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LOCALLY CONFORMALLY FLAT MANIFOLDS WITH CONSTANT SCALAR CURVATURE

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ABSTRACT. Let (M^n, g) be an n-dimensional $(n \geq 4)$ compact locally conformally flat Riemannian manifold with constant scalar curvature and constant squared norm of Ricci curvature. Applying the moving frame method, we prove that such a Riemannian manifold does not exist if its Ricci curvature tensor has three distinct eigenvalues.

1. INTRODUCTION

Let (M^n, g) be a compact locally conformally flat Riemannian manifold. Let R denote the scalar curvature and let S denote the squared norm of the Ricci curvature tensor of M^n.

Yamabe, Trudinger, Aubin, and Schoen (see [1], [12], [14], and [17]) proved the following: any compact Riemannian manifold can be deformed into a Riemannian manifold with constant scalar curvature by a conformal transformation. In [7], S. I. Goldberg established that every complete conformally flat Riemannian manifold M^n with positive constant scalar curvature R is a space form if the squared norm S of the Ricci curvature tensor of M^n satisfies the inequality

$$\sup S < \frac{R^2}{n-1}.$$

We should remark that the condition of R being positive is essential in the proof of Goldberg's Theorem. In [13], Schoen and Yau investigated the global properties of the locally conformally flat Riemannian manifolds by using the Yamabe equation.

There are many results about compact locally conformally flat Riemannian manifolds with constant scalar curvature in the literature. The following theorem is well known.

Theorem A (cf. Tani [15], Wegner [16], and Cheng [5]). *Let (M^n, g) be a compact locally conformally flat Riemannian manifolds with constant scalar curvature. If the Ricci tensor is semi-positive definite, then (M^n, g) is either a space form or the Riemannian product $M_1^{n-1}(c) \times N^1$, $c \geq 0$, where M_1^{n-1} is a space form.*

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The following problem is interesting:

Problem 1.1. Whether there exists an n-dimensional ($n \geq 3$) compact locally conformally flat Riemannian manifold M^n with the following conditions:

(1) constant scalar curvature,
(2) constant squared norm of the Ricci curvature tensor,
(3) the Ricci curvature tensor has three distinct eigenvalues everywhere.

In [3], Q.-M. Cheng, S. Ishikawa, and K. Shiohama gave an affirmative answer for Problem 1.1 when $n = 3$ by proving the following theorem:

Theorem B ([3])**.** *There exists no* 3*-dimensional compact locally conformally flat Riemannian manifold if scalar curvature* R *and squared norm* S *of the Ricci curvature tensor are constants and the Ricci curvature tensor has three distinct eigenvalues.*

Remark 1.2. In fact, in [3], Cheng-Ishikawa-Shiohama also proved that if M^3 is a 3-dimensional complete locally conformally flat Riemannian manifold with constant nonnegative scalar curvature and constant squared norm of the Ricci curvature tensor, then M^3 is either isometric to a space form or the Riemannian product $M_1^2(c) \times N^1,\ c \geq 0$. And in [4], Cheng-Ishikawa-Shiohama proved that there exists no complete locally conformally flat 3-dimensional Riemannian manifolds with negative constant scalar curvature R and Ricci curvature tensor with constant squared norm S satisfying $\frac{R^2}{3} + (\frac{7}{113})^2 \cdot \frac{2R^2}{3} < S < \frac{R^2}{3} + (\frac{35}{137})^2 \cdot \frac{R^2}{6}$.

In this paper, we give an affirmative answer for Problem 1.1 when $n \geq 4$ by proving the following Main Theorem:

Main Theorem. *There exists no n-dimensional* ($n \geq 4$) *compact locally conformally flat Riemannian manifold if scalar curvature* R *and squared norm* S *of the Ricci curvature tensor are constants and the Ricci curvature tensor has three distinct eigenvalues.*

Remark 1.3. We note that Z.Q. Li [11] and S. Chang [2] proved independently that a compact hypersurface M^n ($n \geq 4$) with constant mean curvature H and constant scalar curvature R in $\mathbb{S}^{n+1}$ is isoparametric if the hypersurface has three distinct principal curvatures everywhere.

The organization of the paper is as follows. In section 2, we collects some facts about locally conformally flat Riemannian manifolds with constant scalar curvature. In section 3, we will prove two key propositions by studying the structure of compact locally conformally flat manifolds with constant scalar curvature and Ricci curvature tensor with three distinct eigenvalues. In section 4, we give the proof of the Main Theorem.

