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REMARKS ON THE ABELIAN CONVEXITY THEOREM

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ABSTRACT. This paper contains some observations on abelian convexity theorems. Convexity along an orbit is established in a very general setting using Kempf–Ness functions. This is applied to give short proofs of the Atiyah–Guillemin–Sternberg theorem and of abelian convexity for the gradient map in the case of a real analytic submanifold of complex projective space. Finally we give an application to the action on the probability measures.

1. Introduction

1.1. Let U be a compact connected Lie group, and let $U^{\mathbb{C}}$ be its complexification. Let (Z,ω) be a Kähler manifold on which $U^{\mathbb{C}}$ acts holomorphically. Assume that U acts in a Hamiltonian fashion with momentum map $\mu : Z \longrightarrow \mathfrak{u}^*$. This means that ω is U-invariant, μ is equivariant, and for any $\beta \in \mathfrak{u}$ we have

$$d\mu^\beta = i_{\beta_Z}\omega,$$

where $\mu^\beta = \langle \mu, \beta \rangle$ and β_Z denotes the fundamental vector field on Z induced by the action of U. It is well known that the momentum map represents a fundamental tool in the study of the action of $U^{\mathbb{C}}$ on Z. Of particular importance are convexity theorems [1, 15, 16, 27], which depend on the fact that the functions μ^β are Morse–Bott with even indices.

1.2. More recently the momentum map has been generalized to the following setting [19–23]. Let $G \subset U^{\mathbb{C}}$ be a closed connected subgroup of $U^{\mathbb{C}}$ that is *compatible* with respect to the Cartan decomposition of $U^{\mathbb{C}}$. This means that G is a closed subgroup of $U^{\mathbb{C}}$ such that $G = K \exp(\mathfrak{p})$, where $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ [22, 24]. The inclusion $i\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces by restriction a K-equivariant map $\mu_{i\mathfrak{p}} : Z \longrightarrow (i\mathfrak{p})^*$. Using a fixed U-invariant scalar product $\langle\, , \rangle$ on $\mathfrak{u}$, we identify $\mathfrak{u} \cong \mathfrak{u}^*$. We also denote by $\langle\, , \rangle$ the scalar product on $i\mathfrak{u}$ such that multiplication by i is an isometry of $\mathfrak{u}$ onto $i\mathfrak{u}$. For $z \in Z$ let $\mu_{\mathfrak{p}}(z) \in \mathfrak{p}$ denote $-i$ times the component of $\mu(z)$ in the

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direction of $i\mathfrak{p}$. In other words we require that

$$\langle \mu_{\mathfrak{p}}(z), \beta \rangle = -\langle \mu(z), i\beta \rangle, \tag{1.3}$$

for any $\beta \in \mathfrak{p}$. The map $\mu_{\mathfrak{p}} : Z \to \mathfrak{p}$ is called the *G-gradient map*. Given a compact G-stable subset $X \subset Z$ we consider the restriction $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$. We also set

$$\mu_{\mathfrak{p}}^{\beta} := \langle \mu_{\mathfrak{p}}, \beta \rangle = \mu^{-i\beta}.$$

Many fundamental theorems regarding the momentum map also hold for the gradient map. The functions $\mu_{\mathfrak{p}}^{\beta}$ are Morse–Bott, although in general not with even indices. Even so in [20] (see also [4]) the authors prove the following convexity theorem: let V be a unitary representation of U, and let $Y \subset \mathbb{P}(V)$ be a closed real semialgebraic subset whose real algebraic Zariski closure is irreducible. If $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$ and $\mathfrak{a}_+$ is a positive Weyl chamber, then $\mu_{\mathfrak{p}}(Y) \cap \mathfrak{a}_+$ is a convex polytope. The proof is rather delicate.

One of the goals of the present paper is to give a convexity theorem along an orbit, i.e., to show that the image of an orbit via the gradient map is convex. This will be proved in a very general setting using only so-called Kempf–Ness functions. This allows us to prove the corresponding theorem for the gradient map without using results from the complex case. As applications we get a simple proof of the abelian convexity theorem for the gradient map for real analytic submanifolds and the convexity along an orbit for the gradient map associated to the induced action on probability measures.

