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ZERO, FINITE RANK, AND COMPACT BIG TRUNCATED HANKEL OPERATORS ON MODEL SPACES

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Abstract. In this paper, we obtain sufficient and necessary conditions for big truncated Hankel operators on model spaces to be zero, or of finite rank or compact. Our main tools are the properties of Hardy Hankel operators and function algebras.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, and let L^2 denote the space of square integrable functions on the unit circle $\mathbb{T}$ with respect to the Lebesgue measure. The Hardy space H^2 is the subspace of L^2 consisting of all analytic functions on $\mathbb{D}$. The classical Hankel operators on the Hardy space H^2 are the compositions of the orthogonal projection on the orthogonal complement of H^2 in L^2 and the multiplication operators defined by (2.1). The Hardy–Hankel operators were studied by many authors (see, e.g., [1,24]). Most results about these operators can be found in Peller's book [20]. Indeed, the Hardy–Hankel operator is of finite rank if and only if the conjugate part of its symbol is a rational function and it is compact if and only if the symbol belongs to the Sarason algebra $H^\infty+C$ [22], where H^∞ denotes the algebra of all bounded analytic functions on $\mathbb{D}$ and C denotes the collection of all continuous functions on $\mathbb{T}$.

On the Bergman space or multidimensional Hardy space, the orthogonal complement of them is much larger than the complex conjugate of these spaces. Based on this fact, one can define big Hankel operators (projecting to the orthogonal complement) and little Hankel operators (projecting to the complex conjugate). The properties of big and little Hankel operators on the Bergman space are different [26]. For example, there is no nonzero big Hankel operator with conjugate analytic symbol in the trace class, while lots of little Hankel operators with this kind of symbol can be found to be in the trace class. More results on big Hankel operators and little Hankel operators on the Bergman space are contained in [19]. The big and little Hankel operators on the multidimensional Hardy space were studied by M. Cotlar and C. Sadosky [6], S. Ferguson and C. Sadosky [8], M. Lacey [16], M. Lacey and E. Terwilleger [17], and many others.

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A function $\theta \in H^\infty$ is an inner function if its radial limits are of modulus one almost everywhere on $\mathbb{T}$. For a given nonconstant inner function θ, the model space K_θ^2 [10, 19, 21] is defined by

$$K_\theta^2 = H^2 \ominus \theta H^2,$$

which is the orthogonal complement θH^2 in H^2. Let P_θ denote the orthogonal projection from L^2 onto the model space K_θ^2. For $\varphi \in L^2$, V. Bessonov [2] defined Γ_φ, the truncated Hankel operator with symbol φ, as

$$\Gamma_\varphi f = P_{\overline{\theta}}(\varphi f), f \in K_\theta^2 \cap L^\infty,$$

where $P_{\overline{\theta}}$ denotes the projection from L^2 onto $\overline{zK_\theta^2}$, the complex conjugate of zK_θ^2. This will be called the little truncated Hankel operator to be adapted to the terminology on the Bergman space or multidimensional Hardy space. V. Bessonov showed that Γ_φ is zero if and only if $\varphi \in H^2 + \overline{\theta^2 H^2}$ and Γ_φ is compact if and only if $\varphi \in C + \overline{\theta^2 H^2} + H^2$. Some results about the finite rank and compact little truncated Hankel operators are contained in [3, 18] because of the relationship between little truncated Hankel operators and truncated Toeplitz operators (Lemma 3.3 in [2]).

Recently, C. Gu [11] defined another kind of truncated Hankel operator as the compression of the Hardy–Hankel operators to the model space K_θ^2 and proved a number of algebraic properties of them. For $\varphi \in L^2$, C. Gu defined the truncated Hankel operator B_φ with symbol φ by

$$B_\varphi f = P_\theta J(I - P)(\varphi)f, f \in K_\theta^2 \cap L^\infty,$$

where J denotes the unitary operator on L^2 defined by $Jh(z) = \overline{z}h(\overline{z})$. C. Gu proved that $B_\varphi = 0$ if and only if

$$\varphi(z) = \overline{\theta(z)}\theta(\overline{z})\overline{h(z)} + g(z), \text{ where } h, g \in H^2.$$

D. Kang and H. Kim [15] obtained a sufficient and necessary condition for the product of two such truncated Hankel operators to become a truncated Toeplitz operator when the inner function has a certain symmetric property.

