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LOCAL GRADIENT ESTIMATES FOR HEAT EQUATION ON $RCD^*(k,n)$ METRIC MEASURE SPACES

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ABSTRACT. In this paper, we will establish a local gradient estimate and a Liouville type theorem for weak solutions of the heat equation on $RCD^*(K,N)$ metric measure spaces.

1. Introduction

Let M^n be an n-dimensional complete Riemannian manifold with $Ric(M^n) \geqslant -k$, $k \geqslant 0$. The Li–Yau local gradient estimate states that if u is a positive solution of the heat equation $\Delta u = \partial_t u$ on $B_{2R} \times (0,\infty)$, then

$$(1.1)\quad \sup_{x\in B_R} \big(|\nabla f|^2 - \alpha \cdot \partial_t f\big)(x,t) \leqslant \frac{C_n \cdot \alpha^2}{R^2}\Big(\frac{\alpha^2}{\alpha^2-1} + \sqrt{k}R\Big) + \frac{n\alpha^2 k}{2(\alpha-1)} + \frac{n\alpha^2}{2t}$$

for any $\alpha > 1$, $f := \ln u$. Letting $R \to \infty$ in (1.1), one gets a global estimate as follows:

$$|\nabla f|^2 - \alpha\partial_t f \leqslant \frac{n\alpha^2 k}{2(\alpha-1)} + \frac{n\alpha^2}{2t}.$$

There are many extensions and improvements of Li–Yau's gradient estimate on smooth manifolds, including both the local version and the global version; see, for example, [6, 7, 9, 11, 24, 25, 27, 33, 35, 38, 45, 46] for the related results. The Li–Yau type estimates were generalized to the non-smooth setting; see, for example, [8, 36, 44], and so on.

In 1993, Hamilton [20] proved an elliptic type (global) gradient estimate of the heat equation. If u is a global positive solution of the heat equation with $u \leqslant L$ on M^n, then

$$\frac{|\nabla u|^2}{u^2}(x,t) \leqslant \left(\frac{1}{t} + 2k\right)\ln(L/u).$$

In 2006, Souplet and Zhang [38] proved a sharp local version of the above inequality. They proved that if u is a positive solution of the heat equation with $u \leqslant L$ on $B_R \times [t_0 - T, t_0] \subset M^n \times \mathbb{R}$, then

$$(1.2)\quad \frac{|\nabla u|}{u} \leqslant C_n \cdot \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right)\left(1 + \ln\frac{L}{u}\right) \quad \text{on } B_{R/2} \times [t_0 - T/2, t_0].$$

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Our main purpose in this paper is to study the local gradient estimate of the heat equation on the metric measure spaces with a "lower Ricci curvature bound", so called $RCD^*(K,N)$ spaces.

Given $K \in \mathbb{R}$ and $N \in [1,\infty]$, let (X,d,μ) be a proper (i.e., closed balls with finite radius are compact) complete metric measure space with $\mathrm{supp}(\mu) = X$. In recent years, several notions of "generalized Ricci curvature $\geqslant K$ and dimension $\leqslant N$" have been defined. Sturm [39] and Lott–Villani [28] introduced the curvature-dimension condition $CD(K,N)$ on (X,d,μ) via optimal transportation. In 2010, Bacher-Sturm [5] introduced the reduced curvature-dimension condition $CD^*(K,N)$, which enjoys a local-to-global property. In 2015, for ruling out the Finslerian cases, Ambrosio–Gigli–Savaré [1] introduced the Riemannian curvature-dimension condition $RCD(K,\infty)$ by assuming that the Sobolev space $W^{1,2}(X)$ is a Hilbertian space. Recently, Erbar–Kuwada–Sturm [12] and Ambrosio–Mondino–Savaré [4] introduced a finite-dimensional version of the Riemannian curvature-dimension condition $RCD^*(K,N)$ and proved the equivalence of the Riemannian curvature-dimension condition and the Bochner formula of Bakry–Emery via an abstract Γ_2-calculus. Of course, when X is a Riemannian manifold, the notion $RCD^*(K,N)$ is equivalent to the original $Ric(X) \geqslant K$ and $\dim(X) \leqslant N$. When X is an Alexandrov space with generalized (sectional) curvature bounded below in the sense of Alexandrov, X satisfies the $RCD^*(K,N)$ condition [34, 42].

In recent years, many important results in geometric analysis have been extended to the $RCD^*(K,N)$ metric measure spaces; for example, the local Li–Yau gradient estimate for the heat equation and the local Yau gradient estimate for the harmonic functions [44], Li–Yau–Hamilton estimates [13, 21, 22] and spectral gaps [23, 29, 36], the existence of the universal cover for $RCD^*(K,N)$ metric measure spaces [32], and so on.

In this paper, we will study the local gradient estimate for the local weak solutions of the heat equation on an $RCD^*(K,N)$ metric measure space (X,d,μ). Let $\Omega \subset X$ be a domain. By the $RCD^*(K,N)$ condition, the Sobolev space $W^{1,2}(\Omega)$ is a Hilbertian space. Hence, by polarization in $W^{1,2}(\Omega)$, one can define the inner product $\langle \cdot,\cdot \rangle$; see (2.5) for details. Given an interval $I \subset \mathbb{R}$, we say that $u(x,t) \in W^{1,2}(\Omega \times I)$ is a local weak solution for the heat equation on $\Omega \times I$ if

$$-\int_I \int_\Omega \langle \nabla u, \nabla \phi \rangle d\mu dt = \int_I \int_\Omega \frac{\partial u}{\partial t} \cdot \phi d\mu dt$$

for all Lipschitz functions ϕ with compact support in $\Omega \times I$.

In the previous works [13, 21, 23], the main tool is the Γ_2-calculus for the heat flow. But the local weak solutions $u(x,t)$ do not form a semi-group in general. The method of Γ_2-calculus for the heat flow does not work in this case.

In Zhang-Zhu [44], the authors developed a pointwise maximum principle on metric measure spaces and proved the local Li–Yau estimate on $RCD^*(K,N)$ spaces. Inspired by [38] and [44], we generalized the local gradient estimates (1.2) to the $RCD^*(K,N)$ spaces. Our first main result is the following.

