ is nonnegative; $b > 1$ is to be determined later. Then direct computation yields

$$\nabla \psi = bw_\epsilon^{b-1}\eta^2 \nabla w + 2w_\epsilon^b \eta \nabla \eta.$$

We have

$$\begin{aligned}
&- \int_{B_o(R)} \langle \frac{1}{2} w^{p-2}\nabla w + \frac{1}{2}(p-2)w^{p-3}\langle \nabla v, \nabla w\rangle \nabla v, bw_\epsilon^{b-1}\eta^2\nabla w + 2w_\epsilon^b \eta \nabla \eta\rangle \\
&\geq \int_{B_o(R)} \Big(\frac{\alpha p}{4} w^{p-3}|\nabla w|^2 + \frac{p}{m-1} w^p(1 + chw^{\frac{-p}{2}})^2 \\
&\quad + [\frac{p(p-1)}{m-1}(1 + chw^{\frac{-p}{2}}) - \frac{p^2}{2}] w^{p-2}\langle \nabla v, \nabla w\rangle - p(m-1)Kw^{p-1}\Big) w_\epsilon^b \eta^2.
\end{aligned} \tag{2.3}$$

There are two terms on the left hand side of (2.3) we need to estimate. Under the condition $1 < p \le 2$, we can estimate them as follows:

$$\begin{aligned}\int_{B_o(R)} w^{\frac{p-1}{2}} w_\epsilon^{b-1}|\nabla w|^2 + (p-2)\int_{B_o(R)} w^{\frac{p-2}{2}}\langle \nabla v, \nabla w\rangle w_\epsilon^{b-1} \\ \ge (p-1)\int_{B_o(R)} w^{\frac{p-1}{2}} w_\epsilon^{b-1}|\nabla w|^2,\end{aligned}$$

since $v \in C^{1,\alpha}, w \in C^\alpha$, and $\nabla w \in L^\gamma(\Omega)$ for some $\alpha > 0$ and $\gamma > 1$. Thus, all the other terms in (2.3) converge to the corresponding form without ϵ. By passing ϵ to 0 and letting $\beta = 1 + chw^{\frac{-p}{2}}$, we have

$$\begin{aligned}- & \int_{B_o(R)} \frac{b(p-1)}{2} w^{p+b-3}\eta^2|\nabla w|^2 - \int_{B_o(R)} w^{p+b-2}\eta\langle \nabla w, \nabla \eta\rangle \\ & -\int_{B_o(R)} (p-2)w^{p+b-3}\eta\langle \nabla w, \nabla\eta\rangle\langle\nabla w, \nabla v\rangle \\ \ge & \int_{B_o(R)} \frac{\alpha p}{4} w^{p+b-3}|\nabla w|^2\eta^2 + \int_{B_o(R)} \frac{\beta^2}{m-1} w^{p+b}\eta^2 \\ & + \int_{B_o(R)} (\frac{p-1}{m-1}\beta - \frac{p}{2}) w^{p+b-2}\eta^2\langle\nabla w, \nabla v\rangle - \int_{B_o(R)} (m-1)w^{p+b-1}K\eta^2.\end{aligned}$$

Since

$$\begin{aligned}- & \int_{B_o(R)} w^{p+b-2}\eta\langle\nabla w, \nabla\eta\rangle - \int_{B_o(R)} (p-2)w^{p+b-3}\eta\langle\nabla w, \nabla\eta\rangle\langle\nabla w, \nabla v\rangle \\ \le & \ (1+|p-2|)\int_{B_o(R)} w^{p+b-2}|\nabla w||\nabla\eta|\eta,\end{aligned}$$

with these inequalities we can get

$$\begin{aligned}(1+|p-2|)&\int_{B_o(R)} w^{p+b-2}|\nabla w||\nabla\eta||\eta| + \int_{B_o(R)} \frac{p}{2} w^{p+b-2}\eta^2\langle\nabla w, \nabla v\rangle \\ \ge & \int_{B_o(R)} [\frac{\alpha}{4} + \frac{b(p-1)}{2}] w^{p+b-3}\eta^2|\nabla w|^2 + \int_{B_o(R)} \frac{\beta^2}{m-1} w^{p+b}\eta^2 \\ & + \int_{B_o(R)} (\frac{p-1}{m-1}\beta) w^{p+b-2}\eta^2\langle\nabla w, \nabla v\rangle - \int_{B_o(R)} (m-1)w^{p+b-1}K\eta^2.\end{aligned}$$

From now on we use $a_1, a_1, \cdots$ and $d_1, d_1, \cdots$ to denote constants depending only on p and m. The constant $b > 1$ is to be determined later.