1.4. Using the same notation as above, assume that X is a compact G-invariant submanifold of Z. In [6, 11] the authors and Zedda studied the action of G on the set of probability measures on X. This set is not a manifold, but many features of the action, especially those relating only to a single orbit closure, can be studied with a formalism very similar to the momentum map. We now recall this formalism.

Let $\mathscr{M}$ be a Hausdorff topological space, and let G be a noncompact real reductive group which acts continuously on $\mathscr{M}$. We can write $G = K\exp(\mathfrak{p})$, where K is a maximal compact subgroup of G. Given a function $\Psi : \mathscr{M} \times G \to \mathbb{R}$, consider the following properties:

- (P1) For any $x \in \mathscr{M}$ the function $\Psi(x, \cdot)$ is smooth on G.
- (P2) The function $\Psi(x, \cdot)$ is left–invariant with respect to K, i.e., $\Psi(x, kg) = \Psi(x, g)$.
- (P3) For any $x \in \mathscr{M}$, and any $\xi \in \mathfrak{p}$ and $t \in \mathbb{R}$:

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(t\xi)) \geq 0.$$

  Moreover:

$$\left.\frac{\mathrm{d}^2}{\mathrm{dt}^2}\right|_{t=0} \Psi(x, \exp(t\xi)) = 0$$

  if and only if $\exp(\mathbb{R}\xi) \subset G_x$.
- (P4) For any $x \in \mathscr{M}$, and any $g, h \in G$:

$$\Psi(x, g) + \Psi(g \cdot x, h) = \Psi(x, hg).$$

  This equation is called the *cocycle condition*.

In order to state our fifth condition, let $\langle\, , \,\rangle : \mathfrak{p}^* \times \mathfrak{p} \to \mathbb{R}$ be the duality pairing. For $x \in \mathscr{M}$ define $\mathfrak{F}(x) \in \mathfrak{p}^*$ by requiring that

$$\langle \mathfrak{F}(x), \xi \rangle = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(x, \exp(t\xi)). \tag{1.5}$$

(P5) The map $\mathfrak{F} : \mathscr{M} \to \mathfrak{p}^*$ is continuous.

Definition 1.1. Let G be a noncompact real reductive Lie group, let K be a maximal compact subgroup of G, and let $\mathscr{M}$ be a Hausdorff topological space with a continuous G-action. A *Kempf–Ness function* for $(\mathscr{M}, G, K)$ is a function

$$\Psi : \mathscr{M} \times G \to \mathbb{R}$$

that satisfies conditions (P1)–(P5). The map $\mathfrak{F}$ is called the *gradient map* of $(\mathscr{M}, G, K, \Psi)$.

By [11, Prop. 5] $\mathfrak{F} : \mathscr{M} \to \mathfrak{p}^*$ is a K-equivariant map. Since K is compact, we may fix a K-invariant scalar product $\langle \cdot, \cdot \rangle$ of $\mathfrak{p}$ and we may identify $\mathfrak{p}^* \cong \mathfrak{p}$ by means of $\langle \cdot, \cdot \rangle$. Hence we may think of the gradient map as a $\mathfrak{p}$-valued map $\mathfrak{F} : \mathscr{M} \to \mathfrak{p}$.

Remark 1.1. In [6, 11] a sixth hypothesis is assumed, which is necessary to define the maximal weight and to deal with stability issues. This hypothesis is not needed for the arguments of the present paper.

1.6. The original setting [26] for what we call a Kempf–Ness function is the following: let V be a unitary representation of U. For $x = [v] \in \mathbb{P}(V)$ and $g \in U^{\mathbb{C}}$ set $\Psi(x, g) := \log(|g^{-1}v|/|v|)$. This function satisfies (P1)–(P5) with $\mathfrak{F} = \mu$, the momentum map. Thus the behaviour of the momentum map is encoded in the function Ψ. Functions similar to these exist for rather general actions. The following result has been proven in [18, §2], [2], [29] for $G = U^{\mathbb{C}}$ and in [11, §7] in the general case.

