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AVERAGING ONE-POINT HYPERBOLIC-TYPE METRICS

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ABSTRACT. It is known that the $\tilde{\jmath}$-metric, the half-Apollonian metric, and the scale-invariant Cassinian metric are not Gromov hyperbolic. These metrics are defined as a supremum of *one-point* metrics (i.e., metrics constructed using one boundary point), and the supremum is taken over all boundary points. The aim of this paper is to show that taking the average instead of the supremum yields a metric that is Gromov hyperbolic. Moreover, we show that the Gromov hyperbolicity constant of the resulting metric does not depend on the number of boundary points used in taking the average. We also provide an example to show that the average of Gromov hyperbolic metrics is not, in general, Gromov hyperbolic.

1. INTRODUCTION

The hyperbolic metric is a powerful tool in planar complex analysis and geometric function theory (see [2] and the references therein). In higher-dimensional Euclidean spaces, the hyperbolic metric exists only in balls and half-spaces, and the lack of a hyperbolic metric in general domains has been a primary motivation for introducing the so-called *hyperbolic-type* metrics in geometric function theory. Examples of such metrics include the $\tilde{\jmath}$-metric, the Apollonian metric, Seittenranta's metric, the half-Apollonian metric, the scale-invariant Cassinian metric, and the Möbius-invariant Cassinian metric (see [1, 11–13, 17, 19, 20, 22–24] and the references therein). All these metrics are so-called *point-distance metrics*, meaning that they are defined in terms of distance functions and can be classified into *one-point* metrics or *two-point* metrics based on the number of boundary points used in their definitions. For example, the Apollonian, Seittenranta, and the Möbius-invariant Cassinian metrics are two-point, point-distance metrics. Their corresponding one-point versions, namely, the half-Apollonian metric, the $\tilde{\jmath}$-metric, and the scale-invariant Cassinian metric, are one-point point-distance metrics. In this paper we only consider hyperbolic-type point-distance metrics. There are other hyperbolic-type metrics termed as *hyperbolic-type length metrics* such as the quasihyperbolic metric, Ferrand's metric, and the Kulkarni–Pinkall metric that have been extensively studied by many authors (see [7, 9, 14, 15, 21]).

One of the key features of the hyperbolic-type metrics is their Gromov hyperbolicity. The latter was introduced by Gromov in 1987 as an extension of the concept

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of negative curvature to general metric spaces [10]. This notion has found applications in many areas of mathematics and is widely used in geometric function theory, geometric group theory, and analysis on metric spaces. For more discussion of Gromov hyperbolic spaces the reader is referred to [2–4, 6, 10, 25].

The Apollonian, Seittenranta, and Möbius-invariant Cassinian metrics are roughly similar to each other and, in particular, they are all Gromov hyperbolic (see [20, Theorem 4.8 and Theorem 5.4]). The $\tilde{\jmath}$-metric, the half-Apollonian, and the scale-invariant Cassinian metrics are also roughly similar to each other [19, Theorem 3.3 and Theorem 3.5]. However, they are Gromov hyperbolic if and only if the underlying domain has only one boundary point. In other words, if the domain has more than one boundary point, then these metrics, which are defined as the supremums over all boundary points, are not Gromov hyperbolic.

Recall the following general approach to constructing one-point hyperbolic-type metrics in the setting of Euclidean spaces. Let $D \subset \mathbb{R}^n$ be any domain with nonempty boundary ∂D. To construct a one-point hyperbolic-type metric d_D on D, one first constructs a Gromov hyperbolic metric d_p on the one-punctured space $\mathbb{R}^n \setminus \{p\}$ for each $p \in \mathbb{R}^n$ and then defines d_D by $d_D(x,y) = \sup\{d_p(x,y)\colon\ p \in \partial D\}$. Taking a supremum in this context is very natural since the boundary ∂D is usually uncountable. However, as it turns out, the Gromov hyperbolicity property of d_p is not preserved when taking the supremum.

In this paper we propose an alternative approach to constructing a metric from the one-point metrics mentioned above. Namely, we propose to take the average of these one-point metrics instead of taking their supremum. As mentioned above, these metrics are roughly similar to each other and hence so are their averages. Therefore, here we consider only the one-point scale-invariant Cassinian metrics. The main result of this paper states that the average of finitely many one-point scale-invariant Cassinian metrics is Gromov hyperbolic and, more importantly, its Gromov hyperbolicity constant does not depend on the number of metrics (Lemma 4.1 and Theorem 4.2). Even though here we consider the averages of finitely many metrics, the fact that the Gromov hyperbolicity constant is independent of the number of metrics makes it possible to consider domains which are the complements of certain self-similar sets [16].

To the best of our knowledge, averaging one-point metrics has not been considered before. However, germs of this idea can be traced back to the work of F. W. Gehring and B. Osgood. More precisely, let D be a proper subdomain of $\mathbb{R}^n$. Then the j_D-metric (see, [8, p. 51]),

$$j_D(x,y) = \frac{1}{2}\Big[\log\Big(1 + \frac{|x-y|}{\operatorname{dist}(x,\partial D)}\Big) + \log\Big(1 + \frac{|x-y|}{\operatorname{dist}(y,\partial D)}\Big)\Big],$$

which is an average, is Gromov hyperbolic [11, Theorem 1]. As mentioned above, the $\tilde{\jmath}_D$-metric,

$$\tilde{\jmath}_D(x,y) = \sup\Big\{\log\Big(1 + \frac{|x-y|}{\operatorname{dist}(x,\partial D)}\Big),\ \ \log\Big(1 + \frac{|x-y|}{\operatorname{dist}(y,\partial D)}\Big)\Big\},$$

which is a supremum, is not Gromov hyperbolic [11, Theorem 3]. (Note that in [11] the author denotes the j-metric by $\tilde{\jmath}$ and the $\tilde{\jmath}$-metric by j.)