Naturally, one can define the truncated Hankel operator to be the composition of the orthogonal projection on the orthogonal complement of K_θ^2 in L^2 and the multiplication operators restricted to the model space K_θ^2. We will call these operators the big truncated Hankel operators. Precisely, for $\varphi \in L^2$ we define the big truncated Hankel operator H_φ^θ with symbol φ on $K_\theta^2 \cap L^\infty$ by

$$H_\varphi^\theta f = (I - P_\theta)(\varphi f). \tag{1.1}$$

Since $K_\theta^2 \cap L^\infty$ is dense in K_θ^2, the big truncated Hankel operator H_φ^θ is densely defined on K_θ^2. Clearly, the operator H_φ^θ is bounded if $\varphi \in L^\infty$. It is easy to observe that the range of the little truncated Hankel operators defined by Bessonov is contained in $\theta H^2 \oplus \overline{zH^2}$, the range of the big truncated Hankel operators.

A natural question is to characterize the zero, finite rank, and compact big truncated Hankel operators on model spaces. In this paper we get a block decomposition for the big truncated Hankel operators in terms of Hardy Hankel operators to study this question (see equation (2.6)). Interestingly, the characterizations are different from the case of little truncated Hankel operators. Unlike the result on zero little truncated Hankel operators, there are only trivial zero big truncated Hankel operators.

Theorem 1.1. *For $\varphi \in L^\infty$, $H_\varphi^\theta = 0$ if and only if φ is a constant.*

We characterize the finite rank and compact big truncated Hankel operators which are also different from the little truncated Hankel operators.

Theorem 1.2. *For $\varphi \in L^\infty$, H_φ^θ is of finite rank if and only if either θ is a finite Blaschke product or $\varphi = q_1 + \overline{q_2}$, where q_1, q_2 are rational functions with poles outside of the closure of the unit disk.*

The characterization of compact big truncated Hankel operators needs the notion of support sets which will be defined in Section 3. Using the block decomposition of the big truncated Hankel operator and Axler–Chang–Sarason–Volberg's theorem (see Section 3), we get the following theorem to characterize compact big truncated Hankel operators.

Theorem 1.3. *For $\varphi \in L^\infty$, H_φ^θ is compact if and only if for each support set S_m, either $\theta|_{S_m}$ or $\varphi|_{S_m}$ is a constant.*

This paper is organized as follows. In Section 2, we will give some preliminaries including the block decomposition of the big truncated Hankel operators on model space in terms of Hardy Hankel operators. In Section 3, we will present the proof for these theorems and a corollary of Theorem 1.3.

2. Preliminaries

Let P denote the orthogonal projection from L^2 onto the Hardy space H^2. For $\varphi \in L^\infty$, the multiplication operator M_φ, the Hardy–Toeplitz operator T_φ, and the Hardy–Hankel operator H_φ are defined by

$$M_\varphi f = \varphi f, \; T_\varphi f = P(\varphi f), \; H_\varphi f = (I-P)(\varphi f), \; f \in H^2. \tag{2.1}$$

For $\varphi, \phi \in L^\infty$, a useful relation between the Hardy–Toeplitz operator and the Hardy–Hankel operator is given by

$$T_{\varphi\phi} = T_\varphi T_\phi + H_{\overline{\varphi}}^* H_\phi. \tag{2.2}$$

The following interesting lemma is the key to this paper. Actually, a big truncated Hankel operator can be written as the sum of two operators whose ranges are orthogonal; see formula (2.6). One part is the Hardy–Hankel operator restricted to the model space K_θ^2, and the other is the composition of the orthogonal projection from L^2 onto θH^2 and the Hardy–Toeplitz operator restricted to the model space K_θ^2.