2. Preliminaries

Let (M^n, g) ($n \geq 4$) be an n-dimensional Riemannian manifold. We choose a local orthonormal frame $e_1, \ldots, e_n$ adapted to the Riemannian metric of (M^n, g), with $\omega_1, \ldots, \omega_n$ as dual coframe. Then the structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} = -\omega_{ji}, \tag{2.1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.2}$$

where ω_{ij} is the Levi-Civita connection form and R_{ijkl} are the components of the Riemannian curvature tensor of (M^n, g). Let W_{ijkl} denote the components of the Weyl curvature tensor of (M^n, g). We have

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}), \end{aligned}$$

where $R_{ij} := \sum_{kl} R_{ikjl}g_{kl}$ are components of the Ricci curvature tensor and $R := \sum_{ij} R_{ij}g_{ij}$ is the scalar curvature of M^n. The Schouten tensor A_{ij} is a symmetric (0,2)-tensor, defined by

$$A_{ij} = \frac{1}{n-2}(R_{ij} - \frac{R}{2(n-1)}\delta_{ij}). \tag{2.3}$$

Then, we can express the Riemannian curvature tensor of a locally conformally flat Riemannian manifold by

$$R_{ijkl} = A_{ik}\delta_{jl} - A_{il}\delta_{jk} + A_{jl}\delta_{ik} - A_{jk}\delta_{il}. \tag{2.4}$$

Denote by ∇ the covariant derivation on (M^n, g) and write, e.g., $R_{ij,k} = \nabla_k R_{ij}$, $R_{ij,kl} = \nabla_l \nabla_k R_{ij}$, $A_{ij,k} = \nabla_k A_{ij}$, $A_{ij,kl} = \nabla_l \nabla_k A_{ij}$, and so on.

Since M^n is locally conformally flat and $n \geq 4$, it is well-known that A_{ij} is a Codazzi tensor, that is (see [8]),

$$A_{ij,k} = A_{ik,j}. \tag{2.5}$$

For an arbitrary $(0,2)$-tensor T we have the following Ricci identities:

$$T_{ij,kl} - T_{ij,lk} = \sum_m T_{mj}R_{mikl} + \sum_m T_{im}R_{mjkl}. \tag{2.6}$$

If M^n $(n \geq 4)$ is a locally conformally flat Riemannian manifold with constant scalar curvature R and constant squared norm S of the Ricci curvature tensor, by use of (2.4) and (2.6) we have the following well-known calculation (see [3], [8]):

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k} R_{ij,k}^2 + \sum_{i,j} R_{ij}\Delta R_{ij} \\ &= \sum_{i,j,k} R_{ij,k}^2 + \sum_{i,j,k} R_{ij}R_{ik,jk} \\ &= \sum_{i,j,k} R_{ij,k}^2 + \sum_{i,j,k,m} R_{ij}(R_{ik,kj} + R_{mk}R_{mijk} + R_{im}R_{mkjk}) \\ &= \sum_{i,j,k} R_{ij,k}^2 + \sum_{i,j,k,m} R_{ij}R_{mk}R_{mijk} + \sum_{i,j,m} R_{ij}R_{im}R_{mj} \\ &= \sum_{i,j,k} R_{ij,k}^2 + \frac{1}{n-2}(n\sum_{i,j,k} R_{ij}R_{jk}R_{ki} - \frac{2n-1}{n-1}RS + \frac{R^3}{n-1}). \end{aligned} \tag{2.7}$$

Combining (2.7) with (2.3), we have the following proposition, which will be used in the proof of Proposition 3.1.

Proposition 2.1. *Let M^n $(n \geq 4)$ be an n-dimensional locally conformally flat Riemannian manifold with constant scalar curvature R and constant squared norm*

S of the Ricci curvature tensor. We have the following formula:

$$\sum_{i,j,k} A_{ij,k}^2 = -\frac{1}{(n-2)^3}\Big(n\sum_{i,j,k} R_{ij}R_{jk}R_{ki} - \frac{2n-1}{n-1}RS + \frac{R^3}{n-1}\Big). \tag{2.8}$$

3. Two key propositions and their proofs

In this section, we will prove two key propositions by studying the structure of compact locally conformally flat manifolds with constant scalar curvature and Ricci curvature tensor with three distinct eigenvalues.

Proposition 3.1. *Let M^n $(n \geq 4)$ be a compact locally conformally flat Riemannian manifold with constant scalar curvature R and constant squared norm S of the Ricci curvature tensor. If the Ricci curvature tensor of M^n has three distinct eigenvalues everywhere, then all eigenvalue functions are constants on M^n.*

In order to prove Proposition 3.1, we first prove the following lemma.