Theorem 1.3. *Given $K \geqslant 0$ and $N \in (1,\infty)$, let (X,d,μ) be a metric measure space satisfying $RCD^*(-K,N)$. Let $T \in (0,\infty)$, let $B_{2R} \subset X$ be a geodesic ball of radius $2R$, let $B_{2R,T} = B_{2R} \times (0,T)$, and let $u(x,t) \in W^{1,2}(B_{2R,T})$ be a positive local weak solution of the heat equation on $B_{2R,T}$. Suppose also that $u \leqslant M$. Then,* *we have*

$$\sup_{B_R\times(3T/4,T]}\frac{|\nabla f(x,t)|}{1-f(x,t)}\leqslant C\cdot\left(\frac{\sqrt{N}}{R}+\frac{1}{\sqrt{T}}+\sqrt{K}\right), \tag{1.4}$$

where $f=\ln(u/M)$. *Here and in what follows,* $\sup_U f$ *means* $\operatorname{ess\,sup}_U f$, *and* C *is a universal constant (independent of* N, K *and* R*).*

The constant in (1.4) is more precise than [38, eq. (1.4)]. As a consequence, our estimate even holds for any positive solution, bounded or not, when $K=0$. Precisely,

Corollary 1.5. *Given* $N\in(1,\infty)$, *let* (X,d,μ) *be a metric measure space satisfying* $RCD^*(0,N)$. u *is any positive solution of the heat equation on* $X\times(0,\infty)$. *Then there exists a constant* C_N *such that*

$$\frac{|\nabla u|}{u}(x,t)\leqslant C_N\frac{1}{t^{1/2}}\left(C_N+\ln\frac{u(x,2t)}{u(x,t)}\right)$$

for almost all $(x,t)\in X\times(0,\infty)$.

As a consequence of Theorem 1.3, we have the Liouville type theorem of the heat equation. Our second main result is the following.

Theorem 1.6. *Given* $N\in[1,\infty)$, *let* (X,d,μ) *be a metric measure space satisfying* $RCD^*(0,N)$. *Then, we have the following conclusions.*

(1) *Let* $u(x,t)$ *be a positive weak solution of the heat equation on* $X\times(-\infty,0]$. *If* $u=\exp(o(d(x)+\sqrt{|t|}))$ *near infinity, then* u *must be a constant.*
(2) *Let* $u(x,t)$ *be a weak solution of the heat equation on* $X\times(-\infty,0]$. *If* $u=o(d(x)+\sqrt{|t|})$ *near infinity, then* u *must be a constant.*

We remark that the growth condition in the second statement of Theorem 1.6 is sharp in the spatial direction, due to the example $u=x$.

2. Preliminaries

Let (X,d) be a proper (i.e., closed balls of finite radius are compact) complete metric space and let μ be a Radon measure on X with $\operatorname{supp}(\mu)=X$. Denote by $B_r(x)$ the open ball centered at x with radius r. For any open subset $\Omega\subset X$ and any $p\in[1,\infty]$, we denote by $L^p(\Omega):=L^p(\Omega,\mu)$.

2.1. The curvature-dimension conditions.

Let $\mathscr{P}(X)$ be the set of all Borel probability measures on X. Let $\mathscr{P}_2(X,d)$ be the L^2-Wasserstein space over (X,d), that is,

$$\mathscr{P}_2(X,d)=\{\nu\in\mathscr{P}(X):\int_X d^2(x_0,x)d\nu(x)<\infty,\text{ for some (hence, for all) } x_0\in X\}.$$

Given $\nu_1,\nu_2\in\mathscr{P}_2(X,d)$, the L^2-Wasserstein distance $W^2(\nu_0,\nu_1)$ is defined by

$$W^2(\nu_0,\nu_1):=\inf\int_{X\times X}d^2(x,y)dq(x,y), \tag{2.1}$$

where the infimum is taken over all couplings q of ν_1 and ν_2. Here, we say that q is a coupling of ν_1 and ν_2 if q is in $\mathscr{P}(X\times X)$ with marginals ν_0 and ν_1. A coupling q that realizes the inf in (2.1) is called an optimal coupling of ν_0 and ν_1. Let

$$\mathscr{P}_2(X,d,\mu)=\{\nu\in\mathscr{P}_2(X,d):\nu\text{ is absolutely continuous w.r.t. }\mu\}$$

and

$$\mathscr{P}_\infty(X,d,\mu) = \{\nu \in \mathscr{P}_2(X,d,\mu) : \nu \text{ has bounded support}\}.$$

For any $\nu \in \mathscr{P}_\infty(X,d,\mu)$, according to the Radon–Nikodym Theorem, there exists a Borel measurable function ρ such that, for any μ-measurable set A, we have $\nu(A) = \int_A \rho d\mu$. We write $\nu = \rho \cdot \mu$ in the above sense.

Definition 2.2. Given $K \in \mathbb{R}$ and $N \in [1,\infty)$, we say that a metric measure space (X,d,μ) satisfies the reduced curvature-dimension condition $CD^*(K,N)$ if, for each pair $\nu_0 = \rho_0 \cdot \mu, \nu_1 = \rho_1 \cdot \mu \in \mathscr{P}_\infty(X,d,\mu)$, there exist an optimal coupling q of them and a geodesic $(\nu_t := \rho_t \cdot \mu)_{t\in[0,1]}$ in $\mathscr{P}_\infty(X,d,\mu)$ connecting them such that for all $t \in [0,1]$ and all $N' \geqslant N$:

$$\begin{aligned}&\int_X \rho_t^{-1/N'} d\nu_t \\ &\quad\geqslant \int_{X\times X} \Big[\sigma_{K/N'}^{(1-t)}\big(d(x_0,x_1)\big)\rho_0^{-1/N'}(x_0) + \sigma_{K/N'}^{(t)}\big(d(x_0,x_1)\big)\rho_1^{-1/N'}(x_1)\Big] dq(x_0,x_1),\end{aligned}$$

where

$$\sigma_k^{(t)}(\theta) := \begin{cases} \frac{\sin(\sqrt{k}\cdot t\theta)}{\sin(\sqrt{k}\cdot\theta)}, & 0 < k\theta^2 < \pi^2, \\ t, & k\theta^2 = 0, \\ \frac{\sinh(\sqrt{-k}\cdot t\theta)}{\sinh(\sqrt{-k}\cdot\theta)}, & k\theta^2 < 0, \\ \infty, & k\theta^2 \geqslant \pi^2. \end{cases}$$

Given $f \in C(X)$, the pointwise Lipschitz constant of f at x is defined by

$$\mathrm{Lip} f(x) := \limsup_{y\to x} \frac{|f(y)-f(x)|}{d(x,y)},$$

where we put $\mathrm{Lip} f(x) = 0$ if x is isolated. Clearly $\mathrm{Lip} f$ is μ-measurable on X.

Definition 2.3 ([3])**.** The Cheeger energy $\mathrm{Ch} : L^2(X) \to [0,\infty]$ is defined by

$$\mathrm{Ch}(f) := \inf\Big\{ \liminf_{j\to\infty} \frac{1}{2}\int_X (\mathrm{Lip} f_j)^2 d\mu \Big\},$$

where the infimum is taken over all sequences of Lipschitz functions $(f_j)_{j\in\mathbb{N}}$ such that $f_j \to f$ in $L^2(X)$.

Definition 2.4. We say that a metric measure space (X,d,μ) is infinitesimally Hilbertian if the Cheeger energy Ch is quadratic; i.e., for any $f,g \in L^2(X)$, we have $\mathrm{Ch}(f+g) + \mathrm{Ch}(f-g) = 2\mathrm{Ch}(f) + 2\mathrm{Ch}(g)$. We say that (X,d,μ) satisfies the $RCD^*(K,N)$ condition, for some $K \in \mathbb{R}$ and $N \in [1,\infty)$, if (X,d,μ) is infinitesimally Hilbertian and satisfies the $CD^*(K,N)$ condition.

