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ISOTROPIC MEASURES AND MAXIMIZING ELLIPSOIDS: BETWEEN JOHN AND LOEWNER

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Abstract. We define a one-parametric family of positions of a centrally symmetric convex body K which interpolates between the John position and the Loewner position: for $r > 0$, we say that K is in *maximal intersection position of radius* r if $\mathrm{Vol}_n(K \cap rB_2^n) \geq \mathrm{Vol}_n(K \cap rTB_2^n)$ for all $T \in \mathrm{SL_n}$. We show that under mild conditions on K, each such position induces a corresponding isotropic measure on the sphere, which is simply the normalized Lebesgue measure on $r^{-1}K \cap S^{n-1}$. In particular, for r_M satisfying $r_M^n \kappa_n = \mathrm{Vol}_n(K)$, the maximal intersection position of radius r_M is an M-position, so we get an M-position with an associated isotropic measure. Lastly, we give an interpretation of John's theorem on contact points as a limit case of the measures induced from the maximal intersection positions.

1. Introduction and main results

Given a convex body (that is, a compact convex set with non-empty interior) K in $\mathbb{R}^n$, the *John ellipsoid* $J(K)$ is the maximum-volume ellipsoid contained in K. The body K is in *John position* if $J(K) = B_2^n$, the Euclidean unit ball. Dually, the *Loewner ellipsoid* $L(K)$ is the minimum-volume ellipsoid containing K, and K is in *Loewner position* if $L(K) = B_2^n$. Every convex body has a John and a Loewner position, and these positions are unique up to orthogonal transformations. They are dual in the sense that if $0 \in \mathrm{int}K$, then $J(K^\circ) = (L(K))^\circ$, where $A^\circ = \{y : \langle x, y\rangle \leq 1 \ \forall x \in A\}$ is the *dual body* of A (see [1] for more details).

A finite Borel measure μ on S^{n-1} is called *isotropic* if

$$\int_{S^{n-1}} \langle x, \theta\rangle^2 \, d\mu(x) = \frac{\mu(S^{n-1})}{n}$$

for all $\theta \in S^{n-1}$. In 1948, Fritz John [10] showed the following:

Theorem 1.1 (John)**.** *Let $K \subset \mathbb{R}^n$ be a convex body in John position. Then there exists an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$. Moreover, there exists such a measure whose support is at most $n(n+3)/2$ points.*

A reverse result was given by K. Ball [2], who showed that if $B_2^n \subseteq K$ and there is an isotropic measure supported on $\partial K \cap S^{n-1}$, then K is in John position. By duality, the same result holds for a body in Loewner position.

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It turns out that John's theorem is a special case of a general phenomenon. Given a convex body, one may consider the family of *positions* of K, given by $\{TK : T \in \mathrm{SL}_n\}$. Giannopoulos and Milman [8] showed that solutions to extremal problems over the positions of a convex body often give rise to isotropic measures and demonstrated this fact for, among others, the John position, the isotropic position, the minimal surface area position, and an M-position. John-type results were also extended to the general case, where ellipsoids are replaced by affine transformations of a convex body; see [9]. For more information we refer to the books [5] and [1].

In this work, we consider a one-parametric family of extremal positions which seems not to have been considered before:

Definition 1.2. For a centrally symmetric convex body $K \subset \mathbb{R}^n$, the ellipsoid $\mathcal{E}_r$ of volume $r^n\kappa_n$ is a *maximum intersection ellipsoid of volume-radius* r if

$$\mathrm{Vol}_n\left(K \cap \mathcal{E}_r\right) \geq \mathrm{Vol}_n\left(K \cap \mathcal{E}\right)$$

for all ellipsoids $\mathcal{E}$ of volume $r^n\kappa_n$, where $\kappa_n = \mathrm{Vol}_n(B_2^n)$. We say that K is in *maximal intersection position of radius* r if rB_2^n is a maximum intersection ellipsoid of (volume)-radius r.

In the following, $\mathcal{E}_r$ will always denote a maximum intersection ellipsoid of volume-radius r of K, a centrally symmetric convex body. The set of maximal intersection positions interpolates between the John and Loewner positions: indeed, let r_J be a positive number satisfying $\mathrm{Vol}_n(J(K)) = r_J^n\kappa_n$, and let r_L be such that $\mathrm{Vol}_n(L(K)) = r_L^n\kappa_n$. Then K is in maximal intersection position of radius r_J if and only if $r_J^{-1}K$ is in John position, and similarly for the Loewner position. In other words, up to a scaling, the maximal intersection position of radius r_J is the John position, and the maximal intersection position of radius r_L is the Loewner position.