Lemma 3.2. *Let M^n $(n \geq 4)$ be a compact locally conformally flat Riemannian manifold with constant scalar curvature R and constant squared norm S of the Ricci curvature tensor. For each point $x \in M^n$, let $\lambda(x), \mu(x)$, and $\nu(x)$ be the three distinct eigenvalues of A_{ij} of multiplicities $m_1(x), m_2(x)$, and $m_3(x)$, respectively. Then m_1, m_2, and m_3 are constants.*

Proof. From (2.3), we have the following system of linear equations of m_1, m_2, m_3:

$$m_1 + m_2 + m_3 = n, \tag{3.1}$$

$$m_1\lambda + m_2\mu + m_3\nu = \frac{R}{2(n-1)}, \tag{3.2}$$

$$m_1\lambda^2 + m_2\mu^2 + m_3\nu^2 = \frac{1}{(n-2)^2}\Big(S + \frac{4-3n}{4(n-1)^2}R^2\Big). \tag{3.3}$$

Since m_1, m_2, and m_3 are all integers, from the fact that λ, μ, and ν are distinct continuous functions on M^n and R, S are constants, we have that m_1, m_2, and m_3 are constants. □

We shall make use of the following convention on the ranges of indices:

$$1 \leq i, j, k, \ldots \leq n; \qquad 1 \leq a, b, c, \ldots \leq m_1;$$
$$m_1 + 1 \leq \alpha, \beta, \gamma, \ldots \leq m_1 + m_2; \qquad m_1 + m_2 + 1 \leq r, s, t, \ldots \leq m_1 + m_2 + m_3.$$

Then A_{ij} can be expressed as follows:

$$(A_{ij})_{n\times n} = \begin{pmatrix} \lambda(\delta_{ab})_{m_1\times m_1} & & \\ & \mu(\delta_{\alpha\beta})_{m_2\times m_2} & \\ & & \nu(\delta_{rs})_{m_3\times m_3} \end{pmatrix}_{n\times n.}$$

By a direct calculation, we have the following covariant derivation of A_{ij} (see [6]):

$$\begin{aligned} \sum_k A_{ab,k}\omega_k &= dA_{ab} + \sum_k A_{kb}\omega_{ka} + \sum_k A_{ak}\omega_{kb} \\ &= d(\lambda\delta_{ab}) + A_{bb}\omega_{ba} + A_{aa}\omega_{ab} \\ &= \delta_{ab}d\lambda \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\sum_k A_{a\beta,k}\omega_k &= dA_{a\beta} + \sum_k A_{k\beta}\omega_{ka} + \sum_k A_{ak}\omega_{k\beta} \\ &= A_{\beta\beta}\omega_{\beta a} + A_{aa}\omega_{a\beta} \\ &= (\lambda - \mu)\omega_{a\beta}.\end{aligned} \tag{3.5}$$

Similarly, we get

$$\sum_k A_{\alpha\beta,k}\omega_k = \delta_{\alpha\beta}d\mu, \tag{3.6}$$

$$\sum_k A_{rs,k}\omega_k = \delta_{rs}d\nu, \tag{3.7}$$

$$\sum_k A_{ar,k}\omega_k = (\lambda - \nu)\omega_{ar}, \tag{3.8}$$

$$\sum_k A_{\alpha r,k}\omega_k = (\mu - \nu)\omega_{\alpha r}. \tag{3.9}$$

Define

$$f = \sum_{i,j,k} R_{ij}R_{jk}R_{ki}. \tag{3.10}$$

Then by using (2.3), we have

$$f = (n-2)^3(m_1\lambda^3 + m_2\mu^3 + m_3\nu^3) + \frac{3RS}{2(n-1)} - \frac{(5n-6)R^3}{8(n-1)^3}. \tag{3.11}$$

Differentiating (3.2), (3.3), and (3.11), and using $R =$ constant, $S =$ constant, we have

$$m_1 d\lambda + m_2 d\mu + m_3 d\nu = 0, \tag{3.12}$$

$$m_1\lambda d\lambda + m_2\mu d\mu + m_3\nu d\nu = 0, \tag{3.13}$$

$$m_1\lambda^2 d\lambda + m_2\mu^2 d\mu + m_3\nu^2 d\nu = \frac{1}{3(n-2)^3}df. \tag{3.14}$$

It follows that

$$\frac{m_1 d\lambda}{\nu - \mu} = \frac{m_2 d\mu}{\lambda - \nu} = \frac{m_3 d\nu}{\mu - \lambda} = \frac{df}{3(n-2)^3 D}, \tag{3.15}$$

where $D = (\nu - \mu)(\nu - \lambda)(\mu - \lambda)$.

To finish our proof, we now present three lemmas in different cases of m_1, m_2, and m_3.