It is easy to see that

$$\begin{aligned}(1+|p-2|)&\int_{B_o(R)} w^{p+b-2}|\nabla w||\nabla\eta||\eta| \\ & \le \frac{(p-1)b}{6}\int_{B_o(R)} w^{p+b-3}|\nabla w|^2\eta^2 + \int_{B_o(R)} \frac{a_1}{b}\int_{B_o(R)} w^{p+b-1}|\nabla\eta|^2.\end{aligned}$$

We also have

$$\begin{aligned}\int_{B_o(R)} \frac{p}{2} w^{p+b-2}\eta^2\langle\nabla w, \nabla v\rangle & \le \frac{p}{2}\int_{B_o(R)} w^{p+b-\frac{3}{2}}\eta^2|\nabla w| \\ & \le \frac{(p-1)b}{6}\int_{B_o(R)} w^{p+b-3}|\nabla w|^2\eta^2 + \frac{a_2}{b}\int_{B_o(R)} w^{p+b}\eta^2,\end{aligned}$$

and the following inequality holds:

$$\begin{aligned}\int_{B_o(R)} (\frac{p-1}{m-1}\beta) w^{p+b-2}\eta^2\langle\nabla w, \nabla v\rangle \\ \geq -\int_{B_o(R)} (\frac{p-1}{m-1}\beta) w^{p+b-\frac{3}{2}}\eta^2|\nabla w| \\ \geq -\frac{(p-1)b}{6}\int_{B_o(R)} w^{p+b-3}|\nabla w|^2\eta^2 - \frac{a_3}{b}\int_{B_o(R)}\beta^2 w^{p+b}\eta^2.\end{aligned}$$

Combining these inequalities, we derive

$$\begin{aligned}&-\int_{B_o(R)}\frac{\alpha}{4}w^{p+b-3}\eta^2|\nabla w|^2 + \frac{a_2}{b}\int_{B_o(R)} w^{p+b}\eta^2 + (\frac{a_3}{b} - \frac{1}{m-1})\int_{B_o(R)}\beta^2 w^{p+b}\eta^2 \\ &+\int_{B_o(R)}\frac{a_1}{b}\int_{B_o(R)} w^{p+b-1}|\nabla\nabla\eta|^2 \\ &\quad \geq -\int_{B_o(R)}(m-1)w^{p+b-1}K\eta^2\end{aligned}$$

by requiring that

$$\frac{a_3}{b} - \frac{1}{m-1} \leq 0. \tag{2.4}$$

Since $\beta = 1 + chw^{\frac{-p}{2}} \geq 1$, we get

$$(\frac{a_3}{b} - \frac{1}{m-1})\int_{B_o(R)}\beta^2 w^{p+b}\eta^2 \leq (\frac{a_3}{b} - \frac{1}{m-1})\int_{B_o(R)} w^{p+b}\eta^2,$$

which yields

$$\begin{aligned}&\int_{B_o(R)}\frac{\alpha}{4}w^{p+b-3}\eta^2|\nabla w|^2 + (\frac{1}{m-1} - \frac{a_2}{b} - \frac{a_3}{b})\int_{B_o(R)} w^{p+b}\eta^2 \\ &\quad + \int_{B_o(R)}\frac{a_1}{b}\int_{B_o(R)} w^{p+b-1}|\nabla\nabla\eta|^2 \leq \int_{B_o(R)}(m-1)w^{p+b-1}K\eta^2.\end{aligned} \tag{2.5}$$

For the first term on the LHS of (2.5), we use

$$|\nabla(w^{\frac{p+b-1}{2}}\eta)|^2 \leq \frac{(b+p-1)^2}{2}w^{b+p-3}|\nabla w|^2\eta^2 + 2w^{b+p-1}|\nabla\eta|^2.$$

Substituting it into the above inequality, we have

$$\begin{aligned}\int_{B_o(R)}|\nabla(w^{\frac{p+b-1}{2}}\eta)|^2 + bd_1\int_{B_o(R)} w^{p+b}\eta^2 \\ \leq a_4\int_{B_o(R)} w^{p+b-1}|\nabla\eta|^2 + Kbd_2\int_{B_o(R)}(m-1)w^{p+b-1}\eta^2.\end{aligned}$$

In order to prove the main theorem, we need the following two lemmas.

