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NONDEGENERACY OF HALF-HARMONIC MAPS FROM $\mathbb{R}$ INTO $\mathbb{S}^1$

YANNICK SIRE, JUNCHENG WEI, AND YOUQUAN ZHENG

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Abstract. We prove that the standard half-harmonic map $U : \mathbb{R} \to \mathbb{S}^1$ defined by

$$x \to \begin{pmatrix} \frac{x^2-1}{x^2+1} \\ \frac{-2x}{x^2+1} \end{pmatrix}$$

is nondegenerate in the sense that all bounded solutions of the linearized half-harmonic map equation are linear combinations of three functions corresponding to rigid motions (dilation, translation, and rotation) of U.

1. Introduction

Due to their importance in geometry and physics, the analysis of critical points of conformal invariant Lagrangians has attracted much attention since the 1950s. A typical example is the Dirichlet energy which is defined on two-dimensional domains and whose critical points are harmonic maps. This definition can be generalized to even-dimensional domains whose critical points are called polyharmonic maps. In recent years, people have been very interested in the analog of Dirichlet energy in odd-dimensional case; for example, [2], [3], [4], [5], [13], [14], and the references therein. Among these works, a special case is the so-called half-harmonic maps from $\mathbb{R}$ into $\mathbb{S}^1$ which are defined as critical points of the line energy

$$\mathcal{L}(u) = \frac{1}{2}\int_{\mathbb{R}} |(-\Delta_{\mathbb{R}})^{\frac{1}{4}} u|^2 dx. \tag{1.1}$$

Note that the functional $\mathcal{L}$ is invariant under the trace of conformal maps keeping invariant the half-space $\mathbb{R}^2_+$: the Möbius group. Half-harmonic maps have close relations with harmonic maps with partially free boundary and minimal surfaces with free boundary; see [12] and [13]. Computing the associated Euler–Lagrange equation of (1.1), we obtain that if $u : \mathbb{R} \to \mathbb{S}^1$ is a half-harmonic map, then u satisfies the following equation:

$$(-\Delta_{\mathbb{R}})^{\frac{1}{2}} u(x) = \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|^2} dy\right) u(x) \text{ in } \mathbb{R}. \tag{1.2}$$

The following proposition was proved in [13].

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Proposition 1.1 ([13]). *Let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{S}^1)$ be a nonconstant entire half-harmonic map into $\mathbb{S}^1$, and let u^e be its harmonic extension to $\mathbb{R}^2_+$. Then there exist $d \in \mathbb{N}$, $\vartheta \in \mathbb{R}$, $\{\lambda_k\}_{k=1}^d \subset (0, \infty)$, and $\{a_k\}_{k=1}^d \subset \mathbb{R}$ such that $u^e(z)$ or its complex conjugate equals*

$$e^{i\vartheta} \prod_{k=1}^{d} \frac{\lambda_k(z - a_k) - i}{\lambda_k(z - a_k) + i}.$$

Furthermore,

$$\mathcal{E}(u, \mathbb{R}) = [u]^2_{H^{1/2}(\mathbb{R})} = \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla u^e|^2 dz = \pi d.$$

This proposition shows that the map $U : \mathbb{R} \to \mathbb{S}^1$,

$$x \to \begin{pmatrix} \frac{x^2-1}{x^2+1} \\ \frac{-2x}{x^2+1} \end{pmatrix},$$

is a half-harmonic map corresponding to the case $\vartheta = 0$, $d = 1$, $\lambda_1 = 1$, and $a_1 = 0$. In this paper, we prove the nondegeneracy of U which is a crucial ingredient when analyzing the singularity formation of half-harmonic map flow. Note that U is invariant under translation, dilation, and rotation, i.e., for $Q = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in O(2)$, $q \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$, the function

$$QU\left(\frac{x-q}{\lambda}\right) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} U\left(\frac{x-q}{\lambda}\right)$$