First of all, we consider the case of $m_1 \geq 2$, $m_2 \geq 2$, and $m_3 \geq 2$.

Lemma 3.3. *With the same notation as above, if $m_1 \geq 2$, $m_2 \geq 2$, and $m_3 \geq 2$, then all eigenvalue functions of the Ricci curvature tensor are constants on M^n.*

Proof. Since $m_1 \geq 2$ and A is a codazzi tensor, for an arbitrary d, we can choose $c \neq d$ such that

$$A_{cc,d} = A_{cd,c}. \tag{3.16}$$

On the one hand, from (3.4), let $a = b = c$, and let e_d act on both sides of the equation. Then we have

$$A_{cc,d} = \lambda_{,d}. \tag{3.17}$$

On the other hand, let $a = c,\ b = d$ in (3.4). Since $c \neq d$, $\delta_{cd} = 0$, we get

$$A_{cd,c} = 0. \tag{3.18}$$

Combining (3.16), (3.17), and (3.18), we have $\lambda_{,d} = 0, \forall d$. For the same reason, we obtain $\mu_{,\beta} = 0$ and $\nu_{,r} = 0, \forall \beta, r$.

From (3.15) we have

$$d\mu = \frac{m_1(\lambda - \nu)}{m_2(\mu - \nu)} d\lambda, \quad d\nu = \frac{m_1(\lambda - \mu)}{m_3(\mu - \nu)} d\lambda.$$

Since $\lambda_{,d} = 0$ and $\mu \neq \nu$, we have $\lambda_{,d} = \mu_{,d} = \nu_{,d} = 0, \forall d$.

Similarly, we have $\lambda_{,\beta} = \mu_{,\beta} = \nu_{,\beta} = 0, \forall \beta$, and $\lambda_{,r} = \mu_{,r} = \nu_{,r} = 0, \forall r$. That is, λ, μ, and ν are constants. Hence we complete the proof of Lemma 3.3. □

Next, we consider the case of $m_1 = m_3 = 1$ and $m_2 \geq 2$.

Lemma 3.4. *With the same notation as above, if* $m_1 = m_3 = 1$ *and* $m_2 \geq 2$, *then* $A_{1\alpha,n} = 0, \forall \alpha$.

Proof. Since $m_2 \geq 2$, by making use of a method similar to the proof of Lemma 3.3, we have $\mu_{,\alpha} = 0, \forall \alpha$.

Since $m_1 = m_3 = 1$, from (3.15) we have $d\lambda = m_2 \frac{\mu - \nu}{\nu - \lambda} d\mu$ and $d\nu = m_2 \frac{\mu - \lambda}{\lambda - \nu} d\mu$. Then we get

$$\lambda_{,\alpha} = \mu_{,\alpha} = \nu_{,\alpha} = 0, \quad \forall \alpha. \tag{3.19}$$

From (3.4), (3.6), and (3.7), we have

$$A_{11,k} = \lambda_{,k}, \quad A_{\alpha\alpha,k} = \mu_{,k}, \quad A_{nn,k} = \nu_{,k}, \quad \forall \alpha, k, \tag{3.20}$$

$$A_{\alpha\beta,k} = 0, \quad \forall \alpha \neq \beta. \tag{3.21}$$

From (3.5), (3.8), and (3.9) and by use of (3.19), (3.20), and (3.21) we get

$$\begin{aligned} \omega_{1\alpha} &= \frac{1}{\lambda - \mu} \sum_k A_{1\alpha,k} \omega_k \\ &= \frac{1}{\lambda - \mu} [A_{1\alpha,1} \omega_1 + \sum_{\beta \neq \alpha} A_{1\alpha,\beta} \omega_\beta + A_{1\alpha,\alpha} \omega_\alpha + A_{1\alpha,n} \omega_n] \\ &= \frac{1}{\lambda - \mu} (A_{1\alpha,n} \omega_n + \mu_{,1} \omega_\alpha), \end{aligned} \tag{3.22}$$

$$\begin{aligned} \omega_{1n} &= \frac{1}{\lambda - \nu} \sum_k A_{1n,k} \omega_k \\ &= \frac{1}{\lambda - \nu} [A_{1n,1} \omega_1 + \sum_\beta A_{1n,\beta} \omega_\beta + A_{1n,n} \omega_n] \\ &= \frac{1}{\lambda - \nu} [\lambda_{,n} \omega_1 + \sum_\beta A_{1n,\beta} \omega_\beta + \nu_{,1} \omega_n], \end{aligned} \tag{3.23}$$

and

$$\omega_{\alpha n} = \frac{1}{\mu - \nu} (A_{1\alpha,n} \omega_1 + \mu_{,n} \omega_\alpha). \tag{3.24}$$