Lemma 2.1. *For $\varphi \in L^\infty$ and $f \in K_\theta^2$, we have*

$$H_\varphi^\theta f = H_\varphi H_{\overline{\theta}}^* H_{\overline{\theta}} f + M_\theta H_{\overline{\varphi}}^* H_{\overline{\theta}} f.$$

Proof. It is well known that the orthogonal projection from L^2 onto θH^2 equals $M_\theta P M_{\overline{\theta}}$. Since $K_\theta = H^2 \ominus \theta H^2$,

$$P_\theta = P - M_\theta P M_{\overline{\theta}}. \tag{2.3}$$

Since $PM_{\overline{\theta}}|_{H^2} = T_{\overline{\theta}}$,

$$P_\theta|_{H^2} = I - T_\theta T_{\overline{\theta}} = H_{\overline{\theta}}^* H_\theta. \tag{2.4}$$

Combining (2.3) and the definition of H_φ^θ (see (1.1)) gives

$$H_\varphi^\theta f = (I - P_\theta)(\varphi f) = (I-P)M_\varphi f + M_\theta P M_{\overline{\theta}} M_\varphi f, f \in K_\theta^2.$$

Thus the first component of H_φ^θ is the Hardy–Hankel operator restricted to the model space K_θ^2. By (2.4),

$$\begin{aligned}(I-P)M_\varphi f &= (I-P)M_\varphi P_\theta f\\ &= (I-P)M_\varphi H_{\overline{\theta}}^* H_{\overline{\theta}} f\\ &= H_\varphi H_{\overline{\theta}}^* H_{\overline{\theta}} f.\end{aligned}$$

For the second part of H_φ^θ, we observe that

$$M_\theta P M_{\overline{\theta}} M_\varphi|_{K_\theta^2} = M_\theta P T_{\overline{\theta}\varphi}|_{K_\theta^2},$$

which is the composition of the orthogonal projection from L^2 into θH^2 and the Hardy–Toeplitz operator restricted to the model space K_θ^2. We will show that it is basically a Hankel-type operator. For any f in K_θ^2 and any g in H^2,

$$\langle \overline{\theta} f, g\rangle = \langle f, \theta g\rangle = 0.$$

This immediately gives

$$T_{\overline{\theta}} f = P(\overline{\theta} f) = 0 \tag{2.5}$$

for $f \in K_\theta^2$. Combining (2.2) with (2.5) gives

$$\begin{aligned}M_\theta P M_{\overline{\theta}} M_\varphi f &= M_\theta T_{\overline{\theta}\varphi} f - M_\theta T_\varphi T_{\overline{\theta}} f\\ &= M_\theta (T_{\overline{\theta}\varphi} - T_\varphi T_{\overline{\theta}}) f\\ &= M_\theta H_{\overline{\varphi}}^* H_{\overline{\theta}} f.\end{aligned}$$

Therefore,

$$H_\varphi^\theta f = H_\varphi H_{\overline{\theta}}^* H_{\overline{\theta}} f + M_\theta H_{\overline{\varphi}}^* H_{\overline{\theta}} f.$$

□

As we know, the kernel of the Hardy–Hankel operator $H_{\overline{\theta}}$ is exactly equal to θH^2. If we extend H_φ^θ on θH^2 to be a zero operator, by Lemma 2.1 and above fact, an interesting observation is that for $\varphi \in L^\infty$ the big truncated Hankel operator H_φ^θ can be viewed as a block operator

$$H_\varphi^\theta = \begin{bmatrix} H_\varphi H_{\overline{\theta}}^* H_{\overline{\theta}} \\ M_\theta H_{\overline{\varphi}}^* H_{\overline{\theta}} \end{bmatrix} : H^2 \to (K_\theta^2)^\perp = \begin{bmatrix} \overline{zH^2} \\ \oplus \\ \theta H^2 \end{bmatrix}. \tag{2.6}$$

Lemma 2.2. *For $\varphi \in L^\infty$, the big truncated Hankel operator H_φ^θ is zero (resp., of finite rank, compact) if and only if $H_{\overline{\theta}}^* H_\varphi$ and $H_{\overline{\varphi}}^* H_{\overline{\theta}}$ are zero (resp., of finite rank, compact).*

Proof. The above observation (2.6) shows that the big truncated Hankel operator H_φ^θ is zero (resp., of finite rank, compact) if and only if the operators $H_\varphi H_{\overline{\theta}}^* H_{\overline{\theta}}$ and $M_\theta H_{\overline{\varphi}}^* H_{\overline{\theta}}$ are both zero (resp., of finite rank, compact).