Now we are ready to formulate the main results of the paper. Here and throughout the paper, we let (X,d) be an arbitrary metric space containing at least four points. For each $p \in X$, we define a distance function τ_p on $X \setminus \{p\}$ by

$$\tau_p(x,y) = \log\Big(1 + 2\frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}}\Big). \tag{1.1}$$

For $p_1, p_2, \dots, p_k \in X$ and $D = X \setminus \{p_1, p_2, \dots, p_k\}$, we define a metric τ_D on D by taking the simple average of the metrics τ_{p_i}, namely,

$$\hat{\tau}_D(x,y) = \frac{1}{k}\sum_{i=1}^{k} \tau_{p_i}(x,y) = \frac{1}{k}\sum_{i=1}^{k} \log\Big(1 + 2\frac{d(x,y)}{\sqrt{d(x,p_i)d(y,p_i)}}\Big).$$

We prove that for each $p \in X$, the metric τ_p is Gromov hyperbolic with $\delta = \log 3 + \log 2$ (Lemma 4.1) and that for any $k \geq 1$, the metric $\hat{\tau}_D(x,y)$ is Gromov hyperbolic with $\delta = 3\log 3 + 2\log 2$ (Theorem 4.2). The latter result is unexpected since we also provide an example to demonstrate that the average of two Gromov hyperbolic metrics is not necessarily Gromov hyperbolic (Lemma 4.4).

2. One-point scale-invariant Cassinian metric on general metric spaces

In this section we define the one-point scale-invariant Cassinian metrics in the context of arbitrary metric spaces, and in Section 4 we study Gromov hyperbolicity of the average of finitely many such metrics. Let (X,d) be a metric space. For each $p \in X$, we define a distance function τ_p on $X \setminus \{p\}$ by

$$\tau_p(x,y) = \log\Big(1 + 2\frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}}\Big). \tag{2.1}$$

Theorem 2.1. *Let (X,d) be an arbitrary metric space, and let $p \in X$ be an arbitrary point. Then the distance function τ_p is a metric on $X \setminus \{p\}$.*

Proof. Clearly, $\tau_p(x,y) \geq 0$, $\tau_p(x,y) = \tau_p(y,x)$, and $\tau_p(x,y) = 0$ if and only if $x = y$. So it is enough to show that the triangle inequality holds. That is,

$$\tau_p(x,y) \leq \tau_p(x,z) + \tau_p(z,y) \tag{2.2}$$

for all $x, y, z \in D$. Inequality (2.2) is equivalent to

$$\frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}} \leq \frac{d(x,z)}{\sqrt{d(x,p)d(z,p)}} + \frac{d(z,y)}{\sqrt{d(z,p)d(y,p)}} + 2\frac{d(x,z)d(z,y)}{d(z,p)\sqrt{d(x,p)d(y,p)}}$$

or, equivalently,

$$\frac{d(x,y)d(z,p)}{d(x,z)d(y,z)} \leq \frac{\sqrt{d(x,p)d(z,p)}}{d(x,z)} + \frac{\sqrt{d(y,p)d(z,p)}}{d(y,z)} + 2. \tag{2.3}$$

Since

$$\frac{d(x,y)d(z,p)}{d(x,z)d(y,z)} \leq \frac{d(y,z)d(z,p)}{d(x,z)d(y,z)} + \frac{d(x,z)d(z,p)}{d(x,z)d(y,z)} = \frac{d(z,p)}{d(x,z)} + \frac{d(z,p)}{d(y,z)},$$

it suffices to show that

$$\frac{d(z,p)}{d(x,z)} \leq \frac{\sqrt{d(x,p)d(z,p)}}{d(x,z)} + 1 \quad \text{and} \quad \frac{d(z,p)}{d(y,z)} \leq \frac{\sqrt{d(y,p)d(z,p)}}{d(y,z)} + 1.$$

Due to symmetry, it suffices to prove the first inequality. If $d(z,p) \leq d(x,p)$, then

$$\frac{d(z,p)}{d(x,z)} \leq \frac{\sqrt{d(x,p)d(z,p)}}{d(x,z)} < \frac{\sqrt{d(x,p)d(z,p)}}{d(x,z)} + 1.$$

If $d(x,p) \le d(z,p)$, then

$$\frac{d(z,p)}{d(x,z)} \le \frac{d(x,z)+d(x,p)}{d(x,z)} \le \frac{d(x,z)+\sqrt{d(x,p)d(z,p)}}{d(x,z)} = \frac{\sqrt{d(x,p)d(z,p)}}{d(x,z)} + 1,$$

completing the proof. □

Remark 2.2. Inequality (2.3) implies that the constant 2 in equation (2.1) can be replaced with any constant $c \ge 2$ (see also [5, Theorem 1.1]).

One can easily see that for all $x, y \in X \setminus \{p\}$ we have

$$\tilde{\tau}_p(x,y) \le \tau_p(x,y) \le \tilde{\tau}_p(x,y) + \log 2. \tag{2.4}$$

Here

$$\tilde{\tau}_p(x,y) = \log\Big(1 + \frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}}\Big) = \log\frac{\mu_p(x,y)}{\sqrt{d(x,p)d(y,p)}}. \tag{2.5}$$

The distance function $\tilde{\tau}_p$ was introduced and studied in the context of Euclidean spaces in [19], where it was referred to as a one-point scale-invariant Cassinian metric. However, $\tilde{\tau}_p$ is not a metric in the context of general metric spaces. Indeed, let $X = \{p, x, y, z\}$ and define $d(p,x) = d(y,z) = 2$ and $d(p,y) = d(p,z) = d(x,y) = d(x,z) = 1$. Clearly, d is a metric on X. One can easily see that $\tilde{\tau}_p(y,z) > \tilde{\tau}_p(x,y) + \tilde{\tau}_p(x,z)$. Therefore, $\tilde{\tau}_p$ is not a metric on $X \setminus \{p\}$ justifying the introduction of its modified version τ_p. However, it turns out that if (X,d) is a Ptolemaic metric space, then $\tilde{\tau}_p$ is a metric on $X \setminus \{p\}$ for each $p \in X$. Recall that a metric space (X,d) is called *Ptolemaic* if

$$d(x,y)d(z,w) \le d(x,z)d(y,w) + d(x,w)d(y,z) \tag{2.6}$$

for all $x, y, z, w \in X$.