Lemma 2.2. *Let $(M, g, d\mu)$ be an n-dimensional complete noncompact smooth metric measure space. If $Ric_f^m \geq -(m-1)K$ for some nonnegative constant K and $m > n \geq 2$, then there exists a constant C, depending on m, such that for all $B_o(R) \subset M$ we have for $\phi \in C_0^\infty(B_o(R))$,*

$$\int_{B_o(R)}(|\phi|^{\frac{2m}{m-2}})^{\frac{m-2}{m}} \leq e^{C(1+\sqrt{K}R)}V^{-\frac{2}{m}}R^2\int_{B_o(R)}(|\nabla\phi|^2 + R^{-2}\phi^2)d\mu,$$

where V means weighted volume of $B_o(R)$.

Remark 2.3. The proof is analogous to that of Lemma 2.5 in [16].

Lemma 2.4. *For $b_0 > 0$ large enough and $R > 0$, there exists $d_3 = d_3(m,p) > 0$ such that*

$$\|w\|_{L^{(b_0+p-1)\frac{m}{m-2}}(B_o(\frac{3}{4}R))} \leq d_3 b_0^{\frac{b_0+p}{b_0+p-1}} V^{\frac{m-2}{m(b_0+p-1)}} R^{-2}.$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
&\Big(\int_{B_o(R)} w^{(p+b-1)\frac{m}{m-2}}\eta^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}} \\
&\quad\leq e^{C(1+\sqrt{K}R)}V^{-\frac{2}{m}}\big(R^2\int_{B_o(R)}(|\nabla(w^{\frac{b+p-1}{2}}\eta)|^2+\int_{B_o(R)} w^{b+p-1}\eta^2\big) \\
&\quad\leq e^{c_2 b_0}V^{-\frac{2}{m}}\big(a_4\int_{B_o(R)} w^{p+b-1}|\nabla\eta|^2 + Kbd_2\int_{B_o(R)}(m-1)w^{p+b-1}\eta^2 \\
&\qquad - bd_1\int_{B_o(R)} w^{p+b}\eta^2 + \int_{B_o(R)} w^{b+p-1}\eta^2\big),
\end{aligned}$$

where $b_0 = c_1(m,p)(1+\sqrt{K}R)$ with c_1 large enough to make b_0 satisfy (2.4). Then we have

$$\begin{aligned}
&\Big(\int_{B_o(R)} w^{(p+b-1)\frac{m}{m-2}}\eta^{\frac{2m}{m-2}}\Big)^{\frac{m-2}{m}} + bd_1 e^{c_2 b_0}V^{-\frac{2}{m}}\int_{B_o(R)} w^{p+b}\eta^2 \\
&\quad\leq a_4 R^2 e^{c_2 b_0}V^{-\frac{2}{m}}\int_{B_o(R)} w^{p+b-1}|\nabla\eta|^2 \\
&\qquad + Kbd_2R^2e^{c_2b_0}V^{-\frac{2}{m}}\int_{B_o(R)}(m-1)w^{p+b-1}\eta^2 + e^{c_2b_0}V^{-\frac{2}{m}}\int_{B_o(R)} w^{b+p-1}\eta^2 \\
&\quad\leq a_4R^2e^{c_2b_0}V^{-\frac{2}{m}}\int_{B_o(R)} w^{p+b-1}|\nabla\eta|^2 + a_5b_0^2be^{c_2b_0}V^{-\frac{2}{m}}\int_{B_o(R)} w^{b+p-1}\eta^2.
\end{aligned} \tag{2.6}$$

We note that $a_5b_0^2b < \frac{1}{2}bd_1R^2w^{p+b}$ when $w > a_6b_0^2R^2$. Thus in the evaluation of the second term on the right hand side of inequality (2.6), we decompose ω into subregions: one over the places $w > a_6b_0^2R^2$ and the second region being the complement of the first region. With this decomposition we have

$$\begin{aligned}
&a_5b_0^2be^{c_2b_0}V^{-\frac{2}{m}}\int_{B_o(R)} w^{b+p-1}\eta^2 \\
&\quad\leq \frac{1}{2}bd_1e^{c_2b_0}V^{-\frac{2}{m}}\int_{B_o(R)} w^{p+b}\eta^2 + a_5b_0^2be^{c_2b_0}V^{1-\frac{2}{m}}\Big(\frac{a_6b_0^2}{R^2}\Big)^{b+p-1} \\
&\quad\leq \frac{1}{2}bd_1e^{c_2b_0}V^{-\frac{2}{m}}\int_{B_o(R)} w^{p+b}\eta^2 + a_7^{b_0}b_0^3V^{1-\frac{2}{m}}\Big(\frac{b_0}{R}\Big)^{2b+2p-2}.
\end{aligned}$$