Our first result is the following:

Theorem 1.3. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body such that* $\mathrm{Vol}_{n-1}(\partial K \cap rS^{n-1}) = 0$ *and* $\mathrm{Vol}_{n-1}(K \cap rS^{n-1}) > 0$. *If K is in maximal intersection position of radius r, then the restriction of the surface area measure on the sphere to $S^{n-1} \cap r^{-1}K$ is an isotropic measure.*

Remark 1.4. Note that the condition $\mathrm{Vol}_{n-1}(\partial K \cap rS^{n-1}) = 0$ cannot be omitted. As an example, consider the convex hull of a ball and two points, e.g., $K = \mathrm{conv}\{B_2^2 \cup (\pm\sqrt{2}, 0)\} \subset \mathbb{R}^2$. Here one may check that K is in John position, and so it is in maximal intersection position of radius 1. However, the restriction of the surface area measure to $K \cap S^{n-1}$ is clearly not isotropic, as it has more weight in the direction of the y axis than in the direction of the x axis.

We will denote the surface area measure on the sphere by σ, and for a Borel set $A \subset \mathbb{R}^n$ with $\sigma(A \cap S^{n-1}) > 0$ we let μ_A denote the restriction of σ to A, i.e.,

$$\mu_A(B) = \frac{\sigma(B \cap A \cap S^{n-1})}{\sigma(A \cap S^{n-1})}.$$

Note that if μ_A is isotropic and $\sigma(S^{n-1} \backslash A) > 0$, then $\mu_{S^{n-1}\backslash A}$ is isotropic as well. Theorem 1.3 shows that as in [8], an extremal position induces an isotropic measure. Contrary to John's Theorem 1.1, in our case we have an explicit description of the isotropic measure, which is uniform on $r^{-1}K \cap S^{n-1}$; namely, it is $\mu_{r^{-1}K}$.

Theorem 1.3 does not formally include the result of Theorem 1.1 in the case $r = r_J = 1$, since for K in John position we have $S^{n-1} \subset K$, so Theorem 1.3 merely states that σ is isotropic, a triviality. Nevertheless, our second result gives a new interpretation to John's theorem. We show that when K is in John position, the isotropic measure which is guaranteed to exist by Theorem 1.1 may be constructed as a limit of the isotropic measures from Theorem 1.3. In other words, as r approaches r_J, the corresponding induced measures approach a measure of the type described in John's theorem:

Theorem 1.5. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. For every $r > 1$, denote by μ_r the uniform probability measure on $S^{n-1} \setminus r^{-1}T_rK$, where T_rK is in maximal intersection position of radius r. Then there exists a sequence $r_j \searrow 1$ such that the sequence of measures μ_{r_j} weakly converges to an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$.*

A similar result holds for the Loewner position:

Theorem 1.6. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. For every $r < 1$, denote by ν_r the uniform probability measure on $S^{n-1} \cap r^{-1}T_rK$, where T_rK is in maximal intersection position of radius r. Then there exists a sequence $r_j \nearrow 1$ such that the sequence of measures ν_{r_j} weakly converges to an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$.*

In the range $[r_J, r_L]$ there is a special radius which we denote r_M, defined so that $\mathrm{Vol}_n(K) = r_M^n \kappa_n$, and for this special radius the maximal intersection position of radius r_M is an M-position. To explain what this means we need a few more definitions and background.

In the mid-1980s, V. Milman [12] discovered the existence of a position for convex bodies which enabled him, and the researchers following, to prove many new results and which had a major influence on the field. This position, now called M-position, can be described in many different and equivalent ways. We choose one such way; for an extensive description and the many equivalences see [1].

Theorem 1.7 (Milman)**.** *There exists a universal constant $C > 0$ such that for every $n \in \mathbb{N}$ and any centrally symmetric convex body $K \subset \mathbb{R}^n$, there exists a centrally symmetric ellipsoid $\mathcal{E}$ with $\mathrm{Vol}_n(\mathcal{E}) = \mathrm{Vol}_n(K)$ such that*

$$\frac{\mathrm{Vol}_n(K^\circ + \mathcal{E}^\circ)}{\mathrm{Vol}_n(K^\circ \cap \mathcal{E}^\circ)} \frac{\mathrm{Vol}_n(K + \mathcal{E})}{\mathrm{Vol}_n(K \cap \mathcal{E})} \leq C^n. \tag{1.1}$$

In fact, one may show that if an ellipsoid of the same volume as K satisfies any of the four inequalities

$$\mathrm{Vol}(K^\circ + \mathcal{E}^\circ) \leq c_1^n \,\mathrm{Vol}_n(K), \qquad \mathrm{Vol}_n(K^\circ \cap \mathcal{E}^\circ) \geq c_1^{-n} \,\mathrm{Vol}_n(K),$$
$$\mathrm{Vol}(K + \mathcal{E}) \leq c_1^n \,\mathrm{Vol}_n(K), \qquad \mathrm{Vol}_n(K \cap \mathcal{E}) \geq c_1^{-n} \,\mathrm{Vol}_n(K),$$

then it must satisfy inequality (1.1) with some constant $C = C(c_1)$ depending only on c_1 and not on the body K or on the dimension. For this reason, we shall use the following simple definition for M-position:

Definition 1.8. A centrally symmetric convex body K is in *M-position* with constant C if the centrally symmetric Euclidean ball of radius $\lambda = \left(\frac{\mathrm{Vol}(K)}{\kappa_n}\right)^{1/n}$ satisfies

$$\mathrm{Vol}_n(K \cap \lambda B_2^n) \geq C^{-n} \,\mathrm{Vol}_n(K).$$

Since Milman's theorem implies that there exists some universal C for which any body has an affine image in M-position with constant C, we shall usually omit the words "with constant C" and mention simply an "M-position", by which we mean an M-position with respect to the constant C guaranteed by Milman's Theorem 1.7.

Clearly, when we maximize the volume of the intersection of K and an ellipsoid of volume $\mathrm{Vol}_n(K)$, we get an M-ellipsoid, and when it is a Euclidean ball we get that K is in M-position. We have then:

Corollary 1.9. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body such that $\mathrm{Vol}_{n-1}(\partial K \cap r_M S^{n-1}) = 0$, where $r_M = \left(\frac{\mathrm{Vol}(K)}{\kappa_n}\right)^{1/n}$. If K is in maximal intersection position of radius r_M, then K is in M-position, and the restriction of the surface area measure on the sphere to $S^{n-1} \cap r_M^{-1} K$ is an isotropic measure.*

This paper is organized as follows: in Section 2 we provide some basic results regarding the maximal intersection position. The section concludes with a detailed proof of the main ingredient for the proof of Theorem 1.3. In Section 3 we prove Theorems 1.3, 1.5, and 1.6. In the last section we discuss briefly the question of uniqueness of the maximum intersection position, a question which remains open. We show that uniqueness would follow from a strong variant of the (B) conjecture.

2. Preliminaries

In this section we provide some results needed for the proof of the main theorems. We start by showing that for $r > 0$, the maximal intersection position of radius r does in fact exist. We will make frequent use of the following function:

Definition 2.1. For a centrally symmetric convex body $K = -K \subset \mathbb{R}^n$, define for every $r > 0$,

$$m(r) = \sup \left\{ \mathrm{Vol}_n(K \cap \mathcal{E}) : \mathcal{E} \text{ is an ellipsoid of volume } r^n \kappa_n \right\}. \tag{2.1}$$

Our first lemma shows that a maximal intersection ellipsoid always exists:

Lemma 2.2. *For every centrally symmetric convex body $K \subset \mathbb{R}^n$ and every $r > 0$, the supremum in* (2.1) *is attained.*

Proof. First note that since $K = -K$, the Brunn-Minkowski inequality implies that for every $x \in \mathbb{R}^n$ and every $T \in \mathrm{SL}_n$, we have

$$\mathrm{Vol}_n\left(K \cap (TB_2^n + x)\right) \leq \mathrm{Vol}_n\left(K \cap TB_2^n\right), \tag{2.2}$$

and so if the supremum is attained, it is attained on a centered ellipsoid. Note that the supremum may also be attained on a non-centered ellipsoid only if we have equality in (2.2), which is only possible if $K \cap (TB_2^n + x)$ and $K \cap (TB_2^n - x)$ are homothetic. This occurs, for instance, in the case $(TB_2^n + x) \subset K$ or $K \subset (TB_2^n + x)$, i.e., when $r < r_J$ or $r > r_L$.

Let $\mathcal{E}_j = T_j B_2^n$ be a sequence of centrally symmetric ellipsoids where T_j is positive definite with $\det T_j = r^n$ and $\mathrm{Vol}_n(K \cap T_j B_2^n) \to m(r)$. If the sequence of operator norms $\|T_j\|_{op}$ grows to infinity, then, as $\det(T_j)$ is fixed, we have $\mathrm{Vol}_n(K \cap T_j B_2^n) \to 0 \neq m(r)$, so the set of eigenvalues of $\{T_j\}_{j=1}^{\infty}$ must be bounded, which implies that the ellipsoids $T_j B_2^n$ are all contained in a compact set. It now follows from Blaschke's selection theorem that there exists a subsequence of ellipsoids converging in the Hausdorff distance to a centered ellipsoid $\mathcal{E}$ of volume $r^n\kappa_n$, and, since the map $T \mapsto \mathrm{Vol}_n(K \cap TB_2^n)$ is continuous on SL_n, we have $\mathrm{Vol}_n(K \cap \mathcal{E}) = m(r)$. □

It will be useful to note the following simple properties of $m(r)$:

Lemma 2.3. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. Then*

(1) *For $0 < r \le r_J$ we have $m(r) = r^n\kappa_n$ and for $r \ge r_L$ we have $m(r) = \mathrm{Vol}_n(K)$.*

(2) *The function $m(r)$ is strictly monotone increasing in $[r_J, r_L]$.*

(3) *The function $m(r)$ is continuous and moreover satisfies for $t \le s$ that*

$$m(t) \le m(s) \le \left(\frac{s}{t}\right)^n m(t).$$

Proof. Fact (1) is trivial. For (2) let $r_J \le t < s \le r_L$ and let $\mathcal{E}_t$ be a maximum intersection ellipsoid of volume radius t . Then

$$m(t) = \mathrm{Vol}_n(K \cap \mathcal{E}_t) \le \mathrm{Vol}_n\left(K \cap \frac{s}{t}\mathcal{E}_t\right) \le \mathrm{Vol}_n\left(K \cap \mathcal{E}_s\right).$$

If the last inequality is an equality, then $K \cap \mathcal{E}_t = K \cap \frac{s}{t}\mathcal{E}_t$, which is only possible if $K \subset \mathcal{E}_t$ (which is impossible since $t < r_L$) or if $\frac{s}{t}\mathcal{E}_t \subset K$ (which is impossible since $s > r_J$).

To prove (3) it is enough to show the right hand side inequality and to this end simply note that

$$\begin{aligned} m(t) &= \mathrm{Vol}_n(K \cap \mathcal{E}_t) \ge \mathrm{Vol}_n\left(K \cap \frac{t}{s}\mathcal{E}_s\right) \\ &\ge \mathrm{Vol}_n\left(\frac{t}{s}K \cap \frac{t}{s}\mathcal{E}_s\right) = \left(\frac{t}{s}\right)^n \mathrm{Vol}_n\left(K \cap \mathcal{E}_s\right) = \left(\frac{t}{s}\right)^n m(s). \end{aligned}$$

□

By continuity of $m(r)$, we have:

Lemma 2.4. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. As $r \searrow r_J$ the ellipsoids $\mathcal{E}_r$ converge to $\mathcal{E}_{r_J} = J(K)$ in the Hausdorff distance.*

Proof. Since $\mathrm{Vol}_n(K \cap J(K)) = \mathrm{Vol}_n(J(K))$ then by the continuity of $m(r)$, both $\mathrm{Vol}_n(K \cap \mathcal{E}_r)$ and $\mathrm{Vol}_n(\mathcal{E}_r)$ approach $m(r_J) = r_J^n\kappa_n$ as $r \searrow r_J$. Let T_r be a sequence of transformations such that $T_r\mathcal{E}_r = B_2^n$. As in the proof of Lemma 2.3, since $\mathrm{Vol}_n(K \cap T_r^{-1}B_2^n) \to m(r_J)$ then the set $\mathcal{E}_r$ is contained in a compact set. We thus have a converging subsequence $\mathcal{E}_{r_j} \to \mathcal{E}$ with $\mathrm{Vol}_n(\mathcal{E}) = \mathrm{Vol}_n(K \cap \mathcal{E}) = r_J^n\kappa_n$, so $\mathcal{E}$ is an ellipsoid contained in K with the same volume as $J(K)$, which is unique. It follows that $\mathcal{E} = J(K)$. Since this was true for any converging subsequence, we get that $\mathcal{E}_r$ converges to $J(K)$ as $r \searrow r_J$. □

We will make use of the following fact. The proof is a simple exercise; see e.g., [1, Lemma 2.1.13]:

Lemma 2.5. *A Borel measure μ on S^{n-1} is isotropic if and only if every $A \in M_n(\mathbb{R})$ with $\mathrm{Tr}(A) = 0$ has*

$$\int_{S^{n-1}} \langle x, Ax\rangle \, d\mu(x) = 0.$$

The following theorem is the main ingredient in the proof of Theorem 1.3:

Theorem 2.6. *Let $K \subset \mathbb{R}^n$ be a convex body such that* $\mathrm{Vol}_{n-1}(\partial K \cap S^{n-1}) = 0$ *and* $\mathrm{Vol}_{n-1}(K \cap S^{n-1}) > 0$*. Let $A \in M_n(\mathbb{R})$ with $\mathrm{Tr}(A) = 0$, and let $V(t) : \mathbb{R} \to \mathbb{R}$ be defined by $V(t) = \mathrm{Vol}_n(K \cap e^{tA} B_2^n)$. The derivative of $V(t)$ satisfies*

$$\left.\frac{dV(t)}{dt}\right|_{t=0} = \int_{S^{n-1}\cap K} \langle x, Ax\rangle \, d\sigma(x),$$

where $\sigma = \mathrm{Vol}_{n-1}$ is the Lebesgue surface area measure on the sphere.