Theorem 2.3. *Let (X,d) be a Ptolemaic metric space, and let $p \in X$ be an arbitrary point. Then the distance function $\tilde{\tau}_p$ is a metric on $X \setminus \{p\}$.*

Proof. Clearly, it is enough to show that the triangle inequality holds. That is,

$$\tilde{\tau}_p(x,y) \le \tilde{\tau}_p(x,z) + \tilde{\tau}_p(z,y) \tag{2.7}$$

for all $x, y, z \in X \setminus \{p\}$. Inequality (2.7) is equivalent to

$$\Big(1 + \frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}}\Big) \le \Big(1 + \frac{d(x,z)}{\sqrt{d(x,p)d(z,p)}}\Big)\Big(1 + \frac{d(z,y)}{\sqrt{d(z,p)d(y,p)}}\Big),$$

which is equivalent to

$$\begin{aligned} \frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}} \le& \frac{d(x,z)}{\sqrt{d(x,p)d(z,p)}} \\ &+ \frac{d(z,y)}{\sqrt{d(z,p)d(y,p)}} + \frac{d(x,z)d(z,y)}{d(z,p)\sqrt{d(x,p)d(y,p)}}. \end{aligned} \tag{2.8}$$

Without loss of generality we can assume that $d(x,p) \le d(y,p)$.

If $d(z,p) \le d(x,p) \le d(y,p)$, then

$$\sqrt{d(x,p)d(y,p)} \ge \sqrt{d(x,p)d(z,p)} \quad \text{and} \quad \sqrt{d(x,p)d(y,p)} \ge \sqrt{d(z,p)d(y,p)}.$$

By the triangle inequality we then obtain

$$\begin{aligned}\frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}} &\le \frac{d(x,z)}{\sqrt{d(x,p)d(y,p)}} + \frac{d(z,y)}{\sqrt{d(x,p)d(y,p)}}\\ &\le \frac{d(x,z)}{\sqrt{d(x,p)d(z,p)}} + \frac{d(z,y)}{\sqrt{d(z,p)d(y,p)}},\end{aligned}$$

establishing (2.8).

If $d(x,p) \le d(y,p) \le d(z,p)$, then

$$d(z,p)d(x,y) \le d(y,p)d(x,z) + d(x,p)d(z,y)$$

by Ptolemy's Inequality. Since $d(x,p) \le d(z,p)$ and $d(y,p) \le d(z,p)$, we have

$$d(x,p) \le \sqrt{d(x,p)d(z,p)} \qquad \text{and} \qquad d(y,p) \le \sqrt{d(y,p)d(z,p)}.$$

Hence

$$d(z,p)d(x,y) \le \sqrt{d(y,p)d(z,p)}d(x,z) + \sqrt{d(x,p)d(z,p)}d(z,y).$$

Consequently,

$$\frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}} \le \frac{d(x,z)}{\sqrt{d(x,p)d(z,p)}} + \frac{d(z,y)}{\sqrt{d(z,p)d(y,p)}},$$

establishing (2.8).

Finally, if $d(x,p) \le d(z,p) \le d(y,p)$, then $d(x,p) \le \sqrt{d(x,p)d(z,p)}$ since $d(x,p) \le d(z,p)$. By the triangle inequality we have $d(z,p) \le d(x,p) + d(x,z)$. Hence

$$d(z,p) \le \sqrt{d(x,p)d(z,p)} + d(x,z),$$

or, equivalently,

$$\frac{1}{\sqrt{d(x,p)}} \le \frac{1}{\sqrt{d(z,p)}} + \frac{d(x,z)}{d(z,p)\sqrt{d(x,p)}}.$$

Thus,

$$\frac{d(z,y)}{\sqrt{d(x,p)d(y,p)}} \le \frac{d(z,y)}{\sqrt{d(z,p)d(y,p)}} + \frac{d(x,z)d(z,y)}{d(z,p)\sqrt{d(x,p)d(y,p)}}. \tag{2.9}$$

Now by the triangle inequality we have

$$\frac{d(x,y)}{\sqrt{d(x,p)d(y,p)}} \le \frac{d(x,z)}{\sqrt{d(x,p)d(y,p)}} + \frac{d(z,y)}{\sqrt{d(x,p)d(y,p)}}. \tag{2.10}$$

Also, since $d(z,p) \le d(y,p)$, we have

$$\frac{d(x,z)}{\sqrt{d(x,p)d(y,p)}} \le \frac{d(x,z)}{\sqrt{d(x,p)d(z,p)}}. \tag{2.11}$$

Therefore, combining inequalities (2.9), (2.10), and (2.11), we see that inequality (2.8) holds also in this case. The proof is complete. □

Definition 2.4. In the context of a general metric space (X,d), the metrics τ_p, $p \in X$, are called one-point scale-invariant Cassinian metrics.