Let (X,d,μ) be an $RCD^*(K,N)$ metric measure space. For each $f \in D(\mathrm{Ch}) := \{f \in L^2(X) : \mathrm{Ch}(f) < \infty\}$, it is shown in §4 of [3] that

$$\mathrm{Ch}(f) = \frac{1}{2}\int_X |\nabla f|^2 d\mu,$$

where $|\nabla f|$ is the minimal relaxed gradient of f. Given $f,g \in D(\mathrm{Ch})$, it was proved [14] that the limit

$$\langle \nabla f, \nabla g\rangle := \lim_{\epsilon\to 0} \frac{|\nabla(f+\epsilon\cdot g)|^2 - |\nabla f|^2}{2\epsilon} \tag{2.5}$$

exists in $L^1(X)$. This inner product (2.5) is bi-linear and satisfies the Cauchy–Schwarz inequality, the chain rule, and the Leibniz rule [14].

2.2. **Sobolev spaces.**

Given $K \in \mathbb{R}$ and $N \in [1, \infty)$, let (X, d, μ) be an $RCD^*(K, N)$ metric measure space. Let $\Omega \subset X$ be a domain. We denote by $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ the set of locally Lipschitz continuous functions on Ω, and by $\mathrm{Lip}(\Omega)$ (resp. $\mathrm{Lip}_0(\Omega)$) the set of Lipschitz continuous functions on Ω (resp. with compact support in Ω).

For any $1 \leqslant p \leqslant +\infty$ and $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, its $W^{1,p}(\Omega)$-norm is defined by

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\mathrm{Lip} f\|_{L^p(\Omega)}.$$

The Sobolev space $W^{1,p}(\Omega)$ is defined by the closure of the set

$$\left\{f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) : \ \|f\|_{W^{1,p}(\Omega)} < +\infty\right\}$$

under the $W^{1,p}(\Omega)$-norm. The space $W^{1,p}_0(\Omega)$ is defined by the closure of $\mathrm{Lip}_0(\Omega)$ under the $W^{1,p}(\Omega)$-norm. We say $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ if $f \in W^{1,p}(\Omega')$ for every open subset $\Omega' \Subset \Omega$.

It is well known that $D(\mathrm{Ch}) = W^{1,2}(X)$; see, for example, [44, Lemma 2.5].

We remark that several different notions of Sobolev spaces on metric measure space have been established in [10, 15–17, 37]. They coincide with each other on $RCD^*(K, N)$ metric measure spaces (see, for example, [2]).

2.3. **The weak Laplacian and a local version of the Bochner formula.** Given $K \in \mathbb{R}$ and $N \in [1, \infty)$, let (X, d, μ) be an $RCD^*(K, N)$ metric measure space. Fix any domain $\Omega \subset X$. We will denote $H^1_0(\Omega) := W^{1,2}_0(\Omega)$, $H^1(\Omega) := W^{1,2}(\Omega)$, and $H^1_{\mathrm{loc}}(\Omega) := W^{1,2}_{\mathrm{loc}}(\Omega)$.

Definition 2.6. For each $f \in H^1_{\mathrm{loc}}(\Omega)$, $\mathscr{L} f$ is a functional defined on $H^1_0(\Omega) \cap L^\infty(\Omega)$ by

$$\mathscr{L} f(\phi) := -\int_\Omega \langle \nabla f, \nabla \phi \rangle d\mu \qquad \forall\ \phi \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

For any $g \in H^1(\Omega) \cap L^\infty(\Omega)$, the distribution $g \cdot \mathscr{L} f$ is defined by

$$g \cdot \mathscr{L} f(\phi) := \mathscr{L} f(g\phi) \qquad \forall\ \phi \in H^1_0(\Omega) \cap L^\infty(\Omega). \tag{2.7}$$

By the linearity of inner product $\langle \nabla f, \nabla g \rangle$, this distributional Laplacian is linear.

Definition 2.8. A function $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is said to satisfy the inequality

$$\mathscr{L} f \leqslant (\geqslant, =) h$$

in the sense of distributions if the inequality

$$\mathscr{L} f(\varphi) \leqslant (\geqslant, =) \int_\Omega h\varphi d\mu$$

holds for all $0 \leqslant \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$. In this case, $\mathscr{L} f$ is a signed Radon measure [18].

$\mathscr{L}$ satisfies the following chain rule and Leibniz rule [14]; see also [44].

Lemma 2.9 ([44]). *Given $K \in \mathbb{R}$ and $N \in [1, \infty)$, let Ω be an open domain of an $RCD^*(K, N)$ metric measure space (X, d, μ). Then, we have*

(i) (Chain rule) *Let $f \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\eta \in C^2(\mathbb{R})$. Then we have*

$$\mathscr{L}[\eta(f)] = \eta'(f) \cdot \mathscr{L}f + \eta''(f) \cdot |\nabla f|^2. \tag{2.10}$$

(ii) (Leibniz rule) *Let $f, g \in H^1(\Omega) \cap L^\infty(\Omega)$. Then we have*

$$\mathscr{L}(f \cdot g) = f \cdot \mathscr{L}g + g \cdot \mathscr{L}f + 2\langle \nabla f, \nabla g \rangle. \tag{2.11}$$

Erbar–Kuwada–Sturm [12] and Ambrosio–Mondino–Savaré [4] proved that the $RCD^*(K, N)$ condition is equivalent to a Bakry–Emery Bochner inequality for the heat flow on X. Hence, the $RCD^*(K, N)$ condition implies a global version of the Bochner formula. By using a good cut-off function in [4, 19, 31] and an argument in [19], one can localize the global version of the Bochner formula in [4, 12] to a local one; see, for example, §2 in [44] for details. In the following, a local Bochner formula is given.