Proposition 1.1. *Let X, G, K be as in* §1.2. *Then there exists a Kempf–Ness function Ψ for (X, G, K) satisfying the conditions* (P1)–(P5) *such that $\mathfrak{F} = \mu_{\mathfrak{p}}$.*

1.7. In the present paper we study abelian convexity theorems. In §2 we give an easy proof of convexity for the image of an orbit of an abelian group in the setting of Kempf–Ness functions; see Theorem 2.1. In §3 we apply this to the setting of the gradient map as in §1.2.

If $G = A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an abelian subalgebra, we are able to prove that the image of the gradient map of an A-orbit is convex (Theorem 3.1) without using the convexity results available in the complex setting (see [23, p. 5]). Our proof only uses the existence of Kempf–Ness functions.

We also give a new proof of the Atiyah–Guillemin–Sternberg convexity theorem. Indeed, consider the case where $X = Z$ is compact, T is a compact torus, and $G = T^{\mathbb{C}}$. Atiyah [1] suggested that the convexity of $\mu(T^{\mathbb{C}} \cdot p)$ (for $p \in Z$) could be used to give an alternative proof of the abelian convexity theorem showing that there always exists $p \in Z$ such that $\overline{\mu(T^{\mathbb{C}} \cdot p)} = \mu(Z)$. Duistermaat [14] proved that the set of points p with $\overline{\mu(T^{\mathbb{C}} \cdot p)} = \mu(Z)$ is nonempty and dense (see also [13]) We give a new proof of this result and we also show that this set is open. More importantly, we believe that the abstract approach that we follow adds to the understanding of some basic results in the subject.

In the case of real analytic submanifolds of $\mathbb{P}^n(\mathbb{C})$ our method yields the following.

Theorem 1.1. *Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a compact connected real analytic submanifold that is invariant by $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset i\mathfrak{su}(n+1)$ is an abelian subalgebra. Then*

(a) *$\mu_{\mathfrak{a}}(X)$ is a convex polytope with vertices in $\mu(X^A)$;*
(b) *the set $\{x \in X : \mu_{\mathfrak{a}}(\overline{A \cdot x}) = \mu_{\mathfrak{a}}(X)\}$ is open and dense;*
(c) *for any face $\sigma \subset \mu_{\mathfrak{a}}(X)$, there is an A-orbit Y such that $\mu_{\mathfrak{a}}(\overline{Y}) = \sigma$.*

This result is weaker than the one obtained by Heinzner and Schützdeller [20] (even in the abelian case). Nevertheless the proof in this note is very simple, and (b) and (c) are new. So we think that this might be of some interest.

In the last section we apply the result of §2 to the action of G on the set of probability measures on X (with the notation of §1.4). This yields a simpler and more natural proof of the convexity theorem for measures obtained in [12].

2. Abstract abelian convexity

The following proposition contains the key idea and is basic to the whole paper. Let $\mathscr{M}, G, K, \mathfrak{p}, \Psi$, and $\mathfrak{F}$ be as in §1.4. Let $\mathfrak{a} \subset \mathfrak{p}$ be an abelian subalgebra. Then $A := \exp(\mathfrak{a}) \subset G$ is a compatible abelian subgroup.

Proposition 2.1. *Let $\Psi : \mathscr{M} \times A \to \mathbb{R}$ be a Kempf–Ness function for $(\mathscr{M}, A, \{e\})$, and let $\mathfrak{F} : \mathscr{M} \to \mathfrak{a}$ be the corresponding gradient map. Let $x \in \mathscr{M}$, and let A_x be the stabilizer of x and $\mathfrak{a}_x$ its Lie algebra. Let $\pi : \mathfrak{a} \to \mathfrak{a}_x^\perp$ be the orthogonal projection. Then $\pi(\mathfrak{F}(A \cdot x))$ is an open convex subset of $\mathfrak{a}_x^\perp$. Moreover, $\mathfrak{F}(A \cdot x)$ is an open convex subset of $\mathfrak{F}(x) + \mathfrak{a}_x^\perp$.*