Let $\eta_1 \in C_0^\infty(\Omega)$ satisfy $0 \le \eta_1 \le 1, \eta_1 \equiv 1$ in $B_o(\frac{3}{4}R), |\nabla\eta_1| \le \frac{\widetilde{C}}{R}$, and let $\eta = \eta_1^{p+b}$. The first term on the right hand side of (2.6) satisfies

$$\begin{aligned} a_4R^2e^{c_2b_0}\int_{B_o(R)} w^{p+b-1}|\nabla\eta|^2 &\le a_8b^2\int_{B_o(R)} w^{p+b-1}\eta^{\frac{2(b+p-1)}{b+p}} \\ &\le a_8b^2(\int_{B_o(R)} w^{p+b}\eta^2)^{\frac{2(b+p-1)}{b+p}}V^{\frac{1}{b+p}} \\ &\le \frac{1}{2}bd_1R^2\int_{B_o(R)} w^{p+b}\eta^2 + b^{b+p}d_1^{p+b-1}R^{-2(p+b-1)}V. \end{aligned}$$

Hence

$$(\int_{B_o(R)} w^{(p+b-1)\frac{m}{m-2}}\eta^{\frac{2m}{m-2}})^{\frac{m-2}{m}} \le d_2b_0^{b_0+p}d_1^{b+p-1}R^{-2(p+b-1)}.$$

Assuming that $b_0 = b$ and $b_1 = (p+b_0-1)\frac{m}{m-2}$, we can get

$$\|w\|_{L^{(b_0+p-1)\frac{m}{m-2}}(B_o(\frac{3}{4}R))} \le d_3b_0^{\frac{b_0+p}{b_0+p-1}}V^{\frac{m-2}{m(b_0+p-1)}}R^{-2}.$$

□

Now we begin to prove the main theorem. From (2.6), by ignoring the second item on the left hand side, we have

$$(\int_{B_o(R)} w^{(p+b-1)\frac{m}{m-2}}\eta^{\frac{2m}{m-2}})^{\frac{m-2}{m}} \quad \le \quad a_9\frac{e^{c_2}b_0}{V^{\frac{2}{m}}}\int_{B_o(R)}(b_0^2b\eta^2 + R^2|\nabla\eta|^2)w^{p+b-1}.$$

Set

$$b_{l+1} = b_l\frac{m-2}{m}, \qquad b_l = b+p-1, \qquad \omega_l = B(o, \frac{R}{2}+\frac{R}{4l}), \qquad l = 1, 2\cdots$$

and choose $\eta_l \equiv 1$ in $\omega_{l+1}, \eta_l \equiv 0$ in $B_o(R)\setminus\Omega_l, |\nabla\eta_l| \le \frac{C4^l}{R}$.

We have

$$(\int_{B_o(R)} w^{b_{l+1}})^{\frac{1}{b_{l+1}}} \le (a_9\frac{e^{c_2}b_0}{V^{\frac{2}{m}}})^{\frac{1}{b_l}}(\int_{B_o(R)}(b_0^2b\eta^2 + R^2|\nabla\eta|^2)w^{p+b-1})^{\frac{1}{b_l}},$$

which yields

$$\|w\|_{L^{b_{l+1}}(\Omega_{l+1})} \le (a_{10}\frac{e^{c_2}b_0}{V^{\frac{2}{m}}})^{\frac{1}{b_l}}\big(b_0^2b_l + 16^l\big)^{\frac{1}{b_l}}\|w\|_{L^{b_l}(\Omega_l)}.$$

Noting that $\sum_{l=1}^{l=\infty}\frac{1}{b_l} = \frac{m}{2b_1}, \sum_{i=1}^{\infty}\frac{i}{b_i} = \frac{m^2}{4b_1}$, we obtain

$$\|w\|_{L^\infty(B_o(\frac{R}{2}))} \le (a_{10}\frac{e^{c_2}b_0}{V^{\frac{2}{m}}})^{\frac{M}{2b_l}}\big(17\big)^{\frac{m}{4b_l}}\|w\|_{L^{(b_0+p-1)\frac{m}{m-2}}(B_o(\frac{3R}{4}))}.$$

By Lemma 2.4, we have

$$\|w\|_{L^\infty(B_o(\frac{R}{2}))} \le a_{11}b_0^{\frac{(b_0+p)}{b_0+p-1}}\frac{1}{R^2}.$$

We can choose b_0 large enough to make $\frac{(b_0+p)}{b_0+p-1} \le \frac{3}{2}$, which yields

$$\|w\|_{L^\infty(B_o(\frac{R}{2}))} \le a_{11}\big(c_1(m,p)(1+\sqrt{K}R)\big)^{\frac{3}{2}}\frac{1}{R^2}.$$

We have completed the proof of Theorem 1.1. For the proof of Corollary 1.3, we leave it to the reader.

