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# RATIONAL APPROXIMATION OF $\mathbf{x}^n$

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Abstract. Let $E_{kk}^{(n)}$ denote the minimax (i.e., best supremum norm) error in approximation of $x^n$ on $[0,1]$ by rational functions of type $(k,k)$ with $k<n$. We show that in an appropriate limit $E_{kk}^{(n)} \sim 2H^{k+1/2}$ independently of $n$, where $H \approx 1/9.28903$ is Halphen's constant. This is the same formula as for minimax approximation of $e^x$ on $(-\infty,0]$.

## 1. Introduction

We consider minimax approximation of $x^n$ on $[0,1]$, that is, best approximation with respect to the supremum norm $\|\cdot\|$ on $[0,1]$. Although $n$ is usually thought of as an integer, we permit it to be any nonnegative real number. If $n$ is an even integer, approximation of $x^n$ on $[-1,1]$ is equivalent to approximation of $x^{n/2}$ on $[0,1]$, and results will be stated for both intervals.

For each integer $k \geq 0$, there is a unique minimax approximant $p_k^{(n)}$ of $x^n$ on $[0,1]$ among polynomials of degree at most $k$ [14]. Let $E_k^{(n)} = \|x^n - p_k^{(n)}\|$ denote the associated error, which will be nonzero whenever $k<n$. In 1976 Newman and Rivlin [10] published theorems showing that

$$E_k^{(n)} \approx \tfrac{1}{2}\operatorname{erfc}(k/\sqrt{n}), \tag{1.1}$$

where $\operatorname{erfc}(s) = 2\pi^{-1/2}\int_s^\infty \exp(-t^2)dt$ is the complementary error function. (The constant $1/2$ is our own, based on numerical experiments.) This formula implies that a degree $k = O(\sqrt{n})$ suffices for polynomial approximation of $x^n$ to high accuracy. To illustrate this effect, Figure 1 plots $E_k^{(n)}$ against $k^2$ for the cases $n = 250$ and $1000$, showing good agreement with (1.1). The data in our two figures have been computed with the `minimax` command in Chebfun [3, 9].

We have found that rational functions are far more effective at approximating $x^n$ than polynomials. To be precise, consider approximation by real rational functions of type $(k,k)$, that is, functions that can be written in the form $r(x) = p(x)/q(x)$ where $p$ and $q$ are real polynomials of degree at most $k$. Again, standard theory shows that for each nonnegative real number $n$ and each nonnegative integer $k$, there exists a unique minimax approximant $r_{kk}^{(n)}$ [14]; we denote the error by $E_{kk}^{(n)} = \|x^n - r_{kk}^{(n)}\|$. Here we will prove that, as illustrated in Figure 2, the errors are closely approximated by the formula

$$E_{kk}^{(n)} \approx 2H^{k+1/2}, \quad H = 1/9.2890254919208\ldots, \tag{1.2}$$

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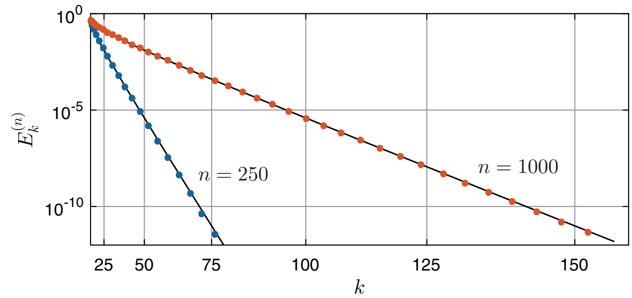


FIGURE 1. Errors in minimax approximation of $x^n$ by polynomials of degree $k$; the solid lines show the approximation (1.1). The convergence is exponential as a function of $k^2/n$. The horizontal axis is scaled quadratically, so this behavior shows up as straight lines.

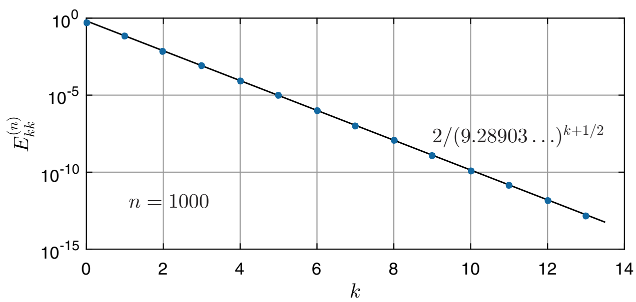


FIGURE 2. With rational functions of type $(k, k)$, the convergence is much faster: exponential as a function of $k$, approximately independent of $n$. The solid line shows the approximation (1.2). In this experiment $n = 1000$, but the data would be approximately the same for any other large value of $n$.

which has the remarkable property of being independent of $n$. The number $H$, known as Halphen's constant, appears in the problem of approximation of $\exp(x)$ for $x \in (-\infty, 0]$, where the minimax errors are asymptotic to exactly the same expression (1.2). Chapter 25 of [14] gives a review of this famous problem of rational approximation theory, a story that among others has involved Aptekarev, Carpenter, Cody, Gonchar, Gutknecht, Magnus, Meinardus, Rakhmanov, Ruttan, Trefethen, and Varga. Halphen first identified the number now named after him in 1886 [5], though not in connection with approximation theory.