We will see in the next section that Theorem 1.3 is almost a direct corollary of Theorem 2.6. However, Remark 1.4 shows that some caution is needed, and the assumption $\mathrm{Vol}_{n-1}(\partial K \cap S^{n-1}) = 0$ should be used. Therefore, while the following proof is basically a direct application of fundamental results in calculus, we provide it in detail.

Proof of Theorem 2.6. Let $\{\varphi_j\}_{j=1}^{\infty}$ be a sequence of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ which approximate the indicator of K:

$$\varphi_j(x) = \begin{cases} 1 & ||x||_K \le 1 - \frac{1}{j}, \\ g_j(x) & 1 - \frac{1}{j} \le ||x||_K \le 1, \\ 0 & ||x||_K \ge 1, \end{cases}$$

where $g_j(x) : \mathbb{R}^n \to [0,1]$ is chosen so that $\varphi_j(x)$ is continuously differentiable. We have then

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} V(t) &= \left.\frac{d}{dt}\right|_{t=0} \int_{B_2^n} \mathbf{1}_{\mathrm{int}K}\left(e^{-tA}x\right) dx \\ &= \left.\frac{d}{dt}\right|_{t=0} \int_{B_2^n} \lim_{j\to\infty} \varphi_j\left(e^{-tA}x\right) dx. \end{aligned}$$

We will show that the following hold in a neighborhood of $t = 0$:

$$\int_{B_2^n} \lim_{j\to\infty} \varphi_j\left(e^{-tA}x\right) dx = \lim_{j\to\infty} \int_{B_2^n} \varphi_j\left(e^{-tA}x\right) dx, \tag{2.3}$$

$$\frac{d}{dt}\lim_{j\to\infty} \int_{B_2^n} \varphi_j\left(e^{-tA}x\right) dx = \lim_{j\to\infty} \frac{d}{dt} \int_{B_2^n} \varphi_j\left(e^{-tA}x\right) dx, \tag{2.4}$$

$$\frac{d}{dt} \int_{B_2^n} \varphi_j\left(e^{-tA}x\right) dx = \int_{B_2^n} \left\langle \nabla\varphi_j(x), -Ae^{-tA}x\right\rangle dx, \tag{2.5}$$

$$\begin{aligned} \lim_{j\to\infty} \int_{B_2^n} \left\langle \nabla\varphi_j(x), -Ae^{tA}x\right\rangle dx &= \int_{S^{n-1}\cap K} \left\langle x, Ae^{-tA}x\right\rangle d\sigma \\ &+ \int_{B_2^n\cap K} \mathrm{Tr}(Ae^{-tA}) dx. \end{aligned} \tag{2.6}$$

Setting $t = 0$ in the equalities above proves the theroem.

The equality (2.3) is a direct consequence of Lebesgue's dominated convergence theorem, and (2.5) follows from Leibniz's integral rule. To prove (2.4) and (2.6), we will show the following:

Proposition 2.7. *There is a neighborhood of $t = 0$ where the function*

$$\frac{d}{dt} \int_{B_2^n} \varphi_j\left(e^{-tA}x\right) dx = \int_{B_2^n} \left\langle \nabla\varphi_j(x), -Ae^{-tA}x\right\rangle dx$$

converges uniformly to

$$\int_{B_2^n \cap K} \operatorname{Tr}(Ae^{-tA})dx + \int_{S^{n-1} \cap K} \left\langle x, Ae^{-tA}x \right\rangle d\sigma.$$

Then, we may exchange limit and derivative and arrive at the needed conclusion. To this end, integrate by parts to get

$$\int_{B_2^n} \left\langle \nabla\varphi_j(x), -Ae^{-tA}x \right\rangle dx \tag{2.7}$$

$$= \int_{B_2^n} \varphi_j(x) \operatorname{Tr}(Ae^{-tA})dx + \int_{S^{n-1}} \varphi_j(x) \left\langle x, -Ae^{-tA}x \right\rangle d\sigma. \tag{2.8}$$

The first term in (2.8) satisfies

$$\left| \int_{B_2^n} \varphi_j(x) \operatorname{Tr}(Ae^{-tA})dx - \int_{B_2^n \cap K} \operatorname{Tr}(Ae^{-tA})dx \right| \tag{2.9}$$

$$= \left| \operatorname{Tr}(Ae^{-tA}) \right| \left| \int_{B_2^n} \left(\varphi_j(x) - \mathbf{1}_K(x) \right) dx \right|. \tag{2.10}$$