still satisfies (1.2). Differentiating with α, q, and λ, respectively, and then setting $\alpha = 0$, $q = 0$, and $\lambda = 1$, we obtain that the following three functions:

$$Z_1(x) = \begin{pmatrix} \frac{2x}{x^2+1} \\ \frac{x^2-1}{x^2+1} \end{pmatrix}, \quad Z_2(x) = \begin{pmatrix} \frac{-4x}{(x^2+1)^2} \\ \frac{2(1-x^2)}{(x^2+1)^2} \end{pmatrix}, \quad Z_3(x) = \begin{pmatrix} \frac{-4x^2}{(x^2+1)^2} \\ \frac{2x(1-x^2)}{(x^2+1)^2} \end{pmatrix} \tag{1.3}$$

satisfy the linearized equation at the solution U of (1.2) defined as

$$\begin{aligned} (-\Delta_{\mathbb{R}})^{\frac{1}{2}} v(x) &= \left(\frac{1}{2\pi} \int_{\mathbb{R}} \frac{|U(x) - U(y)|^2}{|x-y|^2} dy\right) v(x) \\ &\quad + \left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{(U(x) - U(y)) \cdot (v(x) - v(y))}{|x-y|^2} dy\right) U(x) \quad \text{in } \mathbb{R} \end{aligned} \tag{1.4}$$

for $v : \mathbb{R} \to T_U\mathbb{S}^1$. Our main result is the following.

Theorem 1.1. *The half-harmonic map $U : \mathbb{R} \to \mathbb{S}^1$,*

$$x \to \begin{pmatrix} \frac{x^2-1}{x^2+1} \\ \frac{-2x}{x^2+1} \end{pmatrix},$$

is nondegenerate in the sense that all bounded solutions of equation (1.4) *are linear combinations of Z_1, Z_2, and Z_3 defined in* (1.3).

In the case of harmonic maps from two-dimensional domains into $\mathbb{S}^2$, the nondegeneracy of bubbles is a consequence of the computations in linear theory part of [7]. Integro-differential equations have attracted substantial research in recent years. The nondegeneracy of ground state solutions for the fractional nonlinear Schrödinger equations has been proved by Frank and Lenzmann [10], Frank, Lenzmann, and Silvestre [11], Fall and Valdinoci [9], and the corresponding result in the case of the fractional Yamabe problem was obtained by Dávila, del Pino, and Sire in [6].

2. Proof of Theorem 1.1

The rest of this paper is devoted to the proof of Theorem 1.1. For convenience, we identify $\mathbb{S}^1$ with the complex unit circle. Since Z_1, Z_2, and Z_3 are linearly independent and belong to the space $L^\infty(\mathbb{R})\cap \mathrm{Ker}(\mathcal{L}_0)$, we only need to prove that the dimension of $L^\infty(\mathbb{R})\cap \mathrm{Ker}(\mathcal{L}_0)$ is 3. Here the operator $\mathcal{L}_0$ is defined as

$$\begin{aligned}\mathcal{L}_0(v) = (-\Delta_{\mathbb{R}})^{\frac{1}{2}}v(x) &- \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right)v(x)\\ &-\left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{(U(x)-U(y))\cdot(v(x)-v(y))}{|x-y|^2}dy\right)U(x),\end{aligned}$$

for $v:\mathbb{R}\to T_U\mathbb{S}^1$. Let us come back to equation (1.4); for $v:\mathbb{R}\to T_U\mathbb{S}^1$, $v(x)\cdot U(x)=0$ holds pointwisely. Using this fact and the definition of $(-\Delta_{\mathbb{R}})^{\frac{1}{2}}$ (see [8]), we have

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{R}}\frac{(U(x)-U(y))}{|x-y|^2}dy\cdot v(x) &= \left((-\Delta_{\mathbb{R}})^{\frac{1}{2}}U(x)\right)\cdot v(x)\\ &= \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right)U(x)\cdot v(x) = 0.\end{aligned}$$