3. Technical results

In this section we establish several results needed in Section 4. Throughout this section we let (X,d) be an arbitrary metric space. Fix a point $p\in X$ and define

$$\mu_p(x,y)=d(x,y)+\sqrt{d(x,p)d(y,p)} \qquad \text{for} \quad x,y\in X.$$

In this section we study some properties of μ_p, especially Lemmas 3.1 and 3.5, which will be needed in Section 4. In what follows, we set

$$a\wedge b=\min\{a,b\} \qquad \text{and} \qquad a\vee b=\max\{a,b\}$$

for nonnegative real numbers a and b. Observe that

$$(a\vee b)(c\vee d)=ac\vee ad\vee bc\vee bd \tag{3.1}$$

for all nonnegative real numbers a,b,c,d.

Lemma 3.1. *For all $x,y,z,w\in X$ we have*

$$\mu_p(x,y)\mu_p(z,w)\le 9\big[\mu_p(x,z)\mu_p(y,w)\vee \mu_p(x,w)\mu_p(y,z)\big]. \tag{3.2}$$

Proof. Since $d(x,y)\le d(x,p)+d(y,p)\le 2(d(x,p)\vee d(y,p))$ and since

$$\sqrt{d(x,p)d(y,p)}\le \frac{d(x,p)+d(y,p)}{2}\le d(x,p)\vee d(y,p),$$

we have

$$\mu_p(x,y)\le \frac{3}{2}\big[d(x,p)+d(y,p)\big]\le 3\big[d(x,p)\vee d(y,p)\big] \tag{3.3}$$

for all $x,y\in X$. Also, since $d(x,y)\ge d(x,p)\vee d(y,p)-d(x,p)\wedge d(y,p)$ and since $\sqrt{d(x,p)d(y,p)}\ge d(x,p)\wedge d(y,p)$, we have

$$\mu_p(x,y)\ge d(x,p)\vee d(y,p)\ge \frac{1}{2}\big[d(x,p)+d(y,p)\big] \tag{3.4}$$

for all $x,y\in X$. Using (3.1), (3.3), and (3.4) we have

$$\begin{aligned}
&\frac{1}{9}\mu_p(x,y)\mu_p(z,w)\le \big[d(x,p)\vee d(y,p)\big]\big[d(z,p)\vee d(w,p)\big]\\
&\quad = d(x,p)d(z,p)\vee d(x,p)d(w,p)\vee d(y,p)d(z,p)\vee d(y,p)d(w,p)\\
&\quad \le \Big[d(x,p)d(y,p)\vee d(x,p)d(w,p)\vee d(z,p)d(y,p)\vee d(z,p)d(w,p)\Big]\\
&\qquad\quad \vee \Big[d(x,p)d(y,p)\vee d(x,p)d(z,p)\vee d(w,p)d(y,p)\vee d(w,p)d(z,p)\Big]\\
&\quad = \Big[\big(d(x,p)\vee d(z,p)\big)\big(d(y,p)\vee d(w,p)\big)\Big]\vee \Big[\big(d(x,p)\vee d(w,p)\big)\big(d(y,p)\vee d(z,p)\big)\Big]\\
&\quad \le \mu_p(x,z)\mu_p(y,w)\vee \mu_p(x,w)\mu_p(y,z),
\end{aligned}$$

as required. ☐

Note that

$$\mu_p(x,z)+\mu_q(y,z)\ge d(x,z)+d(y,z)\ge d(x,y) \tag{3.5}$$

for all $x,y,z,q\in X$. In particular, for all $x,y,z,q\in X$, we have

$$\mu_p(x,z)\vee \mu_q(y,z)\ge \frac{1}{2}d(x,y). \tag{3.6}$$

Lemma 3.2. *Let $x, y, z \in X$ be arbitrary points. If*

$$\mu_p(x,z) \vee \mu_p(y,z) \geq K\big[\mu_p(x,z) \wedge \mu_p(y,z)\big]$$

for some $K > 3$, then

$$\mu_p(x,z) + \mu_p(y,z) \leq \frac{3(K+3)}{2(K-3)} d(x,y).$$

Proof. Without loss of generality we can assume that $\mu_p(x,z) \geq \mu_p(y,z)$. Using (3.4) we obtain

$$\frac{K}{2}\big(d(y,p) + d(z,p)\big) \leq K\mu_p(y,z) \leq \mu_p(x,z) \leq \frac{3}{2}\big(d(x,p) + d(z,p)\big),$$

which implies $Kd(y,p) + (K-3)d(z,p) \leq 3d(x,p)$. In particular,

$$2d(z,p) \leq \frac{6}{K-3} d(x,p) - \frac{2K}{K-3} d(y,p).$$

The latter, along with (3.3), implies

$$\begin{aligned}
\mu_p(x,z) + \mu_p(y,z) &\leq \frac{3}{2}\big(d(x,p) + d(y,p) + 2d(z,p)\big) \\
&\leq \frac{3}{2}\Big(d(x,p) + d(y,p) + \frac{6}{K-3} d(x,p) - \frac{2K}{K-3} d(y,p)\Big) \\
&= \frac{3(K+3)}{2(K-3)}\big(d(x,p) - d(y,p)\big) \leq \frac{3(K+3)}{2(K-3)} d(x,y),
\end{aligned}$$

completing the proof. □

Suppose now that $p_1, p_2, \ldots, p_k$ are arbitrary points in X and set $P = \{p_1, p_2, \ldots, p_k\}$.