As in [25], define an antiunitary operator V on L^2 by

$$(Vh)(e^{i\theta}) = e^{-i\theta}\overline{h(e^{i\theta})}.$$

It was pointed out in [25] that

$$V^{-1} H_\phi V = H_\phi^*.$$

Thus we have

$$V^{-1} H_{\overline{\theta}}^* H_\varphi V = H_{\overline{\theta}} H_\varphi^* = (H_\varphi H_{\overline{\theta}}^*)^*.$$

Since $P_\theta = H^*_{\overline{\theta}} H_{\overline{\theta}}$ and $P_\theta^2 = P_\theta$, we have

$$\begin{aligned}
[H_\varphi H^*_{\overline{\theta}} H_{\overline{\theta}}][H_\varphi H^*_{\overline{\theta}} H_{\overline{\theta}}]^* &= H_\varphi H^*_{\overline{\theta}} H_{\overline{\theta}} H^*_{\overline{\theta}} H_{\overline{\theta}} H^*_\varphi \\
&= H_\varphi P_\theta P_\theta H^*_\varphi \\
&= H_\varphi P_\theta H^*_\varphi \\
&= H_\varphi H^*_{\overline{\theta}} H_{\overline{\theta}} H^*_\varphi \\
&= [H_\varphi H^*_{\overline{\theta}}][H_\varphi H^*_{\overline{\theta}}]^* \\
&= [V^{-1} H^*_{\overline{\theta}} H_\varphi V][V^{-1} H^*_{\overline{\theta}} H_\varphi V]^* \\
&= V^{-1}[H^*_{\overline{\theta}} H_\varphi][H^*_{\overline{\theta}} H_\varphi]^* (V^{-1})^*.
\end{aligned}$$

Thus $H_\varphi H^*_{\overline{\theta}} H_{\overline{\theta}}$ is zero (of finite rank, compact) if and only if the operator $H^*_{\overline{\theta}} H_\varphi$ is zero (of finite rank, compact, respectively).

On the other hand, $M_\theta H^*_{\overline{\varphi}} H_{\overline{\theta}}$ is zero (of finite rank, compact) if and only if

$$M_{\overline{\theta}} M_\theta H^*_{\overline{\varphi}} H_{\overline{\theta}} = H^*_{\overline{\varphi}} H_{\overline{\theta}}$$

is zero (of finite rank, compact, respectively).

Therefore, H^θ_φ is zero (of finite rank and compact) if and only if $H^*_{\overline{\theta}} H_\varphi$ and $H^*_{\overline{\varphi}} H_{\overline{\theta}}$ are zero (of finite rank, compact, respectively). This completes the proof. $\square$

3. The characterizations of zero, finite rank, and compactness

3.1. **Zero big truncated Hankel operators.** In 1963, A. Brown and P. Halmos [4, Theorem 8] showed that the product $T_\varphi T_\phi$ of two Hardy–Toeplitz operators is a Hardy–Toeplitz operator if and only if either φ is co-analytic or ϕ is analytic and under the condition we have $T_\varphi T_\phi = T_{\varphi\phi}$ (see also [7]). Combining the Brown–Halmos theorem and equation (2.2) gives the following lemma.

Lemma 3.1 ([4])**.** *For $\varphi, \phi \in L^\infty$, then $H^*_\varphi H_\phi = 0$ if and only if either $\varphi \in H^\infty$ or $\phi \in H^\infty$.*

Theorem 1.1 is a consequence of the fact that the multiplier of the model space is constant; see Crofoot [5]. Now we are going to present another proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.2 and Lemma 3.1, $H^\theta_\varphi = 0$ if and only if $H^*_{\overline{\theta}} H_\varphi = 0$ and $H^*_{\overline{\varphi}} H_{\overline{\theta}} = 0$ if and only if $\varphi \in H^\infty$ and $\overline{\varphi} \in H^\infty$ if and only if φ is a constant. $\square$

3.2. **Big truncated Hankel operators with finite rank.** The following theorem is the Kronecker theorem which gives a sufficient and necessary condition for a Hardy–Hankel operator to be of finite rank (see Theorem 3.1 in [20]).