Theorem 2.12 ([44]). *Let (X, d, μ) be an $RCD^*(K, N)$ space for $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let B_R be a geodesic ball with radius R centered at a fixed point x_0.*

Assume that $f \in H^1(B_R)$ satisfies $\mathscr{L}f = g$ on B_R in the sense of distributions with $g \in H^1(B_R) \cap L^\infty(B_R)$. Then we have $|\nabla f|^2 \in H^1(B_{R/2}) \cap L^\infty(B_{R/2})$ and that the distribution $\mathscr{L}(|\nabla f|^2)$ is a signed Radon measure on $B_{R/2}$. If its Radon–Nikodym decomposition w.r.t. μ is denoted by

$$\mathscr{L}(|\nabla f|^2) = \mathscr{L}^{\mathrm{ac}}(|\nabla f|^2) \cdot \mu + \mathscr{L}^{\mathrm{sing}}(|\nabla f|^2),$$

then we have $\mathscr{L}^{\mathrm{sing}}(|\nabla f|^2) \geqslant 0$ and, for μ-a.e. $x \in B_{R/2}$,

$$\frac{1}{2}\mathscr{L}^{\mathrm{ac}}(|\nabla f|^2) \geqslant \frac{g^2}{N} + \langle \nabla f, \nabla g \rangle + K|\nabla f|^2.$$

Furthermore, if $N > 1$, for μ-a.e. $x \in B_{R/2} \cap \{y : \ |\nabla f(y)| \neq 0\}$,

$$\frac{1}{2}\mathscr{L}^{\mathrm{ac}}(|\nabla f|^2) \geqslant \frac{g^2}{N} + \langle \nabla f, \nabla g \rangle + K|\nabla f|^2 + \frac{N}{N-1} \cdot \Big(\frac{\langle \nabla f, \nabla |\nabla f|^2 \rangle}{2|\nabla f|^2} - \frac{g}{N} \Big)^2.$$

2.4. **The maximum principles.** Given $K \in \mathbb{R}$ and $N \in [1, \infty)$, let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$. We need the following maximum principle.

Theorem 2.13 ([44]). *Let Ω be a bounded domain and let $T > 0$. Let $f(x, t) \in H^1(\Omega_T) \cap L^\infty(\Omega_T)$ and suppose that f achieves one of its strict maximums in $\Omega \times (0, T]$ in the sense that: there exist a neighborhood $U \Subset \Omega$ and an interval $(\delta, T] \subset (0, T]$ for some $\delta > 0$ such that*

$$\sup_{U \times (\delta, T]} f > \sup_{\Omega_T \setminus (U \times (\delta, T])} f.$$

Here $\sup_{U \times (\delta, T]} f$ means $\operatorname{ess\,sup}_{U \times (\delta, T]} f$. Assume that, for almost every $t \in (0, T)$, $\mathscr{L}f(\cdot, t)$ is a signed Radon measure with $\mathscr{L}^{\mathrm{sing}}f(\cdot, t) \geqslant 0$. Let $v \in H^1(\Omega_T) \cap L^\infty(\Omega_T)$ with $\partial_t v(x, t) \leqslant C$ for some constant $C > 0$, for almost all $(x, t) \in \Omega_T$. Then, for any $\varepsilon > 0$, we have

$$(\mu \times \mathcal{L}^1)\Big\{ (x, t) : f(x, t) \geqslant \sup_{\Omega_T} f - \varepsilon \quad \text{and} \quad \mathscr{L}^{\mathrm{ac}} f(x, t) + \langle \nabla f, \nabla v \rangle (x, t) - \frac{\partial}{\partial t} f(x, t) \leqslant \varepsilon \Big\} > 0,$$

where $\mathcal{L}^1$ is the 1-dimensional Lebesgue's measure on $(\delta, T]$.

In particular, there exists a sequence of points $\{(x_j, t_j)\}_{j\in\mathbb{N}} \subset U \times (\delta, T]$ such that every x_j is an approximate continuity point of $\mathscr{L}^{\rm ac} f(\cdot, t_j)$ and $\langle \nabla f, \nabla w\rangle(\cdot, t_j)$ and that

$$f(x_j, t_j) \geqslant \sup_{\Omega_T} f - 1/j \quad \text{and} \quad \mathscr{L}^{\rm ac} f(x_j, t_j) + \langle \nabla f, \nabla v\rangle(x_j, t_j) - \frac{\partial}{\partial t} f(x_j, t_j) \leqslant 1/j.$$

3. The local gradient estimates

Given $K \in \mathbb{R}$ and $N \in [1, \infty)$, let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$. In this section, we will prove the local gradient estimates, Theorem 1.3.

3.1. **The heat equations.** Let $\Omega \subset X$ be a domain. Given $T > 0$, we denote by

$$\Omega_T := \Omega \times (0, T].$$

Definition 3.1. A function $u(x, t) \in H^1(\Omega_T)$ $(= W^{1,2}(\Omega_T))$ is called a local weak solution of the heat equation on Ω_T if for any $[t_1, t_2] \subset (0, T)$ and any geodesic ball $B_R \Subset \Omega$, we have

$$\int_{t_1}^{t_2} \int_{B_R} \Big(\partial_t u \cdot \phi + \langle \nabla u, \nabla \phi\rangle\Big) d\mu dt = 0 \tag{3.2}$$

for all $\phi(x, t) \in {\rm Lip}_0\big(B_R \times (t_1, t_2)\big)$. Here and in the sequel, we always denote $\partial_t u := \frac{\partial u}{\partial t}$.

It is well known that the volume doubling property and the L^2-Poincaré inequality hold true in the $RCD^*(K, N)$ metric measure spaces. The local boundedness and the local Hölder continuity for local weak solutions of heat equations have been established in [30, 40, 41].

An equivalent definition of the local weak solution is given in the following.

Lemma 3.3 ([44])**.** *Let $u(x, t)$ be a local weak solution of the heat equation on $\Omega \times (0, T)$. Then, for a.e. $t \in (0, T)$, the function $u(\cdot, t)$ satisfies*

$$\mathscr{L} u = \partial_t u \tag{3.4}$$

in the sense of distributions on Ω. Conversely, if a function $u(x, t) \in H^1\big(\Omega_T\big)$ and (3.4) *holds for a.e. $t \in [0, T]$, then it was shown in* [43, Lemma 6.12] *that $u(x, t)$ is a local weak solution of the heat equation on Ω_T.*

For a local weak solution u of the heat equation on Ω_T, we want to apply the Bochner formula in Theorem 2.12 to (3.4). But in general, $\partial_t u$ is only in L^2. We cannot apply the Bochner formula in Theorem 2.12 to (3.4). Similar to [44], we use the Steklov average to overcome this difficulty.

Definition 3.5. Given $B_R \subset X$ and $u(x, t) \in L^1(B_{R,T})$, where $B_{R,T} := B_R \times (0, T)$, the Steklov average of u is defined as follows. For every $\varepsilon \in (0, T)$ and any $h \in (0, \varepsilon)$,

$$u_h(x, t) := \frac{1}{h} \int_0^h u(x, t + \tau) d\tau, \quad t \in (0, T - \varepsilon]. \tag{3.6}$$

By using the standard theory of L^p spaces, it is well known that if $u \in L^p(B_{R,T})$, then the Steklov average $u_h \to u$ in $L^p(B_{R,T-\varepsilon})$ as $h \to 0$, for every $\varepsilon \in (0, T)$.

We need the following lemmas.

Lemma 3.7 ([44]). *If $u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T})$, then we have, for every $\varepsilon \in (0,T)$, that*

$$u_h \in H^1(B_{R,T-\varepsilon}) \cap L^\infty(B_{R,T-\varepsilon}) \quad \text{and} \quad \partial_t u_h \in H^1(B_{R,T-\varepsilon}) \cap L^\infty(B_{R,T-\varepsilon})$$

for every $h \in (0,\varepsilon)$ and that $\|u_h\|_{H^1(B_{R,T-\varepsilon})}$ is bounded uniformly with respect to $h \in (0,\varepsilon)$.