Lemma 3.3. *For all $x, y, z \in X$ we have*

$$\prod_{i=1}^{k}\Big(\mu_{p_i}(x,z) + \mu_{p_i}(y,z)\Big) \leq 9^k \left(\prod_{i=1}^{k} \mu_{p_i}(x,z) + \prod_{i=1}^{k} \mu_{p_i}(y,z)\right). \tag{3.7}$$

Proof. Let $x, y, z \in X$ be arbitrary points. For simplicity, we set

$$a_i = \mu_{p_i}(x,z) \quad \text{and} \quad b_i = \mu_{p_i}(y,z), \quad i = 1, 2, \ldots, k.$$

By (3.6) we then have

$$a_i \vee b_j \geq \frac{1}{2} d(x,y) \qquad \text{for all} \quad i, j = 1, 2, \ldots, k. \tag{3.8}$$

We will prove the lemma by induction. Assume first that $k = 2$. Hence we need to show that

$$(a_1 + b_1)(a_2 + b_2) \leq 81(a_1 a_2 + b_1 b_2). \tag{3.9}$$

Case 1 ($a_1 \vee b_1 \leq 6(a_1 \wedge b_1)$ or $a_2 \vee b_2 \leq 6(a_2 \wedge b_2)$). Without loss of generality we can assume that $a_1 \vee b_1 \leq 6(a_1 \wedge b_1)$. Then

$$a_1 + b_1 = a_1 \vee b_1 + a_1 \wedge b_1 \leq 7(a_1 \wedge b_1) \quad \text{and} \quad (a_1 \wedge b_1)(a_2 + b_2) \leq a_1 a_2 + b_1 b_2.$$

Hence

$$(a_1 + b_1)(a_2 + b_2) \leq 7(a_1 \wedge b_1)(a_2 + b_2) \leq 7(a_1 a_2 + b_1 b_2)$$

so that (3.9) holds in this case.

Case 2 ($a_1 \vee b_1 \geq 6(a_1 \wedge b_1)$ and $a_2 \vee b_2 \geq 6(a_2 \wedge b_2)$). Without loss of generality we can assume that $a_1 = a_1 \wedge b_1 \wedge a_2 \wedge b_2$. By (3.8) we then have

$$b_1 \geq \frac{1}{2}d(x,y) \quad \text{and} \quad b_2 \geq \frac{1}{2}d(x,y).$$

Hence

$$a_1a_2 + b_1b_2 \geq b_1b_2 \geq \frac{1}{4}\big[d(x,y)\big]^2.$$

Also, by Lemma 3.2 we have

$$a_1 + b_1 \leq \frac{9}{2}d(x,y) \quad \text{and} \quad a_2 + b_2 \leq \frac{9}{2}d(x,y),$$

and hence

$$(a_1 + b_1)(a_2 + b_2) \leq \frac{81}{4}\big[d(x,y)\big]^2.$$

Consequently,

$$(a_1 + b_1)(a_2 + b_2) \leq \frac{81}{4}\big[d(x,y)\big]^2 \leq 81(a_1a_2 + b_1b_2),$$

completing the proof of the lemma for $k = 2$.

Assume now that (3.7) holds for $k = m$. That is,

$$\prod_{i=1}^{m}(a_i + b_i) \leq 9^m\Big(\prod_{i=1}^{m} a_i + \prod_{i=1}^{m} b_i\Big). \tag{3.10}$$

We need to show that it also holds for $k = m + 1$. That is,

$$\prod_{i=1}^{m+1}(a_i + b_i) \leq 9^{m+1}\Big(\prod_{i=1}^{m+1} a_i + \prod_{i=1}^{m+1} b_i\Big). \tag{3.11}$$

Case 1 ($a_i \vee b_i \leq 6(a_i \wedge b_i)$ for some $i \in \{1, 2, \dots, m+1\}$). Note that

$$a_i + b_i = (a_i \vee b_i) + (a_i \wedge b_i) \leq 7(a_i \wedge b_i).$$

Without loss of generality we can assume that $i = 1$. Then

$$\prod_{i=1}^{m+1} a_i + \prod_{i=1}^{m+1} b_i \geq (a_1 \wedge b_1)\Big(\prod_{i=2}^{m+1} a_i + \prod_{i=2}^{m+1} b_i\Big)$$

and hence

$$\begin{aligned}\prod_{i=1}^{m+1}(a_i + b_i) &= (a_1 + b_1)\prod_{i=2}^{m+1}(a_i + b_i) \leq (a_1 + b_1)9^m\Big(\prod_{i=2}^{m+1} a_i + \prod_{i=2}^{m+1} b_i\Big) \\ &\leq 7(a_1 \wedge b_1)9^m\Big(\prod_{i=2}^{m+1} a_i + \prod_{i=2}^{m+1} b_i\Big) < 9^{m+1}\Big(\prod_{i=1}^{m+1} a_i + \prod_{i=1}^{m+1} b_i\Big),\end{aligned}$$

as required.

Case 2 ($a_i \vee b_i \geq 6(a_i \wedge b_i)$ for all $i \in \{1, 2, \dots, m+1\}$). Without loss of generality we can assume that a_1 is the smallest of the numbers a_i and b_i for all $i = 1, 2, \dots, m+1$. By (3.8) we then have

$$b_i \geq \frac{1}{2}d(x,y) \quad \text{for all} \quad i = 1, 2, \dots, m+1.$$

Hence

$$\prod_{i=1}^{m+1} a_i + \prod_{i=1}^{m+1} b_i \geq \prod_{i=1}^{m+1} b_i \geq \frac{1}{2^{m+1}} \big[d(x,y)\big]^{m+1}.$$

Also, by Lemma 3.2 we have $a_i + b_i \leq (9/2)d(x,y)$ for each i. Hence

$$\prod_{i=1}^{m+1} (a_i + b_i) \leq \Big(\frac{9}{2}\Big)^{m+1} \big[d(x,y)\big]^{m+1}.$$

Consequently,

$$\prod_{i=1}^{m+1} (a_i + b_i) \leq \Big(\frac{9}{2}\Big)^{m+1} \big[d(x,y)\big]^{m+1} \leq 9^{m+1}\Big(\prod_{i=1}^{m+1} a_i + \prod_{i=1}^{m+1} b_i\Big),$$

completing the proof of the lemma. □

We need the following lemma. For $K = 1$, this lemma was proved in [18] (see [18, Lemma 3.7]).