Proof. Set $\mathfrak{b} := \mathfrak{a}_x^\perp$ and consider the function $f : \mathfrak{b} \to \mathbb{R}, f(v) = \Psi(x, \exp(v))$. Fix $v, w \in \mathfrak{b}$ with $w \neq 0$ and consider the curve $\gamma(t) = v + tw$. Set $u(t) = f(\gamma(t))$. We claim that $u''(0) > 0$. Using the fact that A is abelian, the cocycle condition yields

$$\begin{aligned} u(t) &= \Psi(x, \exp(v + tw)) = \Psi(x, \exp(tw)\exp(v)) \\ &= \Psi(\exp(v) \cdot x, \exp(tw)) + \Psi(x, \exp(v)), \end{aligned}$$

so

$$u'(t) = \frac{\mathrm{d}}{\mathrm{dt}} \Psi(\exp(v) \cdot x, \exp(tw)).$$

By (P3) we have $u''(0) \geq 0$, and the equality would imply that $w \in \mathfrak{a}_{\exp(v)\cdot x} = \mathfrak{a}_x$, which is impossible since $w \in \mathfrak{a}_x^\perp$. This proves the claim and shows that f is a strictly convex function on $\mathfrak{b}$. Therefore, by a basic result in convex analysis [17, p. 122], $\mathrm{d}f(\mathfrak{b})$ is an open convex subset of $\mathfrak{b} \cong (\mathfrak{b})^*$. Moreover the computation above also shows that

$$(\mathrm{d}f)_v(w) = \langle \mathfrak{F}(\exp(v) \cdot x), w \rangle = \langle \pi(\mathfrak{F}(\exp(v) \cdot x)), w \rangle. \tag{2.1}$$

Using the fact that $A \cdot x = \exp(\mathfrak{b}) \cdot x$ we conclude that

$$\pi(\mathfrak{F}(A \cdot x)) = \pi(\mathfrak{F}(\exp(\mathfrak{b}) \cdot x)) = \mathrm{d}f(\mathfrak{b})$$

is an open convex set of $\mathfrak{b}$. This proves the first assertion. To prove the last assertion it is enough to check that for any $v \in \mathfrak{a}$ and for any $w \in \mathfrak{a}_x$,

$$\langle \mathfrak{F}(\exp(v) \cdot x), w \rangle = \langle \mathfrak{F}(x), w \rangle. \tag{2.2}$$

Using (1.5) and the cocycle condition we have

$$\begin{aligned}\langle \mathfrak{F}(\exp(v)\cdot x), w\rangle &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \Psi(\exp(v)\cdot x, \exp(tw))\\ &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \Psi(x, \exp(tw)\exp(v)).\end{aligned}$$

Using that v and w commute and again the cocycle condition and (1.5) we get

$$\begin{aligned}\langle \mathfrak{F}(\exp(v)\cdot x), w\rangle &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \Psi(x, \exp(v)\exp(tw))\\ &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \Big(\Psi(\exp(tw)\cdot x, \exp(v)) + \Psi(x, \exp(tw))\Big)\\ &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \Psi(x, \exp(v)) + \langle \mathfrak{F}(x), w\rangle = \langle \mathfrak{F}(x), w\rangle.\end{aligned}$$

This proves (2.2). □

Corollary 2.1. *Let $x \in \mathscr{M}$ be such that $A_x = \{e\}$. Then $\mathfrak{F}(A\cdot x)$ is an open convex subset of $\mathfrak{a}$.*

Corollary 2.2. *Set $E := \pi(\mathfrak{F}(\overline{A\cdot x}))$. If $y \in \overline{A\cdot x}$ and $p := \pi(\mathfrak{F}(y)) \in \partial E$, then $\mathfrak{a}_x \subsetneq \mathfrak{a}_y$.*

Proof. Since the A-action on $\mathscr{M}$ is continuous, it follows that $A_x \subset A_y$, and so $\mathfrak{a}_y^\perp \subset \mathfrak{a}_x^\perp$. Assume by contradiction that $\mathfrak{a}_x = \mathfrak{a}_y$ and let $\pi : \mathfrak{a} \longrightarrow \mathfrak{a}_x^\perp$ be the orthogonal projection on $\mathfrak{a}_x^\perp$. By Proposition 2.1 the set $\Omega := \pi(\mathfrak{F}(A\cdot y))$ is an open convex subset of $\mathfrak{a}_x^\perp$. Since $A\cdot y \subset \overline{A\cdot x}$, we have $p \in \Omega \subset E$. But this contradicts the fact that $p \in \partial E$. Thus $\mathfrak{a}_x \subsetneq \mathfrak{a}_y$. □