## 2. THEOREMS

To prove that the errors satisfy an estimate of the form (1.2), we exploit the fact that the set of rational functions of type $(k, k)$ is invariant under Möbius transformation. In particular, we transplant the approximation domain $[0, 1]$ to $(-\infty, 0]$ by the Möbius transformation that maps $x = 0$, $1$ and $1 + 1/(n - 1)$ to

$s = -\infty$, 0, and 1:

$$x = \frac{n}{n-s}, \qquad s = \frac{n(x-1)}{x}.$$

The function $x^n$ transplants to

$$x^n = (n/(n-s))^n = (1-s/n)^{-n}, \tag{2.1}$$

and this establishes our first lemma.

**Lemma 2.1.** *For any real number $n > 0$ and integer $k \geq 0$, the error $E_{kk}^{(n)}$ in type $(k,k)$ minimax approximation of $x^n$ on $[0,1]$ is equal to the error in type $(k,k)$ minimax approximation of $(1-s/n)^{-n}$ on $(-\infty, 0]$.*

Our second lemma quantifies the fact that $(1-s/n)^{-n} \approx e^s$ for $s \in (-\infty, 0]$.

**Lemma 2.2.** *For any $n \in (0,\infty)$ and $s \in (-\infty, 0)$,*

$$0 < (1-s/n)^{-n} - e^s \leq \frac{1}{en}. \tag{2.2}$$

*Proof.* Given $n$, define $g(s) = (1-s/n)^{-n} - e^s$, with $g(-\infty) = g(0) = 0$. From a binomial series we may verify that $(1-s/n)^n < e^{-s}$ for each $s$, and taking reciprocals we establish that $(1-s/n)^{-n} > e^s$, i.e., $g(s) > 0$. The maximum value of $g(s)$ will be attained at a point $s = \sigma$ where the derivative

$$g'(s) = (1-s/n)^{-(n+1)} - e^s$$

is zero, i.e., $(1-\sigma/n)^{-(n+1)} = e^\sigma$. At such a point we calculate

$$g(\sigma) = e^\sigma(1-\sigma/n) - e^\sigma = -\sigma e^\sigma/n,$$

and to complete the proof we note that $0 < -\sigma e^\sigma \leq 1/e$ for $\sigma \in (-\infty, 0)$. □

We can now derive our main result.

**Theorem 2.3.** *The errors in type $(k,k)$ rational minimax approximation of $x^n$ on $[0,1]$ satisfy*

$$\lim_{k\to\infty} \lim_{n\to\infty} E_{kk}^{(n)} \Big/ 2H^{k+1/2} = 1, \tag{2.3}$$

*where $H \approx 1/9.28903$ is Halphen's constant. In this formula $n$ may range over nonnegative real numbers or over nonnegative integers.*

*Proof.* Let $F_{kk}$ denote the error in minimax type $(k,k)$ rational approximation of $e^s$ on $(-\infty, 0]$. Aptekarev [1] established the identity

$$\lim_{k\to\infty} F_{kk} \Big/ 2H^{k+1/2} = 1, \tag{2.4}$$

which had been conjectured earlier by Magnus [6]. On the other hand, Lemmas 2.1 and 2.2 imply that

$$F_{kk} = \lim_{n\to\infty} E_{kk}^{(n)}. \tag{2.5}$$

Equation (2.3) follows from (2.4) and (2.5). □

Equation (2.3) says little about the errors associated with any finite value of $n$. Numerical data such as those plotted in Figure 2 suggest that much sharper estimates are probably valid, with $E_{kk}^{(n)}$ coming much closer to $2H^{k+1/2}$ than is shown by our arguments.

As mentioned at the outset, approximation of $x^n$ on $[-1,1]$ is equivalent to approximation of $x^{n/2}$ on $[0,1]$ when $n$ is an even integer. The equivalence is spelled out in the proof of the following theorem, which uses the same notation $E_{kk}^{(n)}$ for $[-1,1]$ as used previously for $[0,1]$.