Since $\left|\operatorname{Tr}(Ae^{-tA})\right|$ is bounded in a neighborhood of $t = 0$, the sequence

$$\int_{B_2^n} \varphi_j(x) \operatorname{Tr}(Ae^{-tA})dx$$

converges uniformly to $\int_{B_2^n \cap K} \operatorname{Tr}(Ae^{-tA})dx$. As for the second term in (2.8):

$$\left| \int_{S^{n-1}} \varphi_j(x) \left\langle x, -Ae^{-tA}x \right\rangle d\sigma - \int_{S^{n-1} \cap K} \left\langle x, -Ae^{-tA}x \right\rangle d\sigma \right| \tag{2.11}$$

$$= \left| \int_{S^{n-1}} \varphi_j(x) \left\langle x, -Ae^{-tA}x \right\rangle d\sigma - \int_{S^{n-1}} \mathbf{1}_K(x) \left\langle x, -Ae^{-tA}x \right\rangle d\sigma \right| \tag{2.12}$$

$$= \left| \int_{S^{n-1}} \left(\varphi_j(x) - \mathbf{1}_K(x) \right) \left\langle x, -Ae^{-tA}x \right\rangle d\sigma \right|. \tag{2.13}$$

There is $c > 0$ such that, in a neighborhood of $t = 0$, for every $x \in S^{n-1}$, we have $\left|\left\langle x, -Ae^{-tA}x \right\rangle\right| \leq c$. Furthermore, denote

$$M_j = \{x : 1 - \frac{1}{j} \leq ||x||_K \leq 1\} \supset \operatorname{supp}\left(\varphi_j(x) - \mathbf{1}_K(x) \right).$$

Then

$$\left| \int_{S^{n-1}} \left(\varphi_j(x) - \mathbf{1}_K(x) \right) \left\langle x, -Ae^{-tA}x \right\rangle d\sigma \right| \tag{2.14}$$

$$\leq c \int_{S^{n-1}} \left| \varphi_j(x) - \mathbf{1}_K(x) d\sigma \right| \leq c \operatorname{Vol}_{n-1}(S^{n-1} \cap M_j). \tag{2.15}$$

But

$$c \operatorname{Vol}_{n-1}(S^{n-1} \cap M_j) \to c \operatorname{Vol}_{n-1}(S^{n-1} \cap \partial K) = 0,$$

and so the sequence $\int_{S^{n-1}} \varphi_j(x) \left\langle x, -Ae^{-tA}x \right\rangle d\sigma$ converges uniformly to

$$\int_{S^{n-1} \cap K} \left\langle x, -Ae^{-tA}x \right\rangle d\sigma = \int_{S^{n-1} \cap K} \left\langle x, Ae^{-tA}x \right\rangle d\sigma$$

since K is centrally symmetric. Thus Proposition 2.7 is proved and with it Theorem 2.6. □

3. Proof of the main theorems

In this section we use the results of Section 2 to provide short proofs to the three main Theorems 1.3, 1.5, and 1.6.

As we mentioned, the proof of Theorem 1.3 follows almost directly from Theorem 2.6:

Proof of Theorem 1.3. First note that K is in maximal intersection position of radius r if and only if $r^{-1}K$ is in maximal intersection position of radius 1, and so it is enough to prove the theorem in the case $r = 1$.

Let $W : \mathrm{SL}_n \to \mathbb{R}$, $W(T) = \mathrm{Vol}_n(K \cap TB_2^n)$. If I_n is a local maximum of W, then for any $A \in M_n(\mathbb{R})$ such that $\mathrm{tr}A = 0$, the derivative $\frac{dW(e^{tA})}{dt}\Big|_{t=0} = \frac{dV(t)}{dt}\Big|_{t=0}$ is either zero or does not exist. Theorem 2.6 states that the derivative does exist for all A, and it equals $\int_{S^{n-1}\cap K} \langle x, Ax\rangle \, d\sigma(x)$. It follows that

$$\int_{S^{n-1}} \langle x, Ax\rangle \, d\mu_K = \frac{1}{\mathrm{Vol}_{n-1}(S^{n-1}\cap K)} \int_{S^{n-1}\cap K} \langle x, Ax\rangle \, d\sigma = 0$$

for all A such that $\mathrm{tr}A = 0$, and by Lemma 2.5, μ_K is isotropic. □

As we have mentioned, the result of Theorem 1.3 resembles that of John's theorem (Theorem 1.1), but does not include it. However, Theorem 1.3 provides a family of isotropic measures, the limit of which is a John-type measure.

Proof of Theorem 1.5. Let $r \searrow 1$. By Lemma 2.2, we may choose an intersection maximizing ellipsoid $\mathcal{E}_r$ for each r. By Lemma 2.4, $\mathcal{E}_r \to B_2^n$, and so we may choose a sequence of positive definite transformations $T_r \to I_n$ such that $B_2^n = T_r\mathcal{E}_r$. Then T_rK is in maximal intersection position of radius r and $\mathrm{Vol}_{n-1}(\partial T_rK \cap S^{n-1}) = 0$ for almost all r. By Theorem 1.3, the probability measures on the sphere

$$\mu_r(A) = \mu_{S^{n-1}\setminus T_rK}(A) = \frac{\sigma\left(A\setminus T_rK\right)}{\sigma\left(S^{n-1}\setminus T_rK\right)}$$

are isotropic.