Theorem 2.1. *If $\overline{A\cdot x}$ is compact, then*

$$\overline{\mathfrak{F}(A\cdot x)} = \mathfrak{F}(\overline{A\cdot x}) = \operatorname{conv}\big(\mathfrak{F}(\overline{A\cdot x}\cap \mathscr{M}^A)\big).$$

Proof. Since $\overline{A\cdot x}$ is compact, $\mathfrak{F}(\overline{A\cdot x}) = \overline{\mathfrak{F}(A\cdot x)}$. By Proposition 2.1 $E := \mathfrak{F}(A\cdot x)$ is an open convex subset of the affine subspace $L := \mathfrak{F}(x) + \mathfrak{a}_x^\perp$, while $\bar{E} = \mathfrak{F}(\overline{A\cdot x})$ is a compact convex subset. Let $p \in \bar{E}$ be an extreme point, and let $y \in \overline{A\cdot x}$ be such that $\mathfrak{F}(y) = x$. Again by Proposition 2.1 $\mathfrak{F}(A\cdot y)$ is a convex subset of dimension equal to $\dim \mathfrak{a}_y^\perp$. Since p is an extreme point, this dimension must be 0, so $\mathfrak{a}_y^\perp = \{0\}$ and y is a fixed point of A. So the extremal points of E are contained in $\mathfrak{F}(\overline{A\cdot x}\cap \mathscr{M}^A)$. This proves the theorem. □

3. Application to the gradient map

In this section we assume that Z, X, G, and K are as in §1.2. Moreover we assume that $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an abelian subalgebra.

Applying Theorem 2.1 we get a new proof of the following result.

Theorem 3.1. *Assume that $X \subset Z$ is an A-invariant compact submanifold. For any $x \in X$, $\mu_{\mathfrak{a}}(A\cdot x)$ is an open convex subset of $\mu_{\mathfrak{a}}(x) + \mathfrak{a}_x^\perp$, its closure coincides with $\mu_{\mathfrak{a}}(\overline{A\cdot x})$, it is a polytope, and it is the convex hull of $\mu_{\mathfrak{p}}(X^A \cap \overline{A\cdot x})$.*

Proof. By Proposition 1.1, there exists a Kempf–Ness function Ψ for (X, G, K) satisfying the conditions (P1)–(P5) and such that $\mathfrak{F} = \mu_{\mathfrak{p}}$. Now, that $\mu_{\mathfrak{a}}(A \cdot x)$ is an open convex subset of $\mu_{\mathfrak{a}}(x) + \mathfrak{a}_x^{\perp}$ is proven in Proposition 2.1. That $\mu_{\mathfrak{a}}(\overline{A \cdot x}) = \overline{\mu_{\mathfrak{a}}(A \cdot x)} = \operatorname{conv}(\mu_{\mathfrak{a}}(\overline{A \cdot x} \cap X^A))$ is proven in Theorem 2.1 (recall that X is compact by assumption). Next observe that X^A has finitely many connected components, since X is a compact manifold, and $\mu_{\mathfrak{a}}$ is constant on each of them. $\square$

This convexity theorem along the orbits was proven by Atiyah [1] in the case, where $X = Z$ and A is a complex torus. The general case has been proven by Heinzner and Stötzel [23, Prop. 3]. The above proof via Theorem 2.1 is quite short. Note that the first statement in Theorem 3.1, i.e., that $\mu_{\mathfrak{a}}(A \cdot x)$ is an open convex subset of $\mu_{\mathfrak{a}}(x) + \mathfrak{a}_x^{\perp}$, works even if X is not compact. We mention that a simple proof of orbit convexity for an action of a complex torus on a projective manifold can be found in [25]; see also [3, p. 44].