**Theorem 2.4.** *The errors in type* $(k,k)$ *rational minimax approximation of* $x^n$ *on* $[-1,1]$ *satisfy*

$$\lim_{k\to\infty} \lim_{\substack{n\to\infty \\ n \text{ even}}} E_{kk}^{(n)} \Big/ 2H^{\lfloor k/2\rfloor+1/2} = 1. \tag{2.6}$$

*In this formula* $n$ *ranges over nonnegative even integers.*

*Proof.* Let $n$ be a nonnegative even integer. If $k$ is even, then by the change of variables $s = x^2$ (see for example p. 213 of [14]), we find that type $(k,k)$ approximation of $x^n$ on $[-1,1]$ is equivalent to type $(k/2,k/2)$ approximation of $x^{n/2}$ on $[0,1]$. If $k$ is odd, then the uniqueness of best approximants implies that the type $(k,k)$ approximant of $x^n$ on $[-1,1]$ must still be even, hence the same as the type $(k-1,k-1)$ approximant (see e.g. Exercise 24.1 of [14]). These observations justify the floor function $\lfloor k/2\rfloor$ of (2.6). □

If $n$ is odd, the errors are approximately but not exactly the same.

## 3. Discussion

In [14] it is emphasized that rational approximants tend to greatly outperform polynomials in cases where (i) the function to be approximated has a nearby singularity or (ii) the domain of approximation is unbounded. Approximation of $x^n$ on $[0,1]$ is essentially a problem of type (i), with nearly singular behavior at $x \approx 1$ (not technically singular, of course, but one could speak of a "pseudo-singularity"). It is interesting that the proof of Theorem 2.3 proceeds by conversion to an equivalent problem of type (ii).

The $k = O(\sqrt{n})$ effect for polynomial approximation of $x^n$ has practical consequences. For example, Chebfun's method of numerical computation with functions depends on representing them adaptively to approximate machine precision ($\approx$16 digits) by Chebyshev expansions. Table 1 lists the degrees $k$ of the Chebfun polynomials representing various powers $x^n$ on $[0,1]$. We see that for small $n$, the system requires that $k = n$, but for larger values, each quadrupling of $n$ brings approximately just a doubling of $k$.

Another aspect of the $k = O(\sqrt{n})$ effect is discussed by Cornelius Lanczos in a fascinating video recording from 1972 available online [7] (beginning at about time 10:00); for a written discussion see chapter 5 of his book [8]. Lanczos speaks of the monomials $\{x^n\}$ as a "tremendously nonorthogonal system," a fact quantified by the Müntz–Szász theorem [12], and observes that it was this effect that led him to invent what are now called Chebyshev spectral methods for the numerical solution of differential equations [2, 8, 14]. Numerical analysts would rarely cite the Müntz–Szász theorem, but they are well aware that monomials provide exponentially ill-conditioned bases on real intervals, making them nearly useless for

TABLE 1. Chebfun [3] constructs a polynomial of an adaptively determined degree $k$ to represent a function on a given interval to about 16 digits of accuracy. For $x^n$ on $[0, 1]$, $k = n$ is needed for smaller values of $n$, whereas for larger values, $k$ grows at a rate $O(\sqrt{n})$ consistent with (1.1).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | 1 | 4 | 16 | 64 | 256 | 1024 | 4096 |
| $k$ | 1 | 4 | 16 | 44 | 91 | 178 | 349 |

numerical computing, whereas suitably scaled Chebyshev polynomials are excellent for computation because they give well-conditioned bases.

One can explain intuitively why $k = O(\sqrt{n})$ suffices for polynomial approximation of $x^n$ as follows. As is well known, polynomials are intrinsically nonuniform in their representation power on an interval [4, 14]. They have more power near the endpoints than in the middle, an effect quantified by the $O((1-x^2)^{-1/2})$ factor that appears widely in analyses of orthogonal polynomials on $[-1, 1]$. For example, the grids used in Chebyshev spectral methods cluster quadratically near the endpoints and give enhanced resolution of functions with boundary layers. This enhanced resolution is precisely the issue, for $x^n$ is a function with a boundary layer, making its transition from 0 to 1 over a region $x \approx 1$ of a width that scales as $O(1/n)$. Since polynomials have quadratic resolution for $x \approx 1$, degree $k = O(\sqrt{n})$ is the right scaling to resolve this function.

These observations pertain to polynomial approximation of $x^n$, whereas the new results of this paper concern the much greater power of rational approximations. There is some previous literature on rational approximation of $x^n$, and an early survey can be found in [11]. The most developed part of this problem has been the case in which $n$ is a fixed positive number that is not an integer and $k \to \infty$. Here one obtains root-exponential convergence with respect to $k$; see [13] for both sharp results and a survey of earlier work. The more basic phenomenon considered in the present paper of exponential convergence for $k \to \infty$ for large $n$ seems not to have been noted previously nor, in particular, the connection with $\approx 9.28903$. Perhaps there may be applications where this too will have practical consequences.

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