For a local weak solution u, we have the following property of u_h.

Lemma 3.8 ([44]). *Let $u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T})$ be a local weak solution for the heat equation, and fix any two constants ε, h such that $\varepsilon \in (0,T)$ and $h \in (0,\varepsilon)$. Then for almost all $t \in (0, T-\varepsilon)$,*

$$\mathscr{L} u_h = \partial_t u_h$$

on B_R, in the sense of distributions.

We need the following lemma.

Lemma 3.9 ([44]). *Given $K \in \mathbb{R}$ and $N \in [1,\infty)$, let (X,d,μ) be a metric measure space satisfying $RCD^*(K,N)$. Let $u(x,t) \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ be a local weak solution of the heat equation on $B_{2R,T}$. Assume that $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$. Then we have $|\nabla u|^2 \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$.*

3.2. **Proof of Theorems 1.3 and 1.6.** We firstly prove an important elliptic inequality.

Lemma 3.10. *Given $K \geqslant 0$ and $N \in (1,\infty)$, let (X,d,μ) be an $RCD^*(-K,N)$ space. Let $u(x,t) \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ be the local weak solution of the heat equation on $B_{2R,T}$. Assume that $M \geqslant u \geqslant \delta > 0$ and $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$. We put*

$$w(x,t) = \frac{|\nabla f|^2}{(1-f)^2},$$

where $f = \ln(u/M)$. Then, we have $w \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$ and that, for almost every $t \in (0,T)$, the function $w(\cdot,t)$ satisfies

$$\mathscr{L}^{\mathrm{ac}} w \geqslant \partial_t w + \frac{2f}{1-f}\langle \nabla f, \nabla w\rangle + 2\frac{|\nabla f|^4}{(1-f)^3} - 2K\frac{|\nabla f|^2}{(1-f)^2} \qquad \mu\text{-a.e.} \tag{3.11}$$

on $B_{3R/2}$, and

$$\mathscr{L}^{\mathrm{sing}} w \geqslant 0. \tag{3.12}$$

Proof. Without loss of generality, we may assume $M = 1$. By Lemma 3.9, $|\nabla u|^2 \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$. Note that for $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ and $1 \geqslant u \geqslant \delta > 0$, we have $|\nabla f|^2 = \frac{|\nabla u|^2}{u^2} \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$, and hence

$$w = \frac{|\nabla f|^2}{(1-f)^2} \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T}).$$

Moreover, for almost every t, we have $f(\cdot,t)$, $\partial_t f(\cdot,t) \in H^1(B_{2R}) \cap L^\infty(B_{2R})$ and, by Lemma 2.9, for almost every t,

$$\mathscr{L} f = \frac{1}{u}\mathscr{L} u - \frac{|\nabla u|^2}{u^2} = \frac{1}{u}\partial_t u - |\nabla f|^2 = \partial_t f - |\nabla f|^2 \ (\in H^1(B_{2R}) \cap L^\infty(B_{2R})), \tag{3.13}$$

in the sense of distributions on B_{2R}. We denote $g=\partial_t f-|\nabla f|^2$. For almost every $(x,t)\in B_{3R/2,T}$, we have

$$\begin{aligned}\partial_t w &= \frac{(1-f)^2\partial_t|\nabla f|^2+2(1-f)(\partial_t f)|\nabla f|^2}{(1-f)^4}\\ &= \frac{2\langle\nabla f,\nabla f_t\rangle}{(1-f)^2}+\frac{2(g+|\nabla f|^2)|\nabla f|^2}{(1-f)^3}\\ &= \frac{2\langle\nabla f,\nabla g+\nabla|\nabla f|^2\rangle}{(1-f)^2}+\frac{2(g+|\nabla f|^2)|\nabla f|^2}{(1-f)^3}.\end{aligned}\tag{3.14}$$

By Lemma 2.9, we get

$$\mathscr{L}w=\mathscr{L}(|\nabla f|^2)\cdot\frac{1}{(1-f)^2}+|\nabla f|^2\mathscr{L}\left(\frac{1}{(1-f)^2}\right)+2\langle\nabla|\nabla f|^2,\nabla\frac{1}{(1-f)^2}\rangle.\tag{3.15}$$

By the Bochner formula, Theorem 2.12, we have

$$\mathscr{L}^{\mathrm{ac}}(|\nabla f|^2)\geqslant\frac{2g^2}{N}+2\langle\nabla f,\nabla g\rangle-2K|\nabla f|^2+\frac{2N}{N-1}\left(\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle}{2|\nabla f|^2}-\frac{g}{N}\right)^2\tag{3.16}$$

μ-a.e. on $B_{3R/2}\cap\{|\nabla f|\neq0\}$. By Lemma 2.9,

$$\begin{aligned}\mathscr{L}\left(\frac{1}{(1-f)^2}\right)&=\frac{2}{(1-f)^3}\mathscr{L}f+\frac{6}{(1-f)^4}|\nabla f|^2\\ &=\frac{2}{(1-f)^3}g+\frac{6}{(1-f)^4}|\nabla f|^2\end{aligned}\tag{3.17}$$

and

$$2\langle\nabla|\nabla f|^2,\nabla\frac{1}{(1-f)^2}\rangle=\frac{4}{(1-f)^3}\langle\nabla f,\nabla|\nabla f|^2\rangle.\tag{3.18}$$

Combining (3.14)–(3.18), we have,

$$\begin{aligned}\mathscr{L}^{\mathrm{ac}}w-\partial_t w&=\frac{\mathscr{L}^{\mathrm{ac}}(|\nabla f|^2)}{(1-f)^2}+|\nabla f|^2\mathscr{L}^{\mathrm{ac}}\left(\frac{1}{(1-f)^2}\right)\\ &\quad+2\langle\nabla|\nabla f|^2,\nabla\frac{1}{(1-f)^2}\rangle-\partial_t w\\ &\geqslant\frac{1}{(1-f)^2}\left(\frac{2g^2}{N}-2K|\nabla f|^2+\frac{2N}{N-1}\left(\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle}{2|\nabla f|^2}-\frac{g}{N}\right)^2\right)\\ &\quad+\frac{6}{(1-f)^4}|\nabla f|^4+\frac{4}{(1-f)^3}\langle\nabla f,\nabla|\nabla f|^2\rangle-\frac{2}{(1-f)^2}\langle\nabla f,\nabla|\nabla f|^2\rangle\\ &\quad-\frac{2}{(1-f)^3}|\nabla f|^4\end{aligned}\tag{3.19}$$

μ-a.e. on $B_{3R/2}\cap\{|\nabla f|\neq0\}$. On the other hand, for almost every t, we have

$$0=\frac{2}{(1-f)^2}\langle\nabla f,\nabla|\nabla f|^2\rangle-2\langle\nabla f,\nabla w\rangle+\frac{4}{(1-f)^3}|\nabla f|^4\tag{3.20}$$

and

$$0=\frac{-2}{(1-f)^3}\langle\nabla f,\nabla|\nabla f|^2\rangle+\frac{2}{1-f}\langle\nabla f,\nabla w\rangle-\frac{4}{(1-f)^4}|\nabla f|^4.\tag{3.21}$$