Note that S^{n-1} is a compact metric space, and so the family of measures μ_r has a weakly converging subsequence $\mu_j \to \mu$ where μ is a probability measure on S^{n-1}. We will show that the limit measure μ is an isotropic measure whose support lies in $\partial K \cap S^{n-1}$.

First, weak convergence implies that

$$\int_{S^{n-1}} \langle x, \theta\rangle^2 \, d\mu_j(x) \to \int_{S^{n-1}} \langle x, \theta\rangle^2 \, d\mu(x)$$

and

$$\frac{1}{n} = \frac{\mu_j(S^{n-1})}{n} \to \frac{\mu(S^{n-1})}{n},$$

so for every θ we have $\int_{S^{n-1}} \langle x, \theta\rangle^2 \, d\mu(x) = \frac{\mu(S^{n-1})}{n} = \frac{1}{n}$; i.e., μ is isotropic.

Second, let

$$U_k = \left\{ x \in S^{n-1} : d(x, \partial K) > \frac{1}{k} \right\},$$

where $d(\cdot,\cdot)$ is a metric on S^{n-1}. The measure μ_j is supported on $S^{n-1}\setminus T_{r_j}K$ where $T_{r_j}K \to K$, and so there is M such that for any $k > M$ there is some $N(k)$ such that $\mu_j(U_k) = 0$ for all $j > N(k)$. Since U_k is open, weak convergence implies that $\mu(U_k) \leq \liminf \mu_j(U_k) = 0$, so $\mu(U_k) = 0$ for all $k > M$. It follows that $\mu\left(\bigcup_{k=M}^{\infty} U_k\right) = \lim_{k\to\infty} \mu(U_k) = 0$, where

$$\bigcup_{k=M}^{\infty} U_k = \{x \in S^{n-1} : d(x, \partial K) > 0\} = S^{n-1}\backslash \mathrm{cl}\partial K = S^{n-1}\backslash \partial K.$$

It follows that $\mu(S^{n-1}\backslash\partial K) = 0$, and so $\mathrm{supp}\mu \subset S^{n-1} \cap \partial K$. □

The proof of Theorem 1.6 is analogous to that of Theorem 1.5, only here we use

$$\nu_j(A) = \mu_{T_{r_j}K}(A) = \frac{\sigma\left(A \cap T_{r_j}K\right)}{\sigma\left(S^{n-1} \cap T_{r_j}K\right)},$$

which is isotropic by Theorem 1.3. In this case the measures ν_j satisfy $\nu_j(U_k) = 0$ for all $j > N(k)$. In other words, for a John-type measure we use a sequence of uniform measures "outside" $T_{r_j}K$, whereas for a Loewner-type measure we use a sequence of uniform measures "inside" $T_{r_j}K$.

4. Remarks about uniqueness following from the (B) property

We end this note with a short discussion of the possible uniqueness of the maximal intersection positions of a body K. If $0 < r < r_J$ or $r > r_L$, then the maximum intersection ellipsoid $\mathcal{E}_r$ of volume-radius r is clearly not unique. If $r = r_J$ or $r = r_L$, then $\mathcal{E}_r$ is unique by John's theorem. The question of uniqueness remains open for the case $r_J < r < r_L$, but it is implied by a variant of a well-known conjecture which we next discuss:

Conjecture 4.1. For a centrally symmetric convex body $K \subset \mathbb{R}^n$ and a diagonal $n \times n$ matrix Λ, the function

$$\phi(t) = \mathrm{Vol}_n\left(e^{t\Lambda}K \cap B_2^n\right)$$

is log-concave in t; i.e.,

$$\mathrm{Vol}_n\left(e^{\frac{t}{2}\Lambda}K \cap B_2^n\right)^2 \geq \mathrm{Vol}_n\left(e^{t\Lambda}K \cap B_2^n\right)\mathrm{Vol}_n\left(K \cap B_2^n\right) \tag{4.1}$$

for all $t \in \mathbb{R}$ and all diagonal Λ. Furthermore, equality is attained if and only if one of the following holds: $K \subset B_2^n$, $B_2^n \subset K$, or $\Lambda = \lambda I_n$ for some $\lambda \in \mathbb{R}$.