Next we turn to the abelian convexity theorem. Fix an abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and set $A := \exp(\mathfrak{a})$. Given a subset $X \subset Z$ and $\beta \in \mathfrak{a}$ set

$$W_{\max}^{\beta}(X) := \{x \in X : \lim_{t \to +\infty} \mu_{\mathfrak{a}}^{\beta}(\exp(t\beta) \cdot x) = \max_X \mu_{\mathfrak{a}}^{\beta}\}. \tag{3.1}$$

Proposition 3.1. *Assume that Z is compact and let $X \subset Z$ be a closed A-invariant subset. Assume that for any $\beta \in \mathfrak{a}$ the set $W_{\max}^{\beta}(X)$ is open and dense in X. Then:*

(a) *$P = \mu_{\mathfrak{a}}(X)$ is a convex polytope with vertices in $\mu_{\mathfrak{a}}(X^A)$;*
(b) *the set $\{x \in X : \mu_{\mathfrak{a}}(\overline{A \cdot x}) = \mu_{\mathfrak{a}}(X)\}$ is dense and it is also open if X is a smooth submanifold of Z;*
(c) *if $\sigma \subset \mu_{\mathfrak{a}}(X)$ is a face of P there exists an A-orbit Y such that $\mu_{\mathfrak{a}}(\overline{Y}) = \sigma$.*

Proof. The set Z^A has finitely many connected components since Z is compact, and each component is a smooth submanifold of Z. Moreover, $\mu_{\mathfrak{a}}$ is constant on each component. Therefore $\mu_{\mathfrak{a}}(Z^A)$ is a finite set. Since $X^A = X \cap Z^A$, we also conclude that $\mu_{\mathfrak{a}}(X^A)$ is a finite set. Therefore $P := \operatorname{conv}(\mu_{\mathfrak{a}}(X^A))$ is a convex polytope. By Theorem 2.1 if $x \in X$, then $\mu_{\mathfrak{a}}(\overline{A \cdot x}) = \operatorname{conv}(\mu_{\mathfrak{a}}(\overline{A \cdot x} \cap X^A)) \subset P$. Hence $\operatorname{conv}(\mu_{\mathfrak{a}}(X)) \subset P$. The reverse inclusion is obvious, so $P = \operatorname{conv} \mu_{\mathfrak{a}}(X)$. Now let $\xi_1, \dots, \xi_k$ be the vertices of P. Choose $\beta_i \in \mathfrak{a}$ such that

$$\{\xi \in P : \langle \xi, \beta_i \rangle = \max_P \langle \cdot, \beta_i \rangle\} = \{\xi_i\}.$$

By our assumption the set $W_{\max}^{\beta_1} \cap \cdots \cap W_{\max}^{\beta_k}$ is open and dense. Fix $x \in W_{\max}^{\beta_1} \cap \cdots \cap W_{\max}^{\beta_k}$ and set

$$y_i := \lim_{t \to +\infty} \exp(t\beta_i) \cdot x.$$

Then $y_i \in X$, and using (1.3) and (3.1) we get

$$\mu_{\mathfrak{a}}^{\beta_i}(y_i) = \max_X \mu_{\mathfrak{a}}^{\beta_i} = \max_{\mu_{\mathfrak{a}}(X)} \langle \cdot, \beta_i \rangle = \max_P \langle \cdot, \beta_i \rangle.$$

Therefore $\mu_{\mathfrak{a}}(y_i) = \xi_i$. So $\xi_i \in \mu_{\mathfrak{a}}(\overline{A \cdot x})$ for any $i = 1, \dots, k$. But $\mu_{\mathfrak{a}}(\overline{A \cdot x})$ is convex by Theorem 2.1. Since $\mu_{\mathfrak{a}}(\overline{A \cdot x}) \subset \mu_{\mathfrak{a}}(X) \subset P$, we get

$$\mu_{\mathfrak{a}}(\overline{A \cdot x}) = \mu_{\mathfrak{a}}(X) = P.$$

This proves (a). Next set

$$W := \{x \in X : \mu_{\mathfrak{a}}(\overline{A \cdot x}) = \mu_{\mathfrak{a}}(X)\}.$$