Summing up (3.19)–(3.21), for almost every t, we have

$$
\begin{aligned}
\mathscr{L}^{\mathrm{ac}}w-\partial_t w \geqslant & -2K\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^4}{(1-f)^3}+\frac{2f}{1-f}\langle\nabla f,\nabla w\rangle \\
& +\frac{2g^2}{N(1-f)^2}+\frac{2N}{(N-1)(1-f)^2}\left(\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle}{2|\nabla f|^2}-\frac{g}{N}\right)^2 \\
& +\frac{2|\nabla f|^4}{(1-f)^4}+\frac{2\langle\nabla f,\nabla|\nabla f|^2\rangle}{(1-f)^3}
\end{aligned}
\tag{3.22}
$$

μ-a.e. on $B_{3R/2}\cap\{|\nabla f|\neq 0\}$. Note that by the Young inequality,

(3.23)

$$
\begin{aligned}
&\frac{2g^2}{N(1-f)^2}+\frac{2N}{(N-1)(1-f)^2}\left(\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle}{2|\nabla f|^2}-\frac{g}{N}\right)^2+\frac{2|\nabla f|^4}{(1-f)^4}+\frac{2\langle\nabla f,\nabla|\nabla f|^2\rangle}{(1-f)^3} \\
&\quad=\frac{2}{(1-f)^2}\left(\frac{g^2}{N-1}-\frac{g\langle\nabla f,\nabla|\nabla f|^2\rangle}{(N-1)|\nabla f|^2}+\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle^2}{4(N-1)|\nabla f|^4}\right) \\
&\quad\quad+\frac{2}{(1-f)^2}\left(\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle^2}{4|\nabla f|^4}+\frac{|\nabla f|^4}{(1-f)^2}+\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle}{1-f}\right) \\
&\quad\geqslant 0.
\end{aligned}
$$

Hence, by (3.22) and (3.23), we have, for almost every t,

$$
\mathscr{L}^{\mathrm{ac}}w-\partial_t w \geqslant -2K\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^4}{(1-f)^3}+\frac{2f}{1-f}\langle\nabla f,\nabla w\rangle
\tag{3.24}
$$

μ-a.e. on $B_{3R/2}\cap\{|\nabla f|\neq 0\}$. On the other hand, for almost all t, we have, on $B_{3R/2}\cap\{|\nabla f|=0\}$,

$$
\begin{aligned}
\mathscr{L}^{\mathrm{ac}}w-\partial_t w &= \frac{\mathscr{L}^{\mathrm{ac}}(|\nabla f|^2)}{(1-f)^2}+|\nabla f|^2\mathscr{L}^{\mathrm{ac}}\left(\frac{1}{(1-f)^2}\right) \\
&\quad +2\langle\nabla|\nabla f|^2,\nabla\frac{1}{(1-f)^2}\rangle-\partial_t w \\
&= \frac{\mathscr{L}^{\mathrm{ac}}(|\nabla f|^2)}{(1-f)^2}\geqslant 0.
\end{aligned}
\tag{3.25}
$$

Hence, by (3.24) and (3.25), for almost all t, we have

$$
\mathscr{L}^{\mathrm{ac}}w-\partial_t w \geqslant -2K\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^4}{(1-f)^3}+\frac{2f}{1-f}\langle\nabla f,\nabla w\rangle
$$

μ-a.e. on $B_{3R/2}$. The inequality (3.12) follows from (3.15), (3.17) and Theorem 2.12. Hence, we complete the proof. □

We are ready to prove Theorem 1.3 under some additional assumptions that $0<\delta\leqslant u\leqslant M$ and $\partial_t u\in H^1(B_{2R,T})\cap L^\infty(B_{2R,T})$.

Lemma 3.26. *Let $K\geqslant 0$ and $N\in(1,\infty)$, and let (X,d,μ) be a metric measure space satisfying $RCD^*(-K,N)$. Let $T\in(0,\infty)$, let $B_{2R}:=B_{2R}(x_0)$, let $B_{2R,T}=B_{2R}\times(0,T)$, and let $u(x,t)\in W^{1,2}\big(B_{2R,T}\big)$ be a positive local weak solution of the* *heat equation on* $B_{2R,T}$. *Suppose also that* $0 < \delta \leqslant u \leqslant M$ *and* $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$. *Then, we have*

$$\sup_{B_R \times (3T/4,T)} \frac{|\nabla f(x,t)|}{1 - f(x,t)} \leqslant C \cdot \left(\frac{\sqrt{N}}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right),$$

where $f = \ln(u/M)$. *Here and in what follows,* C *is a universal constant* (*independent of* N, K *and* R).

Proof. Let w be as in Lemma 3.10. Then we have $w \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$. Without loss of generality, we assume $M = 1$. We follow the strategies of [38, 44].

Step 1. (Setting up the cut-off functions)
We put

$$M_1 := \sup_{B_R \times (3T/4,T)} w \qquad \text{and} \qquad M_2 := \sup_{B_{3R/2} \times (T/2,T)} w.$$

We now choose $\phi(x) = \phi(r(x))$ to be a function of the distance r to x_0 with the following properties that

$$\left(\frac{M_1}{2M_2}\right)^{1/5} \leqslant \phi \leqslant 1 \text{ on } B_{3R/2}, \quad \phi = 1 \text{ on } B_R, \quad \phi = \left(\frac{M_1}{2M_2}\right)^{1/5} \text{ on } B_{3R/2} \backslash B_{5R/4},$$

and

$$-\frac{C}{R}\phi^{\frac{1}{2}} \leqslant \phi'(r) \leqslant 0 \quad \text{and} \quad |\phi''(r)| \leqslant \frac{C}{R^2} \qquad \forall\, r \in (0, 3R/2)$$

for some universal constant C (independent of N, K, and R).