Lemma 3.4. *Let $r_{ij} \geq 0$ be real numbers such that $r_{ij} = r_{ji}$ and $r_{ij} \leq K(r_{ik} + r_{jk})$ for some $K \geq 1$ and for all $i, j, k \in \{1, 2, 3, 4\}$. Then*

$$\sqrt{r_{12}r_{34}} \leq K(\sqrt{r_{13}r_{24}} + \sqrt{r_{14}r_{23}}).$$

In particular,

$$r_{12}r_{34} \leq 2K^2(r_{13}r_{24} + r_{14}r_{23}) \leq (2K)^2 \max\{r_{13}r_{24},\, r_{14}r_{23}\}.$$

Proof. We can assume, without loss of generality, that r_{13} is the smallest of the numbers $r_{13}, r_{14}, r_{24}, r_{23}$ and that $r_{23} \geq r_{14}$. Clearly, it suffices to show that

$$r_{12}r_{34} \leq K^2(r_{13}r_{24} + r_{14}r_{23} + 2\sqrt{r_{13}r_{24}r_{14}r_{23}}).$$

Equivalently, we need to show that $\alpha \geq 0$, where

$$\alpha = -r_{12}r_{34} + K^2(r_{13}r_{24} + r_{14}r_{23} + 2\sqrt{r_{13}r_{24}r_{14}r_{23}}).$$

By the assumptions we have

$$r_{12} \leq K \min\{r_{13} + r_{23},\, r_{14} + r_{24}\} \quad \text{and} \quad r_{34} \leq K \min\{r_{13} + r_{14},\, r_{23} + r_{24}\}.$$

If $r_{14} + r_{24} \leq r_{13} + r_{23}$, then $r_{23} \geq r_{14} + r_{24} - r_{13}$. Since $r_{24} \geq r_{13}$, we obtain

$$\begin{aligned}\alpha \geq -K^2(r_{14} + r_{24})(r_{13} + r_{14}) + K^2\big(&r_{13}r_{24} + r_{14}(r_{14} + r_{24} - r_{13}) \\ &+ 2\sqrt{r_{13}r_{24}r_{14}(r_{14} + r_{24} - r_{13})}\big) \\ &= 2K^2(\sqrt{r_{13}r_{24}r_{14}(r_{14} + r_{24} - r_{13})} - r_{13}r_{14}) \geq 0.\end{aligned}$$

Now suppose that $r_{14} + r_{24} \geq r_{13} + r_{23}$. Then $r_{23} \leq r_{14} + r_{24} - r_{13}$, and hence

$$\alpha \geq -K^2(r_{13} + r_{23})(r_{13} + r_{14}) + K^2\big(r_{13}r_{24} + r_{14}r_{23} + 2\sqrt{r_{13}r_{24}r_{14}r_{23}}\big) = K^2 f(r_{23}),$$

where

$$f(x) = r_{13}r_{24} + 2\sqrt{r_{13}r_{24}r_{14}}\sqrt{x} - (r_{13})^2 - r_{13}r_{14} - r_{13}x.$$

The function $f(x)$ is increasing on the interval $[r_{14}, r_{14} + r_{24} - r_{13}]$. Indeed, for each $x \in [r_{14}, r_{14} + r_{24} - r_{13}]$ we have $r_{13}x - r_{24}r_{14} \leq r_{13}(r_{14} + r_{24} - r_{13}) - r_{24}r_{14} = (r_{14} - r_{13})(r_{13} - r_{24}) \leq 0$, and hence $r_{13}\sqrt{x} - \sqrt{r_{13}r_{24}r_{14}} \leq 0$. The latter is equivalent to $f'(x) \geq 0$. Since $f(r_{14}) = r_{13}r_{24} + 2r_{14}\sqrt{r_{13}r_{24}} - (r_{13})^2 - 2r_{13}r_{14} = r_{13}(r_{24} - r_{13}) + 2r_{14}(\sqrt{r_{13}r_{24}} - r_{13}) \geq 0$, we obtain $\alpha \geq K^2 f(r_{23}) \geq K^2 f(r_{14}) \geq 0$, completing the proof of the first part. Since $(a+b)^2 \leq 2(a^2+b^2)$ for all real numbers a and b, the second part follows. □

Next, we define a distance function $\mu_P \colon X \times X \to [0, +\infty)$ by

$$\mu_P(x,y) = \prod_{i=1}^{k} \mu_{p_i}(x,y) = \prod_{i=1}^{k} \big[d(x,y) + \sqrt{d(x,p_i)d(y,p_i)}\big]. \tag{3.12}$$

Lemma 3.5. *For all $x, y, z \in X$ we have*

$$\mu_P(x,y) \le \Big(\frac{27}{2}\Big)^k \Big(\mu_P(x,z) + \mu_P(z,y)\Big).$$

Moreover,

$$\mu_P(x,y)\mu_P(z,w) \le 4\Big(\frac{27}{2}\Big)^{2k} \max\Big\{\mu_P(x,z)\mu_P(y,w),\ \mu_P(x,w)\mu_P(y,z)\Big\}.$$

Proof. Using (3.12) and Lemma 3.3 we have

$$\begin{aligned}\mu_P(x,y) &= \prod_{i=1}^{k} \mu_{p_i}(x,y) \le \Big(\frac{3}{2}\Big)^k \prod_{i=1}^{k} \Big(\mu_{p_i}(x,z) + \mu_{p_i}(y,z)\Big) \\ &\le \Big(\frac{3}{2}\Big)^k 9^k \Big(\prod_{i=1}^{k} \mu_{p_i}(x,z) + \prod_{i=1}^{k} \mu_{p_i}(y,z)\Big) \\ &= \Big(\frac{27}{2}\Big)^k \Big(\mu_P(x,z) + \mu_P(y,z)\Big),\end{aligned}$$

completing the proof of the first part. The second part follows from the first part and Lemma 3.4. □