Differentiating both sides of (3.24), on the one hand, we have

$$\begin{aligned}
d\omega_{\alpha n} &= \omega_{\alpha 1}\wedge\omega_{1n}+\sum_\beta \omega_{\alpha\beta}\wedge\omega_{\beta n}+\omega_{\alpha n}\wedge\omega_{nn}-\frac{1}{2}\sum_{k,l}R_{\alpha nkl}\omega_k\wedge\omega_l\\
&= \omega_{\alpha 1}\wedge\omega_{1n}+\sum_\beta \omega_{\alpha\beta}\wedge\omega_{\beta n}\\
&\quad -\frac{1}{2}\sum_{k,l}(A_{\alpha k}\delta_{nl}-A_{\alpha l}\delta_{nk}+A_{nl}\delta_{\alpha k}-A_{nk}\delta_{\alpha l})\omega_k\wedge\omega_l\\
&= \omega_{\alpha 1}\wedge\omega_{1n}+\sum_\beta \omega_{\alpha\beta}\wedge\omega_{\beta n}-(\mu+\nu)\omega_\alpha\wedge\omega_n\\
&= -\frac{1}{\lambda-\mu}(A_{1\alpha,n}\omega_n+\mu_{,1}\omega_\alpha)\wedge\frac{1}{\lambda-\nu}(\lambda_{,n}\omega_1+\sum_\beta A_{1n,\beta}\omega_\beta+\nu_{,1}\omega_n)\\
&\quad +\sum_\beta\omega_{\alpha\beta}\wedge[\frac{1}{\mu-\nu}(A_{1\beta,n}\omega_1+\mu_{,n}\omega_\beta)]-(\mu+\nu)\omega_\alpha\wedge\omega_n.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
&d[\frac{1}{\mu-\nu}(A_{1\alpha,n}\omega_1+\mu_{,n}\omega_\alpha)]\\
=\ &(d\frac{1}{\mu-\nu})\wedge(A_{1\alpha,n}\omega_1+\mu_{,n}\omega_\alpha)\\
&+\frac{1}{\mu-\nu}(dA_{1\alpha,n}\wedge\omega_1+A_{1\alpha,n}d\omega_1+d\mu_{,n}\wedge\omega_\alpha+\mu_{,n}d\omega_\alpha)\\
=\ &(d\frac{1}{\mu-\nu})\wedge(A_{1\alpha,n}\omega_1+\mu_{,n}\omega_\alpha)\\
&+\frac{1}{\mu-\nu}[dA_{1\alpha,n}\wedge\omega_1+A_{1\alpha,n}(\sum_\beta\omega_{1\beta}\wedge\omega_\beta+\omega_{1n}\wedge\omega_n)\\
&+d\mu_{,n}\wedge\omega_\alpha+\mu_{,n}(\omega_{\alpha 1}\wedge\omega_1+\sum_\beta\omega_{\alpha\beta}\wedge\omega_\beta+\omega_{\alpha n}\wedge\omega_n)].
\end{aligned}$$

Then we can get the equation of terms with type of $\omega_\beta\wedge\omega_n$:

$$\begin{aligned}
&-\frac{1}{(\lambda-\mu)(\lambda-\nu)}(\mu_{,1}\omega_\alpha\wedge\nu_{,1}\omega_n+\sum_\beta A_{1\alpha,n}\omega_n\wedge A_{1\beta,n}\omega_\beta)-(\mu+\nu)\omega_\alpha\wedge\omega_n\\
=\ &-\frac{\mu_{,n}-\nu_{,n}}{(\mu-\nu)^2}\omega_n\wedge\mu_{,n}\omega_\alpha+\frac{1}{\mu-\nu}[A_{1\alpha,n}\sum_\beta(\frac{A_{1\beta,n}}{\lambda-\mu}\omega_n\wedge\omega_\beta+\frac{A_{1\beta,n}}{\lambda-\nu}\omega_\beta\wedge\omega_n)\\
&+(\mu_{,nn}\omega_n\wedge\omega_\alpha+\frac{\mu_{,n}}{\mu-\nu}\omega_\alpha\wedge\omega_n)].
\end{aligned}$$