Proposition 4.2. *Assuming Conjecture* 4.1 *is true, if K is a centrally symmetric convex body, the maximum intersection ellipsoid of radius r is unique for $r_J < r < r_L$.*

Proof. Letting $r_J < r < r_L$, assume there are two distinct maximum intersection ellipsoids of radius r. By changing K linearly, we may assume that one of these ellipsoids is B_2^n, and the other is of the form $e^{\Lambda}B_2^n$, where Λ is a diagonal matrix with $\mathrm{tr}\Lambda = 0$. Conjecture 4.1 now gives

$$\mathrm{Vol}_n\left(K \cap e^{\frac{\Lambda}{2}}B_2^n\right) \geq \mathrm{Vol}_n\left(K \cap B_2^n\right),$$

where maximality of B_2^n implies equality in the above. Since $r_J < r < r_L$, we have $K \not\subseteq B_2^n$ and $B_2^n \not\subseteq K$. It follows that Λ is a traceless scalar matrix; i.e., Λ is the zero matrix and $e^{\Lambda} = I_n$. □

Conjecture 4.1 describes a (B)-type property on the Lebesgue measure on B_2^n under the following terminology:

Definition 4.3. Given a measure μ on $\mathbb{R}^n$ and a measurable set $K \subset \mathbb{R}^n$, we say that μ and K have *the weak* (B) *property* if the function

$$t \mapsto \mu(e^t K)$$

is log-concave on $\mathbb{R}$.

Denoting by $\mathrm{diag}(t_1, \ldots, t_n)$ the diagonal matrix with diagonal entries $t_1, \ldots, t_n$, we will say that μ and K have *the strong* (B) *property* if the function

$$(t_1, \ldots, t_n) \mapsto \mu(e^{\mathrm{diag}(t_1,\ldots,t_n)} K)$$

is log-concave on $\mathbb{R}^n$.

The notion of the (B) property arises from a problem proposed by Banaszczyk and described by Latala [11] known as the (B) conjecture (now the (B) theorem), where, in the terminology as above, it was conjectured that the standard Gaussian probability measure γ on $\mathbb{R}^n$ and any centrally symmetric convex body $K \subset \mathbb{R}^n$ have the weak (B) property. The (B) conjecture was solved by Cordero-Erausquin, Fradelizi, and Maurey [6], where it was shown that γ and K have in fact a strong (B) property.

Conjecture 4.1 proposes that the uniform Lebesgue measure on B_2^n and any centrally symmetric convex body have the strong (B) property, with further assumptions on the equality case.

Unfortunately not a lot is known about the (B) property of general measures and even less about the equality case. We will briefly mention what is currently known: Livne Bar-on [3] showed that in $\mathbb{R}^2$, the uniform Lebesgue measure on a centrally symmetric convex body $L \subset \mathbb{R}^2$ has the weak (B) property with any centrally symmetric convex body $K \subset \mathbb{R}^2$. This result was generalized by Saroglou [13], where it was shown that if the log-Brunn-Minkowski inequality holds in dimension n, then the uniform probability measure on the n-dimensional cube has the strong (B) property, and the uniform probability measure of every centrally symmetric convex body has the weak (B) property, with any centrally symmetric convex body K.

The log-Brunn-Minkowski inequality states that for two centrally symmetric convex bodies $K, L \subset \mathbb{R}^n$, and $\lambda \in [0, 1]$,

$$\mathrm{Vol}_n\left((1-\lambda)\, K +_o \lambda L\right) \geq \mathrm{Vol}_n(K)^{1-\lambda}\, \mathrm{Vol}_n(L)^{\lambda}, \tag{4.2}$$

where

$$(1-\lambda)K +_o \lambda L = \bigcap_{u \in S^{n-1}} \left\{x : \langle x, u\rangle \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\right\}.$$

It was shown by Böröczky, Lutwak, Yang, and Zhang [4] that the log-Brunn-Minkowski inequality holds for $n = 2$, and so together with [13] the result of [3] is implied. In [14], Saroglou states that an unconditional log-concave measure μ and an unconditional body K have the strong (B) property. For our purposes, it is enough to mention that the uniform measure on B_2^n is unconditional log-concave. It follows that Conjecture 4.1 (without the equality case) holds whenever K is unconditional; i.e., $(x_1, \ldots, x_n) \in K$ implies $(\delta_1 x_1, \ldots, \delta_n x_n) \in K$ for any choice of $\delta_i \in \{-1, 1\}$ where $i = 1, \ldots, n$.

Still not a lot is known on equality cases in inequalities such as (4.1). In [14], Saroglou expands further on the relationship between the (B) property and the log-Brunn-Minkowski and conjectures that equality in (4.2) is attained if and only if $K = K_1 \times \cdots \times K_m$ for some convex sets $K_1, \ldots, K_m$ that cannot be written as Cartesian products of lower dimensional sets, and $L = c_1K_1 \times \cdots \times c_mK_m$ for some positive numbers $c_1, \ldots, c_m$. It was shown to us by Rotem [7] that the strong (B) property cannot hold in the most general sense for any two log-concave measures (instead of assuming one to be uniform on a ball, say), essentially by considering two Gaussian measures which are not mutually diagonalizable. However, these examples do not seem to give a counterexample to Conjecture 4.1.

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