Lemma 3.2 ([20])**.** *For $\varphi \in L^\infty$, H_φ is of finite rank if and only if $(I-P)\varphi$ is a rational function.*

Lemma 3.3 ([1])**.** *For $\varphi, \phi \in L^\infty$, $H^*_\varphi H_\phi$ is of finite rank if and only if either H_φ or H_ϕ is of finite rank.*

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemmas 2.2, 3.2, and 3.3, H_φ^θ is of finite rank if and only if $H_{\overline{\theta}}^*H_\varphi$ and $H_{\overline{\varphi}}^*H_{\overline{\theta}}$ are both of finite rank if and only if either $H_{\overline{\theta}}$ is of finite rank or H_φ and $H_{\overline{\varphi}}$ are both of finite rank if and only if either θ is a finite Blaschke product or $\varphi = q_1 + \overline{q_2}$, where q_1, q_2 are rational functions with poles outside of the closure of the unit disk. □

3.3. **Compact big truncated Hankel operators.** We will need to make extensive use of the *maximal ideal space* of H^∞. The Gelfand space (the space of nonzero multiplicative linear functionals) of the commutative Banach algebra B will be denoted by $M(B)$. Let m be in the maximal ideal space $M(H^\infty)$. We can identify m with a multiplicative linear functional on H^∞. Moreover, the Gleason–Whitney theorem [7] says that m extends uniquely to a bounded positive linear functional l_m on L^∞. By the Riesz representation theorem, there is a measure $d\mu_m$ called the representing measure for m with support S_m, which is a subset of the maximal ideal space of L^∞ such that

$$l_m(f) = \int_{S_m} f d\mu_m;$$

for more details, see page 181 in [13]. A subset of $M(L^\infty)$ will be called a support set if it is the (closed) support of the representing measure for a functional in $M(H^\infty + C)$. For more details on H^∞ and L^∞ and their maximal ideal spaces, see [9].

S. Axler, S.-Y. Chang, D. Sarason [1], and A. Vol′berg [24] proved the following remarkable result.

Lemma 3.4 ([1])**.** *For $\varphi, \phi \in L^\infty$, $H_{\overline{\varphi}}^*H_\phi$ is compact if and only if for each support set S_m, either $\overline{\varphi}|_{S_m}$ or $\phi|_{S_m}$ is in $H^\infty|_{S_m}$.*

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 2.2, we have that H_φ^θ is compact if and only if $H_{\overline{\theta}}^*H_\varphi$ and $H_{\overline{\varphi}}^*H_{\overline{\theta}}$ are compact. By Lemma 3.4, we have that H_φ^θ is compact if and only if for each support set S_m, either $\overline{\theta}|_{S_m}$ or $\varphi|_{S_m}$ is in $H^\infty|_{S_m}$ and either $\overline{\theta}|_{S_m}$ or $\overline{\varphi}|_{S_m}$ is in $H^\infty|_{S_m}$. This is equivalent to the fact that for each support set S_m, either $\overline{\theta}|_{S_m}$ is in $H^\infty|_{S_m}$ or $\varphi|_{S_m}$ and $\overline{\varphi}|_{S_m}$ are in $H^\infty|_{S_m}$. Since a support set S_m is an antisymmetric set of L^∞ [14], we have that if $\varphi|_{S_m}$ and $\overline{\varphi}|_{S_m}$ are in $H^\infty|_{S_m}$, then $\varphi|_{S_m}$ is a constant, and if $\overline{\theta}|_{S_m}$ is in $H^\infty|_{S_m}$, then $\theta|_{S_m}$ is a constant. This completes the proof. □

For $\varphi \in L^\infty$, Hartman [12] proved that H_φ is compact if and only if $\varphi \in H^\infty + C$ and Sarason [23] proved that $\varphi \in H^\infty + C$ if and only if for each support set S_m, $\varphi|_{S_m} \in H^\infty|_{S_m}$. Thus ϕ is in QC if and only if for each support set S_m, $\phi|_{S_m}$ is a constant where QC denotes $(H^\infty + C) \cap \overline{(H^\infty + C)}$. Theorem 1.3 immediately gives the following corollary.

Corollary 3.5. *If $\varphi \in QC$, then H_φ^θ is compact.*

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