That is, $\forall\alpha,\beta$,

$$\begin{aligned}
&-\frac{\mu_{,1}\nu_{,1}}{(\lambda-\mu)(\lambda-\nu)}\delta_{\alpha\beta}+\frac{A_{1\alpha,n}A_{1\beta,n}}{(\lambda-\mu)(\lambda-\nu)}-(\mu+\nu)\delta_{\alpha\beta}\\
=\ &\frac{\mu_{,n}(\mu_{,n}-\nu_{,n})}{(\mu-\nu)^2}\delta_{\alpha\beta}-\frac{A_{1\alpha,n}A_{1\beta,n}}{(\lambda-\mu)(\lambda-\nu)}-\frac{\mu_{,nn}}{\mu-\nu}\delta_{\alpha\beta}+\frac{\mu_{,n}}{(\mu-\nu)^2}\delta_{\alpha\beta}.
\end{aligned}$$

Hence

$$2A_{1\alpha,n}A_{1\beta,n}=E\delta_{\alpha\beta}, \tag{3.25}$$

where E is a smooth function on M^n defined as

$$\begin{aligned} E &= (\mu+\nu)(\lambda-\mu)(\lambda-\nu)+\mu_{,1}\nu_{,1}+\frac{\mu_{,n}(\mu_{,n}-\nu_{,n})}{(\mu-\nu)^2}(\lambda-\mu)(\lambda-\nu) \\ &\quad -\frac{\mu_{,nn}}{\mu-\nu}(\lambda-\mu)(\lambda-\nu)+\frac{\mu_{,n}}{(\mu-\nu)^2}(\lambda-\mu)(\lambda-\nu). \end{aligned}$$

From (3.25) we have $\forall\alpha,\beta$,

$$A_{1\alpha,n}A_{1\beta,n}=0,\ \alpha\neq\beta, \tag{3.26}$$

$$A^2_{1\alpha,n}=A^2_{1\beta,n}=\frac{E}{2}. \tag{3.27}$$

For any fixed α, choose $\beta\neq\alpha$, from (3.26), if $A_{1\alpha,n}=0$, we finish our proof; if $A_{1\alpha,n}\neq 0$, from (3.26), we have $A_{1\beta,n}=0$, then from (3.27) we get a contradiction. This completes the proof of Lemma 3.4. □

Now, we consider the case of $m_3=1$ and $m_1\geq 2,\ m_2\geq 2$.

Lemma 3.5. *With the same notation as above, if $m_3=1$ and $m_1\geq 2,\ m_2\geq 2$, then $A_{b\alpha,n}=0,\forall b,\alpha$.*

Proof. Similarly to Lemma 3.4, since $m_1\geq 2, m_2\geq 2$, we have

$$\begin{aligned} &\lambda_{,\alpha}=\mu_{,\alpha}=\nu_{,\alpha}=0, \quad \forall\alpha, \\ &\lambda_{,b}=\mu_{,b}=\nu_{,b}=0, \quad \forall b. \end{aligned}$$

And from (3.4)-(3.9), we get

$$\omega_{b\alpha} = \frac{1}{\lambda-\mu}A_{b\alpha,n}\omega_n, \tag{3.28}$$

$$\omega_{bn} = \frac{1}{\lambda-\nu}(\lambda_{,n}\omega_b+\sum_\beta A_{b\beta,n}\omega_\beta), \tag{3.29}$$

$$\omega_{\alpha n} = \frac{1}{\mu-\nu}(\sum_c A_{c\alpha,n}\omega_c+\mu_{,n}\omega_\alpha). \tag{3.30}$$

By differentiating (3.28), on the one hand, we have

$$\begin{aligned} d\omega_{b\alpha} &= \sum_c\omega_{bc}\wedge\omega_{c\alpha}+\sum_\beta\omega_{b\beta}\wedge\omega_{\beta\alpha}+\omega_{bn}\wedge\omega_{n\alpha}-(\lambda+\mu)\omega_b\wedge\omega_\alpha \\ &= \sum_c\omega_{bc}\wedge\frac{1}{\lambda-\mu}A_{c\alpha,n}\omega_n+\sum_\beta\frac{1}{\lambda-\mu}A_{b\beta,n}\omega_n\wedge\omega_{\beta\alpha} \\ &\quad -\frac{1}{\lambda-\nu}(\lambda_{,n}\omega_b+\sum_\beta A_{b\beta,n}\omega_\beta)\wedge\frac{1}{\mu-\nu}(\mu_{,n}\omega_\alpha+\sum_c A_{c\alpha,n}\omega_c) \\ &\quad -(\lambda+\mu)\omega_b\wedge\omega_\alpha. \end{aligned}$$