Therefore

$$\begin{aligned}(-\Delta_{\mathbb{R}})^{\frac{1}{2}}v(x) &= \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right)v(x)\\ &\quad+\left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{(U(x)-U(y))\cdot(v(x)-v(y))}{|x-y|^2}dy\right)U(x)\\ &= \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right)v(x)\\ &\quad+\left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{(U(x)-U(y))}{|x-y|^2}dy\cdot v(x)\right)U(x)\\ &\quad+\left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{(v(x)-v(y))}{|x-y|^2}dy\cdot U(x)\right)U(x)\\ &= \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right)v(x)\\ &\quad+\left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{(v(x)-v(y))}{|x-y|^2}dy\cdot U(x)\right)U(x)\\ &= \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right)v(x)\\ &\quad+\left((-\Delta_{\mathbb{R}})^{\frac{1}{2}}v(x)\cdot U(x)\right)U(x).\end{aligned}$$

Therefore equation (1.4) becomes

$$\begin{aligned}(-\Delta_{\mathbb{R}})^{\frac{1}{2}}v(x) &= \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right)v(x)+\left((-\Delta_{\mathbb{R}})^{\frac{1}{2}}v(x)\cdot U(x)\right)U(x)\\ &= \frac{2}{x^2+1}v(x)+\left((-\Delta_{\mathbb{R}})^{\frac{1}{2}}v(x)\cdot U(x)\right)U(x). \qquad (2.1)\end{aligned}$$

Next, we lift equation (2.1) to $\mathbb{S}^1$ via the stereographic projection from $\mathbb{R}$ to $\mathbb{S}^1 \setminus \{pole\}$:

$$S(x) = \begin{pmatrix} \frac{2x}{x^2+1} \\ \frac{1-x^2}{x^2+1} \end{pmatrix}. \tag{2.2}$$

It is well known that the Jacobian of the stereographic projection is

$$J(x) = \frac{2}{x^2+1}.$$

For a function $\varphi : \mathbb{R} \to \mathbb{R}$, define $\tilde{\varphi} : \mathbb{S}^1 \to \mathbb{R}$ by

$$\varphi(x) = J(x)\tilde{\varphi}(S(x)). \tag{2.3}$$

Then we have

$$\begin{aligned} [(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{\varphi}](S(x)) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\tilde{\varphi}(S(x)) - \tilde{\varphi}(S(y))}{|S(x)-S(y)|^2} dS(y) \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\frac{1+x^2}{2}\varphi(x) - \frac{1+y^2}{2}\varphi(y)}{\frac{4(x-y)^2}{(x^2+1)(y^2+1)}} \frac{2}{1+y^2} dy \\ &= \frac{1+x^2}{4\pi}\int_{\mathbb{R}} \frac{(1+x^2)\varphi(x) - (1+y^2)\varphi(y)}{(x-y)^2} dy \\ &= \frac{1+x^2}{2}(-\Delta_{\mathbb{R}})^{1/2}\left[\frac{x^2+1}{2}\varphi(x)\right] \\ &= \frac{1+x^2}{2}(-\Delta_{\mathbb{R}})^{1/2}\left[\tilde{\varphi}(S(x))\right]. \end{aligned}$$

Therefore,

$$(-\Delta_{\mathbb{R}})^{1/2}\left[\tilde{\varphi}(S(x))\right] = J(x)[(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{\varphi}](S(x)).$$

Denote $v = (v_1, v_2)$ and let $\tilde{v}_1$, $\tilde{v}_2$ be the functions defined by (2.3), respectively. Then the linearized equation (2.1) becomes

$$\begin{cases} J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 = J(x)\tilde{v}_1 + \frac{x^2-1}{x^2+1}\frac{x^2-1}{x^2+1}J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \frac{x^2-1}{x^2+1}\frac{-2x}{x^2+1}J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2, \\ J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2 = J(x)\tilde{v}_2 + \frac{-2x}{x^2+1}\frac{x^2-1}{x^2+1}J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \frac{-2x}{x^2+1}\frac{-2x}{x^2+1}J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2. \end{cases}$$