Acknowledgments

The first author was partially supported by the National Natural Science Foundation of China (11701277) and the Natural Science Foundation of Jiangsu Province (BK20140804, BK20130779). The second author's research was supported by the NSFC, grant No. 11761049. This work was done while the first author was visiting the University of California, San Diego (UCSD) in 2016, and he would like to express his thanks to Professor Lei Ni for his hospitality.

References

[1] Mohammed Benalili and Youssef Maliki, *Solving p-Laplacian equations on complete manifolds*, Electron. J. Differential Equations (2006), No. 155, 9 pp. MR2276580

[2] D. Bakry and Michel Émery, *Diffusions hypercontractives* (French), Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206. MR889476

[3] Yvonne Choquet-Bruhat, James Isenberg, and Daniel Pollack, *The Einstein-scalar field constraints on asymptotically Euclidean manifolds*, Chinese Ann. Math. Ser. B **27** (2006), no. 1, 31–52. MR2209950

[4] Olivier Druet, *Generalized scalar curvature type equations on compact Riemannian manifolds*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), no. 4, 767–788. MR1776675

[5] Yi Li and Xiaorui Zhu, *Harnack estimates for a heat-type equation under the Ricci flow*, J. Differential Equations **260** (2016), no. 4, 3270–3301. MR3434399

[6] Brett Kotschwar and Lei Ni, *Local gradient estimates of p-harmonic functions, $1/H$-flow, and an entropy formula* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 1, 1–36. MR2518892

[7] Li Ma and Xingwang Xu, *Uniform bound and a non-existence result for Lichnerowicz equation in the whole n-space* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **347** (2009), no. 13-14, 805–808. MR2543987

[8] Li Ma, *Liouville type theorem and uniform bound for the Lichnerowicz equation and the Ginzburg-Landau equation* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **348** (2010), no. 17-18, 993–996. MR2721787

[9] Li Ma and Juncheng Wei, *Stability and multiple solutions to Einstein-scalar field Lichnerowicz equation on manifolds*, J. Math. Pures Appl. (9) **99** (2013), no. 2, 174–186. MR3007843

[10] Li Ma, Yuhua Sun, and Xiao Tang, *Heat flow method for Lichnerowicz type equations on closed manifolds*, Z. Angew. Math. Phys. **63** (2012), no. 2, 261–270. MR2912348

[11] Xianfa Song and Lin Zhao, *Gradient estimates for the elliptic and parabolic Lichnerowicz equations on compact manifolds*, Z. Angew. Math. Phys. **61** (2010), no. 4, 655–662. MR2673328

[12] Lin-feng Wang and Yue-ping Zhu, *A sharp gradient estimate for the weighted p-Laplacian*, Appl. Math. J. Chinese Univ. Ser. B **27** (2012), no. 4, 462–474. MR3001092

[13] Lin Feng Wang, Ze Yu Zhang, Liang Zhao, and Yu Jie Zhou, *A Liouville theorem for weighted p-Laplace operator on smooth metric measure spaces*, Math. Methods Appl. Sci. **40** (2017), no. 4, 992–1002. MR3610712

[14] Yu-Zhao Wang and Huai-Qian Li, *Lower bound estimates for the first eigenvalue of the weighted p-Laplacian on smooth metric measure spaces*, Differential Geom. Appl. **45** (2016), 23–42. MR3457386

[15] Xiaodong Wang and Lei Zhang, *Local gradient estimate for p-harmonic functions on Riemannian manifolds*, Comm. Anal. Geom. **19** (2011), no. 4, 759–771. MR2880214

[16] Jia-Yong Wu and Peng Wu, *Heat kernel on smooth metric measure spaces with nonnegative curvature*, Math. Ann. **362** (2015), no. 3-4, 717–742. MR3368080

[17] Liang Zhao, *Harnack inequality for parabolic Lichnerowicz equations on complete noncompact Riemannian manifolds*, Bound. Value Probl. (2013), 2013:190, 10 pp. MR3117294

[18] Liang Zhao, *Liouville theorem for Lichnerowicz equation on complete noncompact manifolds*, Funkcial. Ekvac. **57** (2014), no. 1, 163–172. MR3221843

[19] Liang Zhao, *Gradient estimates for a simple parabolic Lichnerowicz equation*, Osaka J. Math. **51** (2014), no. 1, 245–256. MR3192542

Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, People's Republic of China
Email address: zhaozongliang09@163.com

College of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330022, People's Republic of China
Email address: yangdengyun@126.com