4. Gromov hyperbolicity of the average of one-point scale-invariant Cassinian metrics

We begin by showing that each one-point scale-invariant Cassinian metric is Gromov hyperbolic. Recall that a metric space (X, d) is Gromov hyperbolic if

$$d(x,y) + d(z,v) \le \big[d(x,z) + d(y,v)\big] \vee \big[d(x,v) + d(y,z)\big] + 2\delta \tag{4.1}$$

for all $v, x, y, z \in X$ and for some $\delta \ge 0$. The reader is referred to [6, 10, 25] for a detailed discussion of Gromov hyperbolic metric spaces. Recall that

$$\tilde{\tau}_p(x,y) \le \tau_p(x,y) \le \tilde{\tau}_p(x,y) + \log 2$$

for all $x, y \in X \setminus \{p\}$ (see (2.4)). It follows that if the metric $\tilde{\tau}_p$ satisfies (4.1) with a constant δ, then the metric τ_p satisfies (4.1) with a constant $\delta + \log 2$.

Lemma 4.1. *Let (X, d) be an arbitrary metric space, and let $p \in X$ be any point. Then the space $(X \setminus \{p\}, \tilde{\tau}_p)$ is Gromov hyperbolic with $\delta = \log 3$. In particular, the space $(X \setminus \{p\}, \tau_p)$ is Gromov hyperbolic with $\delta = \log 3 + \log 2$.*

Proof. It suffices to show that $\tilde{\tau}_p$ satisfies (4.1) with $\delta = \log 3$. Let $x, y, z, v \in X \setminus \{p\}$ be arbitrary points. By Lemma 3.1 we have

$$\mu_p(x,y)\mu_p(z,v) \le 9\big[\mu_p(x,z)\mu_p(y,v) \vee \mu_p(x,v)\mu_p(y,z)\big]$$

or, equivalently,

$$\begin{aligned}&\frac{\mu_p(x,y)\mu_p(z,v)}{\sqrt{d(x,p)d(y,p)d(z,p)d(v,p)}} \\ &\le 9\left[\frac{\mu_p(x,z)\mu_p(y,v)}{\sqrt{d(x,p)d(y,p)d(z,p)d(v,p)}} \vee \frac{\mu_p(x,v)\mu_p(y,z)}{\sqrt{d(x,p)d(y,p)d(z,p)d(v,p)}}\right].\end{aligned}$$

The latter implies

$$\tilde{\tau}_p(x,y)+\tilde{\tau}_p(z,v)\le\Big[\tilde{\tau}_p(x,z)+\tilde{\tau}_p(y,v)\Big]\vee\Big[\tilde{\tau}_p(x,v)+\tilde{\tau}_p(y,z)\Big]+2\log 3, \tag{4.2}$$

completing the proof. □

We are now ready to present the main result of the paper. Let (X,d) be any metric space, and let $p_1,p_2,\dots,p_k$ be any points in X. Put $P=\{p_1,p_2,\dots,p_k\}$ and $D=X\setminus\{p_1,p_2,\dots,p_k\}$. We define a new metric $\hat{\tau}_D$ on D by taking the simple average of the one-point scale-invariant Cassinian metrics τ_{p_i}, $i=1,2,\dots,k$. Namely, for $x,y\in D$ we define

$$\hat{\tau}_D(x,y)=\frac{1}{k}\big[\tau_{p_1}(x,y)+\tau_{p_2}(x,y)+\cdots+\tau_{p_k}(x,y)\big]=\frac{1}{k}\sum_{i=1}^{k}\tau_{p_i}(x,y). \tag{4.3}$$

It is clear that the average of any finitely many metrics is again a metric. We have

$$\tilde{\tau}_D(x,y)\le\hat{\tau}_D(x,y)\le\tilde{\tau}_D(x,y)+\log 2 \tag{4.4}$$

for all $x,y\in D$, where

$$\tilde{\tau}_D(x,y)=\frac{1}{k}\sum_{i=1}^{k}\tilde{\tau}_{p_i}(x,y)=\frac{1}{k}\log\left(\prod_{i=1}^{k}\frac{\mu_{p_i}(x,y)}{\sqrt{d(x,p_i)d(y,p_i)}}\right). \tag{4.5}$$

Theorem 4.2. *The space $(D,\hat{\tau}_D)$ is Gromov hyperbolic with $\delta=3\log 3+\log 2$. In particular, if (X,d) is Ptolemaic, then the space $(D,\tilde{\tau}_D)$ is Gromov hyperbolic with $\delta=3\log 3$.*

Proof. It suffices to show that for all $x,y,z,w\in D$ we have

$$\tilde{\tau}_D(x,y)+\tilde{\tau}_D(z,w)\le\max\big\{\tilde{\tau}_D(x,z)+\tilde{\tau}_D(y,w),\ \tilde{\tau}_D(x,w)+\tilde{\tau}_D(y,z)\big\}+6\log 3.$$