We have just proven that W contains $W_{\max}^{\beta_1} \cap \cdots \cap W_{\max}^{\beta_k}$, so it is dense. Assume now that X is a smooth submanifold of Z. Fix one of the vertices of P, say ξ_i, and consider the set

$$\Omega_i := \{x \in X : \overline{A \cdot x} \cap \mu_{\mathfrak{a}}^{-1}(\xi_i) \neq \emptyset\}.$$

We claim that this is an open subset of X. This follows from the stratification theorem in [22]. Indeed, in the abelian case one can shift the gradient map so we can assume that $\xi_i = 0 \in \mathfrak{a}$. Then Ω_i coincides with the stratum corresponding to the minimum of $||\mu_{\mathfrak{a}}||^2$, and as such it is open. This proves the claim. Finally observe that $W = \bigcap_{i=1}^k \Omega_i$. Thus W is also open in X and (b) is proved. Finally let $\sigma \subset P$ be a face of P. It is an exposed face, so there exists $\beta \in \mathfrak{a}$ such that

$$\sigma = \{\xi \in P : \langle \xi, \beta \rangle = \max_{\mu_{\mathfrak{a}}(X)} \langle \cdot, \beta \rangle\}.$$

Hence $\mu_{\mathfrak{a}}^{-1}(\sigma) = \{x \in X : \mu_{\mathfrak{p}}^{\beta}(x) = \max_X \mu_{\mathfrak{p}}^{\beta}\}$. By (b) there is $x \in W_{\max}^{\beta}$ such that $\mu_{\mathfrak{a}}(\overline{A \cdot x}) = \mu_{\mathfrak{a}}(X)$. Define

$$\varphi_\infty : W_{\max}^{\xi} \longrightarrow \mu^{-1}(\sigma), \qquad \varphi_\infty(x) := \lim_{t \mapsto +\infty} \exp(t\xi) \cdot x.$$

Since $\mu_{\mathfrak{a}}^{-1}(\sigma)$ is A-stable, it follows that $\overline{A \cdot \varphi_\infty(x)} \subset \overline{A \cdot x} \cap \mu_{\mathfrak{a}}^{-1}(\sigma)$. On the other hand, let a_n be a sequence of elements of A such that $a_n \cdot x \mapsto \theta \in \overline{A \cdot x} \cap \mu_{\mathfrak{a}}^{-1}(\sigma)$. Since $\varphi_\infty(\theta) = \theta$, it follows that

$$\theta = \lim_{n \mapsto \infty} \varphi_\infty(a_n \cdot x) = \lim_{n \mapsto \infty} a_n \cdot \varphi_\infty(x).$$

Therefore

$$\overline{A \cdot x} \cap \mu^{-1}(\sigma) = \overline{A \cdot \varphi_\infty(x)}.$$

Since $\mu_{\mathfrak{a}}|_{\overline{A \cdot x}} : \overline{A \cdot x} \to P$ is a surjective map, $\mu_{\mathfrak{a}}(\overline{A \cdot x} \cap \mu_{\mathfrak{a}}^{-1}(\sigma)) = \sigma$. Thus $\mu_{\mathfrak{a}}(\overline{A \cdot \varphi_\infty(x)}) = \sigma$. $\square$

Now let $T^{\mathbb{C}}$ be a complex torus acting on the Kähler manifold Z. The functions $\mu^\beta : Z \to \mathbb{R}$ (for $\beta \in \mathfrak{t}$) are Morse–Bott functions with even indices. Atiyah proved that the set of maximum points of each μ^β is a connected critical manifold. Therefore the corresponding unstable manifold, which coincides with the set $W_{\max}^{\beta}$, is an open dense subset of Z. Set $\mathfrak{a} = i\mathfrak{t}$ and $A = \exp(i\mathfrak{t})$. Moreover, $T^{\mathbb{C}} = A \cdot T$, $Z^A = Z^T = Z^{T^{\mathbb{C}}}$ since the action is holomorphic. Finally, $\mu_{\mathfrak{a}} = i\mu$ and $\mu(T^{\mathbb{C}} \cdot x) = -i\mu_{\mathfrak{a}}(A \cdot x)$ since μ is T-invariant. Therefore the following theorem immediately follows from Proposition 3.1.