On the other hand, we get

$$
\begin{aligned}
&d[\frac{1}{\lambda-\mu}A_{b\alpha,n}\omega_n] \\
&=(d\frac{1}{\lambda-\mu})\wedge A_{b\alpha,n}\omega_n+\frac{1}{\lambda-\mu}[dA_{b\alpha,n}\wedge\omega_n \\
&\qquad\qquad\qquad\qquad +A_{b\alpha,n}(\sum_c\omega_{nc}\wedge\omega_c+\sum_\beta\omega_{n\beta}\wedge\omega_\beta)] \\
&=(d\frac{1}{\lambda-\mu})\wedge A_{b\alpha,n}\omega_n+\frac{1}{\lambda-\mu}dA_{b\alpha,n}\wedge\omega_n \\
&+\frac{A_{b\alpha,n}}{\lambda-\mu}(-\frac{1}{\lambda-\nu}\sum_{c,\beta}A_{c\beta,n}\omega_\beta\wedge\omega_c-\frac{1}{\mu-\nu}\sum_{c,\beta}A_{c\beta,n}\omega_c\wedge\omega_\beta) \\
&=(d\frac{1}{\lambda-\mu})\wedge A_{b\alpha,n}\omega_n+\frac{1}{\lambda-\mu}dA_{b\alpha,n}\wedge\omega_n-\sum_{c,\beta}\frac{A_{b\alpha,n}A_{c\beta,n}}{(\lambda-\nu)(\mu-\nu)}\omega_c\wedge\omega_\beta.
\end{aligned}
$$

Then we can get the equation of terms with type $\omega_c\wedge\omega_\beta$:

$$
\begin{aligned}
&-\frac{1}{(\lambda-\nu)(\mu-\nu)}(\lambda_{,n}\omega_b\wedge\mu_{,n}\omega_\alpha+\sum_{c,\beta}A_{b\beta,n}\omega_\beta\wedge A_{c\alpha,n}\omega_c)-(\lambda+\mu)\omega_b\wedge\omega_\alpha \\
&=-\sum_{c,\beta}\frac{A_{b\alpha,n}A_{c\beta,n}}{(\lambda-\nu)(\mu-\nu)}\omega_c\wedge\omega_\beta.
\end{aligned}
$$

That is, $\forall\alpha,\beta,b,c,$

$$
\begin{aligned}
&-\frac{\lambda_{,n}\mu_{,n}}{(\lambda-\nu)(\mu-\nu)}\delta_{bc}\delta_{\alpha\beta}+\frac{A_{c\alpha,n}A_{b\beta,n}}{(\lambda-\nu)(\mu-\nu)}-(\lambda+\mu)\delta_{bc}\delta_{\alpha\beta} \\
&=-\frac{A_{b\alpha,n}A_{c\beta,n}}{(\lambda-\nu)(\mu-\nu)}.
\end{aligned}
$$

Then

$$A_{b\alpha,n}A_{c\beta,n}+A_{c\alpha,n}A_{b\beta,n}=\bar{E}\delta_{bc}\delta_{\alpha\beta},\forall\ b,c,\alpha,\beta, \tag{3.31}$$

where $\bar{E}=\lambda_{,n}\mu_{,n}+(\lambda+\mu)(\lambda-\nu)(\mu-\nu)$. From (3.31), letting $c=b$, we have $\forall b,\alpha,\beta,$

$$A_{b\alpha,n}A_{b\beta,n}=0,\ \ \alpha\neq\beta, \tag{3.32}$$

$$A^2_{b\alpha,n}=A^2_{b\beta,n}=\frac{\bar{E}}{2}. \tag{3.33}$$

For any fixed b,α, choose $\beta\neq\alpha$. From (3.32), if $A_{b\alpha,n}=0$, we finish our proof; if $A_{b\alpha,n}\neq 0$, from (3.32) we have $A_{b\beta,n}=0$. Then from (3.33) we get a contradiction. This completes the proof of Lemma 3.5. □

Proof of Proposition 3.1. Let M^n $(n\geq 4)$ be a compact locally conformally flat Riemannian manifold with constant scalar curvature R and constant squared norm S of the Ricci curvature tensor.

In the case when $m_1,m_2,m_3\geq 2$, we can immediately finish our proof from Lemma 3.3.

Because of $n\geq 4$, we now suppose otherwise that $m_3=1$ and $m_1\cdot m_2\geq 2$.