Similarly as above, we choose $\xi(t)$ to be a cut-off function such that

$$\xi = 1 \text{ on } (3T/4, T), \quad \left(\frac{M_1}{2M_2}\right)^{1/5} \leqslant \xi \leqslant 1 \text{ on } (T/2, T)$$

and

$$\xi = \left(\frac{M_1}{2M_2}\right)^{1/5} \quad \text{on } (T/2, 5T/8),$$

and

$$-\frac{C}{R}\xi^{\frac{1}{2}} \leqslant \xi'(t) \leqslant 0 \qquad \forall\, t \in (T/2, T).$$

Let $\psi = \phi^4 \cdot \xi$. Then, it is easy to check that

$$\psi = 1 \text{ on } B_R \times [3T/4, T], \tag{3.27}$$

ψ is decreasing as a radial function in the spatial variables,

$$|\nabla \psi| \leqslant \frac{C}{R}\psi^{3/4} \leqslant \frac{C}{R}\sqrt{\psi} \quad \text{on } B_{3R/2}, \tag{3.28}$$

$$-\sqrt{\psi}\frac{C}{T} \leqslant \partial_t \psi \leqslant 0 \quad on (T/2, T], \tag{3.29}$$

$$|\psi''(r)| \leqslant \frac{C}{R^2}\sqrt{\psi} \qquad \forall\, r \in (0, 3R/2). \tag{3.30}$$

We now give an estimate for $\mathscr{L}\psi$. By the Laplacian comparison theorem [14, Corollary 5.15] for $RCD^*(-K,N)$ with $N > 1$, $K > 0$, and (3.28) and (3.30), we get

(3.31)

$$\mathscr{L}\psi = \psi'\mathscr{L}r + \psi''|\nabla r|^2 \geqslant -\frac{C}{R}\Big(\sqrt{(N-1)K}\coth\big(r\sqrt{\frac{K}{N-1}}\big)\Big)\sqrt{\psi} - \frac{C}{R^2}\sqrt{\psi}$$

on $B_{3R/2}$, in the sense of distributions. Note that $\psi = 0$ on B_R, and if $r \geqslant R$, we have

$$\coth\big(r\sqrt{\frac{K}{N-1}}\big) \leqslant \coth\big(R\sqrt{\frac{K}{N-1}}\big) \leqslant 1 + \frac{1}{R\sqrt{K/(N-1)}}.$$

Hence, we have

$$\begin{aligned} \mathscr{L}\psi &\geqslant -\frac{C}{R}\Big(\sqrt{(N-1)K} + \frac{N-1}{R}\Big)\sqrt{\psi} - \frac{C}{R^2}\sqrt{\psi} \\ &\geqslant \left(-\frac{C}{R}\sqrt{(N-1)K} - \frac{CN}{R^2}\right)\sqrt{\psi} \end{aligned} \tag{3.32}$$

on $B_{3R/2}$, in the sense of distributions. In fact, the estimate (3.32) still holds for $RCD^*(-K,N)$ with $N \geqslant 1$ and $K \geqslant 0$. Indeed, in the case when $K = 0$ and $N > 1$, the Laplacian comparison theorem states that $\mathscr{L}r \leqslant (N-1)/r$. Then (3.32) still holds.

By (3.32), $\mathscr{L}\psi$ is a signed Radon measure. Then its absolutely continuous part

$$(\mathscr{L}\psi)^{\mathrm{ac}} \geqslant \left(-\frac{C}{R}\sqrt{(N-1)K} - \frac{CN}{R^2}\right)\sqrt{\psi} \quad \mu\text{-a.e. } x \in B_{3R/2}, \tag{3.33}$$

and its singular part

$$(\mathscr{L}\psi)^{\mathrm{sing}} \geqslant 0. \tag{3.34}$$

Step 2. (Maximum principle arguments)

Claim. We have, for almost all t,

$$\begin{aligned} &\mathscr{L}^{\mathrm{ac}}(w\psi) + \frac{2f}{f-1}\langle\nabla f, \nabla(w\psi)\rangle - 2\frac{\langle\nabla\psi, \nabla(w\psi)\rangle}{\psi} - \partial_t(w\psi) \\ &\qquad \geqslant 2\psi(1-f)w^2 + \frac{2fw}{f-1}\langle\nabla f, \nabla\psi\rangle - 2\frac{w|\nabla\psi|^2}{\psi} + w\mathscr{L}^{ac}\psi - w\partial_t\psi - 2Kw\psi \end{aligned}$$

μ-a.e. on $B_{3R/2}$.

Proof of Claim. By Lemmas 2.9 and 3.10,

$$\begin{aligned} \mathscr{L}^{\mathrm{ac}}(\psi w) &= w\mathscr{L}^{\mathrm{ac}}\psi + \psi\mathscr{L}^{\mathrm{ac}}w + 2\langle\nabla w, \nabla\psi\rangle \\ &\geqslant w\mathscr{L}^{\mathrm{ac}}\psi + \psi\left(\partial_t w - \frac{2f}{1-f}\langle\nabla f, \nabla w\rangle + 2(1-f)w^2 - 2Kw\right) + 2\langle\nabla w, \nabla\psi\rangle \end{aligned}$$

μ-a.e. on $B_{3R/2}$. Hence, using the above inequality, we get

$$\begin{aligned} &\mathscr{L}^{\mathrm{ac}}(w\psi) + \frac{2f}{f-1}\langle\nabla f, \nabla(w\psi)\rangle - 2\frac{\langle\nabla\psi, \nabla(w\psi)\rangle}{\psi} - \partial_t(w\psi) \\ &\qquad \geqslant 2\psi(1-f)w^2 + \frac{2fw}{f-1}\langle\nabla f, \nabla\psi\rangle - 2\frac{w|\nabla\psi|^2}{\psi} + w\mathscr{L}^{\mathrm{ac}}\psi - w\partial_t\psi - 2Kw\psi \end{aligned}$$

μ-a.e. on $B_{3R/2}$. □

We now return to the proof of Lemma 3.26. Let $G = w\psi$, and let $\overline{G} = \sup_{B_{3R/2}\times(T/2,T]} G$. Notice that

$$\begin{aligned}&\mathscr{L}(w\psi)+\frac{2f}{f-1}\langle\nabla f,\nabla(w\psi)\rangle-2\frac{\langle\nabla\psi,\nabla(w\psi)\rangle}{\psi}-\partial_t(w\psi)\\&\qquad=\mathscr{L}(w\psi)+\langle\nabla G,\nabla(2f+2\ln(1-f)-2\ln\psi)\rangle-\partial_t(w\psi).\end{aligned}$$

By the definition of ψ, the function G attains its maximum of $B_{3R/2}\times(T/2,T)$ in $B_{5R/4}\times(5T/8,T)$ in the sense of Theorem 2.13. By (3.34) and (3.12), we have

$$\mathscr{L}^{\rm sing}G=w\mathscr{L}^{\rm sing}\psi+\psi\mathscr{L}^{\rm sing}w\geqslant 0$$

for almost every t. By Theorem 2.13 with $v=2f+2\ln(1-f)-2\ln\psi$, there exists $\{(x_j,t_j)\}$, such that

$$G(x_j,t_j)\geqslant\overline{G}-1/j \tag{3.35}$$

and

$$\mathscr{L}^{\rm ac}G(x_j,t_j)+\langle\nabla G,\nabla v\rangle(x_j,t_j)-\partial tG(x_j,t_j)\leqslant 1/j. \tag{3.36}$$