Using Lemma 3.5 we obtain

$$\begin{aligned}
&\tilde{\tau}_D(x,y)+\tilde{\tau}_D(z,w)=\frac{1}{k}\log\left(\prod_{i=1}^{k}\frac{\mu_{p_i}(x,y)\mu_{p_i}(z,w)}{\sqrt{d(x,p_i)d(y,p_i)d(z,p_i)d(w,p_i)}}\right)\\
&=\frac{1}{k}\log\left(\frac{\prod_{i=1}^{k}\mu_{p_i}(x,y)\prod_{i=1}^{k}\mu_{p_i}(z,w)}{\prod_{i=1}^{k}\sqrt{d(x,p_i)d(y,p_i)d(z,p_i)d(w,p_i)}}\right)\\
&=\frac{1}{k}\log\left(\frac{\mu_P(x,y)\mu_P(z,w)}{\prod_{i=1}^{k}\sqrt{d(x,p_i)d(y,p_i)d(z,p_i)d(w,p_i)}}\right)\\
&\le\frac{1}{k}\log\left(\frac{4(27/2)^{2k}\max\big\{\mu_P(x,z)\mu_P(y,w),\ \mu_P(x,w)\mu_P(y,z)\big\}}{\prod_{i=1}^{k}\sqrt{d(x,p_i)d(y,p_i)d(z,p_i)d(w,p_i)}}\right)\\
&=\frac{1}{k}\log\left(\frac{\max\big\{\mu_P(x,z)\mu_P(y,w),\ \mu_P(x,w)\mu_P(y,z)\big\}}{\prod_{i=1}^{k}\sqrt{d(x,p_i)d(y,p_i)d(z,p_i)d(w,p_i)}}\right)+2\log(27/2)+\frac{1}{k}\log 4\\
&=\max\big\{\tilde{\tau}_D(x,z)+\tilde{\tau}_D(y,w),\ \tilde{\tau}_D(x,w)+\tilde{\tau}_D(y,z)\big\}+2(\log(27/2)+\frac{1}{k}\log 2)\\
&\le\max\big\{\tilde{\tau}_D(x,z)+\tilde{\tau}_D(y,w),\ \tilde{\tau}_D(x,w)+\tilde{\tau}_D(y,z)\big\}+6\log 3,
\end{aligned}$$

completing the proof. □

Definition 4.3. In the context of a general metric space (X,d), the metric $\hat{\tau}_D$ will be referred to as the average scale-invariant Cassinian metric.

We end the paper with the following example that shows that the sum of two Gromov hyperbolic metrics is not, in general, Gromov hyperbolic. Consider the two-dimensional Euclidean space $\mathbb{R}^2$ equipped with the Euclidean metric $|-|$. For $x \in \mathbb{R}^2$ we write $x = (x_1, x_2)$. Define metrics d_1 and d_2 on $\mathbb{R}^2$ by

$$d_1(x,y) = |x_1 - y_1| + \tan^{-1}(|x_2 - y_2|) \quad \text{and} \quad d_2(x,y) = |x_2 - y_2| + \tan^{-1}(|x_1 - y_1|).$$

Clearly, both d_1 and d_2 are nonnegative and symmetric, and $d_m(x,y) = 0$ $(m = 1, 2)$ if and only if $x = y$. Since $\tan^{-1}$ is an increasing and concave function on $[0, \infty)$, we see that both d_1 and d_2 obey the triangle inequality. Thus, d_1 and d_2 are indeed metrics on $\mathbb{R}^2$.

Lemma 4.4. *The spaces* $(\mathbb{R}^2, d_1)$ *and* $(\mathbb{R}^2, d_2)$ *are Gromov hyperbolic with* $\delta = \pi/2$, *but the space* $(\mathbb{R}^2, s)$, $d = d_1 + d_2$, *is not Gromov hyperbolic.*

Proof. Due to the similarity between d_1 and d_2 it is enough to show that $(\mathbb{R}^2, d_1)$ is Gromov hyperbolic with $\delta = \pi/2$. First, observe that the Euclidean distance on $\mathbb{R}$ is Gromov hyperbolic with $\delta = 0$. That is, for all $p, q, r, t \in \mathbb{R}$, we have

$$|p - q| + |r - t| \le \big[|p - r| + |q - t|\big] \vee \big[|p - t| + |q - r|\big]. \tag{4.6}$$

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$, and $v = (v_1, v_2)$ be arbitrary points in $\mathbb{R}^2$. Using (4.6) along with the fact that $\tan^{-1}(a) < \pi/2$ for all $a \in [0, +\infty)$, we obtain

$$\begin{aligned} d_1(x,y) + d_1(z,v) &= |x_1 - y_1| + |z_1 - v_1| + \tan^{-1}(|x_2 - y_2|) + \tan^{-1}(|z_2 - v_2|) \\ &\le |x_1 - y_1| + |z_1 - v_1| + \frac{\pi}{2} + \frac{\pi}{2} \\ &\le \big[|x_1 - z_1| + |y_1 - v_1|\big] \vee \big[|x_1 - v_1| + |y_1 - z_1|\big] + 2 \cdot \frac{\pi}{2} \\ &\le \big[d_1(x,z) + d_1(y,v)\big] \vee \big[d_1(x,v) + d_1(y,z)\big] + 2 \cdot \frac{\pi}{2}, \end{aligned}$$

completing the proof of the first part.

Next, we show that $(\mathbb{R}^2, d)$ is not Gromov hyperbolic. Observe that d is roughly similar to the *taxicab* metric. That is,

$$d_T(x,y) \le d(x,y) \le d_T(x,y) + \pi \qquad \text{for all} \qquad x, y \in \mathbb{R}^2. \tag{4.7}$$

Here d_T is the taxicab metric defined by $d_T(x,y) = |x_1 - y_1| + |x_2 - y_2|$. It is known that the taxicab metric is not Gromov hyperbolic. Indeed, for $t > 0$ and

$$x = (0,0), \quad y = (t,t), \quad z = (0,t), \quad v = (t,0)$$

we have

$$d_T(x,y) + d_T(z,v) = 2t, \quad d_T(x,z) + d_T(y,v) = t, \quad \text{and} \quad d_T(x,v) + d_T(y,z) = t.$$

Hence there exist no $\delta \ge 0$ such that

$$d_T(x,y) + d_T(z,v) \le \big[d_T(x,z) + d_T(y,v)\big] \vee \big[d_T(x,v) + d_T(y,z)\big] + 2\delta$$

for all $t > 0$. Finally, it follows from (4.7) that the space $(\mathbb{R}^2, d)$ is not Gromov hyperbolic, completing the proof. ∎

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