Consider a point $p_0 \in M^n$ such that $f(p_0) = \inf f$, where f is defined by (3.10). Then $df(p_0) = 0$. We have

$$d\lambda = \frac{df}{3m_1(n-2)^3(\nu-\lambda)(\mu-\lambda)},$$
$$d\mu = \frac{df}{3m_2(n-2)^3(\nu-\mu)(\lambda-\mu)},$$
$$d\nu = \frac{df}{3m_3(n-2)^3(\mu-\nu)(\lambda-\nu)}.$$

Since λ, μ, and ν are distinct at p_0, we get

$$d\lambda(p_0) = d\mu(p_0) = d\nu(p_0) = 0.$$

Then from (3.4), (3.6), and (3.7), we get

$$A_{ab,k}(p_0) = A_{\alpha\beta,k}(p_0) = A_{rs,k}(p_0) = 0, \forall a, b, \alpha, \beta, r, s, k.$$

From Lemmas 3.4 and 3.5, we know that $A_{a\beta,n} = 0$, and then by using (2.8) and (3.10), we have

$$0 = \sum_{i,j,k} A^2_{ij,k}(p_0) = -\frac{1}{(n-2)^3}(nf - \frac{2n-1}{n-1}RS + \frac{R^3}{n-1})(p_0).$$

It follows that $\inf f = \frac{1}{n}(\frac{2n-1}{n-1}RS - \frac{R^3}{n-1})$ is constant. Similarly, we can have $\sup f = \frac{1}{n}(\frac{2n-1}{n-1}RS - \frac{R^3}{n-1}) = \inf f$, which together prove that f is a constant. This in turn would yield that λ, μ, and ν are constants. This completes the proof of Proposition 3.1. □

To complete the proof of our Main Theorem, now we only need the following proposition for locally conformally flat Riemannian manifolds with constant eigenvalues of the Schouten tensor.

Proposition 3.6. *Let M^n be a locally conformally flat Riemannian manifold. If all eigenvalue functions of the Schouten tensor are constants with constant multiplicities on M^n, then M^n must be one of the following three cases:*

(1) *The Schouten tensor has only one eigenvalue. M^n is isometric to $M^n(c)$.*
(2) *The Schouten tensor has two eigenvalues, and the simplicity of one of the two eigenvalues is one. M^n is isometric to the Riemannian product $M_1^{n-1}(c_0) \times N^1$.*
(3) *The Schouten tensor has two eigenvalues, and the simplicities of both eigenvalues are at least two. M^n is isometric to the Riemannian product $M_1^k(c_1) \times M_2^{n-k}(c_2)$, where $2 \leq k \leq n-2$ and $c_1 + c_2 = 0$.*

Proof. Suppose there are g distinct eigenvalues of the Schouten tensor on M^n. We make the following convention on the range of indices:

$$1 \leq i,\ j,\ k,\ \ldots \leq n,$$
$$1 \leq \zeta,\ \eta,\ \theta,\ \ldots \leq g.$$

Let $A_{ij} = a_i\delta_{ij}$, $a_i = constant$. From the theorem due to H. Li (see the theorem on

p. 47 in [10]), we have the following generalized Cartan identities for any fixed i:

$$\sum_{j\notin[i]} \frac{R_{ijij}}{a_i - a_j} = 0, \tag{3.34}$$

where $[i] = \{j|\ a_i = a_j\}$ and R_{ijij} is the sectional curvature of the plane spanned by e_i and e_j on M^n.

By a direct calculation, from (2.4), for $i \neq j$ we have

$$R_{ijij} \quad = \quad a_i + a_j.$$

Let m_ζ denote the multiplicity of a_ζ. Then for any fixed ζ we get

$$0 = \sum_{j\notin[\zeta]} \frac{R_{\zeta j\zeta j}}{a_\zeta - a_j} = \sum_{\theta\neq\zeta} m_\theta \frac{a_\zeta + a_\theta}{a_\zeta - a_\theta} = \sum_{\theta\neq\zeta} m_\theta \frac{a_\zeta^2 - a_\theta^2}{(a_\zeta - a_\theta)^2}. \tag{3.35}$$

In (3.35), choose ζ such that $a_\zeta^2 = \max_\theta\{a_\theta^2\}$. Then for $\forall\theta \neq \zeta$,

$$a_\zeta^2 = a_\theta^2,$$

that is, $a_\zeta = a_\theta$ *or* $a_\zeta = -a_\theta$. Then there exist at most two distinct eigenvalues of A_{ij}. This completes the proof of Proposition 3.6. □

Remark 3.7. When $n \leq 8$, Proposition 3.6 was proved in [9] by the use of different methods.

4. Proof of Main Theorem

From Proposition 3.6, we can easily get the result that if M^n $(n \geq 4)$ is a compact locally conformally flat Riemannian manifold and eigenvalues of its Ricci curvature tensor are constants, then there exist at most two distinct eigenvalues of the Ricci curvature tensor.

Combining with Proposition 3.1, it follows that there exists no compact locally conformally flat Riemannian manifold M^n $(n \geq 4)$ with constant scalar curvature R and constant squared norm S of the Ricci curvature tensor such that the Ricci curvature tensor on M^n has three distinct eigenvalues everywhere. Therefore, we complete the proof of the Main Theorem.

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