By (3.36) and the above Claim,

$$\begin{aligned}&\left(2\psi(1-f)w^2+\frac{2fw}{f-1}\langle\nabla f,\nabla\psi\rangle-2\frac{w|\nabla\psi|^2}{\psi}+w\mathscr{L}^{\rm ac}\psi-w\partial_t\psi-2Kw\psi\right)(x_j,t_j)\\&\qquad\leqslant 1/j,\end{aligned}$$

and, hence,

$$\begin{aligned}&(2\psi(1-f)w^2)(x_j,t_j)\\&\qquad\leqslant-\left(\frac{2fw}{f-1}\langle\nabla f,\nabla\psi\rangle-2\frac{w|\nabla\psi|^2}{\psi}+w\mathscr{L}^{\rm ac}\psi-w\partial_t\psi-2Kw\psi\right)(x_j,t_j)\\&\qquad\quad+1/j.\end{aligned}\tag{3.37}$$

By Young's inequality,

$$-\frac{2fw}{f-1}\langle\nabla f,\nabla\psi\rangle\leqslant 2w^{3/2}|f|\cdot|\nabla\psi|\leqslant\psi(1-f)w^2+C\frac{f^4\cdot|\nabla\psi|^4}{(\psi(1-f))^3}. \tag{3.38}$$

Thus, by (3.28), we have

$$-\frac{2fw}{f-1}\langle\nabla f,\nabla\psi\rangle\leqslant\psi(1-f)w^2+C\frac{f^4}{R^4(1-f)^3}. \tag{3.39}$$

Using the Young inequality and (3.28), we get

$$\frac{w|\nabla\psi|^2}{\psi}\leqslant\frac{\psi w^2}{8}+C\left(\frac{|\nabla\psi|^2}{\psi^{3/2}}\right)^2\leqslant\frac{\psi w^2}{8}+C/R^4. \tag{3.40}$$

By (3.33),

$$\begin{aligned}w\mathscr{L}^{\rm ac}\psi&\geqslant\left(-\frac{C}{R}\sqrt{(N-1)K}-\frac{CN}{R^2}\right)\sqrt{\psi}w\\&\geqslant-\frac{\psi w^2}{8}-\left(\frac{C}{R}\sqrt{(N-1)K}+\frac{CN}{R^2}\right)^2\\&\geqslant-\frac{\psi w^2}{8}-\frac{C}{R^2}(N-1)K-\frac{CN^2}{R^4}.\end{aligned}\tag{3.41}$$

By (3.29) and the Young inequality,

$$|\partial_t\psi|w \leqslant \psi w^2/8 + C/T^2. \tag{3.42}$$

By the Young inequality and the fact that $0 \leqslant \psi \leqslant 1$,

$$2Kw\psi \leqslant \psi w^2/8 + CK^2. \tag{3.43}$$

Combining (3.37)–(3.43), we have

$$\begin{aligned}(2(1-f)\psi w^2)(x_j,t_j) \leqslant &\left((1-f)\psi w^2 + \frac{\psi w^2}{2} + \frac{Cf^4}{R^4(1-f)^3}\right)(x_j,t_j)\\ &+\frac{1}{j}+\frac{C}{R^4}+\frac{CN^2}{R^4}+\frac{C}{R^2}(N-1)K+\frac{C}{T^2}+CK^2,\end{aligned}$$

and hence

$$\begin{aligned}((1-f)\psi w^2)(x_j,t_j) \leqslant &\frac{\psi w^2}{2}(x_j,t_j)+\frac{1}{j}+\frac{Cf^4}{R^4(1-f)^3}(x_j,t_j)+\frac{CN^2}{R^4}\\ &+\frac{C}{R^2}(N-1)K+\frac{C}{T^2}+CK^2.\end{aligned}$$

By $f \leqslant 0$, and hence $1-f \geqslant 1$, $f/(1-f) \leqslant 1$, and that $0 \leqslant \psi \leqslant 1$,

$$(\psi^2 w^2)(x_j,t_j) \leqslant \frac{2}{j}+\frac{C}{R^4}+\frac{CN^2}{R^4}+\frac{C}{R^2}(N-1)K+\frac{C}{T^2}+CK^2.$$

By the above inequality and (3.35), letting $j \to \infty$,

$$\overline{G}^2 \leqslant \frac{C}{R^4}+\frac{CN^2}{R^4}+\frac{C}{R^2}(N-1)K+\frac{C}{T^2}+CK^2 \leqslant \frac{CN^2}{R^4}+\frac{C}{T^2}+CK^2.$$

Hence,

$$\overline{G} \leqslant \frac{CN}{R^2}+\frac{C}{T}+CK.$$

Note that $\psi = 1$ on $B_R \times (3T/4, T]$ by (3.27). Thus,

$$\sup_{B_R\times(3T/4,T]} w \leqslant \frac{CN}{R^2}+\frac{C}{T}+CK.$$

By definition of w, we have

$$\sup_{B_R\times(3T/4,T]} \frac{|\nabla f|}{1-f} \leqslant \frac{C\sqrt{N}}{R}+\frac{C}{\sqrt{T}}+C\sqrt{K}. \qquad \square$$

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. Given any $\mathbb{Q} \ni \delta > 0$, by [41, Theorem 2.2], $u+\delta \in L^\infty_{\mathrm{loc}}(B_{2R,T})$. Since the desired estimate is a local estimate, without loss of generality, we may assume that $u+\delta \in L^\infty(B_{2R,T})$.

Given any $\mathbb{Q} \ni \varepsilon > 0$ sufficiently small, by Lemma 3.8, for almost all $t \in (0, T-\varepsilon)$,

$$\mathscr{L}\left((u+\delta)_h\right) = \partial_t\left((u+\delta)_h\right)$$

on B_R, in the sense of distributions. By Lemma 3.7, we apply Lemma 3.26 to the Steklov averages $(u+\delta)_h$. Then, we have

$$\sup_{B_R\times(\frac{3(T-\varepsilon)}{4},T-\varepsilon]} \frac{|\nabla f_{\delta,h}(x,t)|}{1-f_{\delta,h}(x,t)} \leqslant C\cdot\left(\frac{\sqrt{N}}{R}+\frac{1}{\sqrt{T-\varepsilon}}+\sqrt{K}\right),$$

where $f_{\delta,h} = \ln(\frac{(u+\delta)_h}{M+\delta})$, $0 < h < \varepsilon$. By letting $\mathbb{Q} \ni h \to 0$, we get

$$\sup_{B_R \times (\frac{3(T-\varepsilon)}{4}, T-\varepsilon]} \frac{|\nabla f_\delta(x,t)|}{1 - f_\delta(x,t)} \leqslant C \cdot \left(\frac{\sqrt{N}}{R} + \frac{1}{\sqrt{T-\varepsilon}} + \sqrt{K} \right),$$

where $f_\delta = \ln(\frac{u+\delta}{M+\delta})$. By letting $\delta \to 0$ and the arbitrariness of ε, we complete the proof. $\square$

Using Theorem 1.3, [44, Corollary 1.5] with $K = 0$, and the same arguments as in the proof of [38, eq. (1.5)], we can prove Corollary 1.5.

Using Theorem 1.3, and by the same arguments as in [38, Theorem 1.2], we conclude Theorem 1.6.

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