Since $J(x) > 0$ and set $U = (\cos\theta, \sin\theta)$, we get

$$\begin{cases} (-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 = \tilde{v}_1 + \cos^2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \cos\theta\sin\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2, \\ (-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2 = \tilde{v}_2 + \cos\theta\sin\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \sin^2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2, \end{cases}$$

which is equivalent to

$$\begin{cases} (-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 = 2\tilde{v}_1 + \cos 2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \sin 2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2, \\ (-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2 = 2\tilde{v}_2 + \sin 2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 - \cos 2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2. \end{cases}$$

Set $w = \tilde{v}_1 + i\tilde{v}_2$, $z = \cos\theta + i\sin\theta$; then we have

$$(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}w = 2w + z^2(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\bar{w}. \tag{2.4}$$

Here $\bar{w}$ is the conjugate of w.

Since $v \in L^\infty(\mathbb{R})$, w is also bounded, so we can expand w into the Fourier series

$$w = \sum_{k=-\infty}^{\infty} a_k z^k.$$

Note that all the eigenvalues for $(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}$ are $\lambda_k = |k| = 0, 1, 2, \ldots$ with $k \in \mathbb{Z}$; see, for example, [1]. Using (2.4), $(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}} z^k = |k| z^k$, and $(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}} \bar{z}^k = |k| \bar{z}^k$, we obtain

$$\begin{cases} (-k-2)a_k = (2-k)\bar{a}_{2-k}, \text{ if } k < 0, \\ (k-2)a_k = (2-k)\bar{a}_{2-k}, \text{ if } 0 \le k \le 2, \\ a_k = \bar{a}_{2-k}, \text{ if } k \ge 3. \end{cases}$$

Furthermore, from the orthogonal condition $v(x) \cdot U(x) = 0$ (so $(\tilde{v}_1, \tilde{v}_2) \cdot (\cos\theta, \sin\theta) = 0$), we have

$$a_k = -\bar{a}_{2-k}, \quad k = \cdots -1, 0, 1, \ldots.$$

Thus

$$a_k = 0, \text{ if } k < 0 \text{ or } k \ge 3$$

and

$$a_0 = -\bar{a}_2, \quad a_1 = -\bar{a}_1$$

hold, which imply that

$$w = -\bar{a}_2 + a_1 z + a_2 z^2 = a(iz) + b\left[\frac{i}{2}(z-1)^2\right] + c\frac{(z^2-1)}{2}.$$

Here a, b, c are real numbers and satisfy the relations

$$i(a-b) = a_1, \quad \frac{c}{2} + \frac{i}{2} b = a_2.$$

Also, it is easy to check that iz, $\frac{i}{2}(z-1)^2$, and $\frac{(z^2-1)}{2}$ are, respectively, Z_1, Z_2, and Z_3 under stereographic projection (2.2). By the one-to-one correspondence of w and v, we know that the dimension of $L^\infty(\mathbb{R}) \cap \mathrm{Ker}(\mathcal{L}_0)$ is 3. This completes the proof.

Remark 2.1. The above proof also shows that the half-harmonic map from $\mathbb{S}^1$ to $\mathbb{S}^1$: $z \to -iz$ is nondegenerate.

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Department of Mathematics, Johns Hopkins University, 404 Krieger Hall, 3400 N. Charles Street, Baltimore, Maryland 21218

Email address: sire@math.jhu.edu

Department of Mathematics, University of British Columbia, Vancouver V6T 1Z2, Canada

Email address: jcwei@math.ubc.ca

School of Mathematics, Tianjin University, Tianjin 300072, People's Republic of China

Email address: zhengyq@tju.edu.cn