Theorem 3.2. *Let T be a compact torus. Let (Z, ω) be a compact Kähler manifold on which $T^{\mathbb{C}}$ acts holomorphically. Assume that T acts in a Hamiltonian fashion with momentum map $\mu : Z \longrightarrow \mathfrak{t}^*$. Then there is a $T^{\mathbb{C}}$-orbit $\mathcal{O}$ such that $\mu(\overline{\mathcal{O}}) = \mu(Z)$. More precisely:*

(a) *the set $\{x \in Z : \mu(\overline{T^{\mathbb{C}} \cdot x}) = \mu(Z)\}$ is nonempty, open, and dense;*
(b) *$\mu(Z)$ is a convex polytope with vertices in $\mu(Z^T)$;*
(c) *if σ is a face of $\mu(Z)$, then there exists a $T^{\mathbb{C}}$-orbit Y such that $\mu(\overline{Y}) = \sigma$.*

One can also apply the method of proof used in Proposition 3.1 in the setting considered by Heinzner and Huckleberry in [18]. In this case Z is a connected Kähler manifold, not necessarily compact, and $X \subset Z$ is a compact irreducible (complex) analytic subset.

Theorem 3.3. *Let $X \subset Z$ be a compact irreducible (complex) analytic subset, which is invariant by the $T^{\mathbb{C}}$-action. Then:*

(a) *$\mu(X)$ is a convex polytope with vertices in $\mu(X^T)$;*
(b) *the set $W := \{x \in X : \mu(\overline{T^{\mathbb{C}} \cdot x}) = \mu(X)\}$ is nonempty, open, and dense.*

Proof. We claim that for $\xi \in \mathfrak{t}$ the set

$$W_\xi := \{x \in X : \overline{T^{\mathbb{C}} \cdot x} \cap \mu^{-1}(\xi) \neq \emptyset\}$$

is either empty or open and dense. Indeed by shifting we can assume that $\xi = 0$. Hence this is the set of semistable points for the action and the claim follows from the results in [18]. Set $P := \operatorname{conv}(\mu(X))$. This is a polytope with vertices in $\mu(X^T)$. Let $\xi_1, \dots, \xi_k$ be the vertices. Set $W' := W_{\xi_1} \cap \dots \cap W_{\xi_k}$. This is an open dense subset of X. Fix $x \in W'$. By Theorem 3.1 $\mu(\overline{T^{\mathbb{C}} \cdot x})$ is a convex subset of P. Since it contains all the vertices we have $\mu(\overline{T^{\mathbb{C}} \cdot x}) = \mu(X) = P$. This proves (b) (which of course was also proved directly in [18]). Moreover, we have just seen that $W' \subset W$. The opposite inclusion is obvious. Hence $W = W'$ and (b) is proved. ☐

One would like to prove convexity for $\mu_{\mathfrak{a}}(X)$ for $X \subset Z$ a general A-invariant closed submanifold of Z. In this setting convexity is unknown in general. Convexity of $\mu_{\mathfrak{a}}(X)$ (and also *nonabelian* convexity) is known to hold if X is a real flag manifold, thanks to the pioneering paper [28], and more generally if Z is a Hodge manifold and X is an irreducible semialgebraic subset of Z whose real algebraic Zariski closure is irreducible, [4, 20].

Using Proposition 3.1 we can give a short argument when X is a compact connected real analytic submanifold of $\mathbb{P}^n(\mathbb{C})$. This class is narrower than the one considered in [20], but it is quite interesting. Above all, we feel that our proof is rather geometric and very clear in its strategy.

Lemma 3.1. *Assume that $Z = \mathbb{P}^n(\mathbb{C})$ and that X is a compact connected A-invariant real analytic submanifold endowed with the restriction of the Fubini–Study form. Then for any $\beta \in \mathfrak{a}$ the set $W^\beta_{\max}(X)$ is open and dense in X.*

Proof. Since $Z = \mathbb{P}^n(\mathbb{C})$, β induces a linear flow on $\mathbb{P}^n(\mathbb{C})$ which restricts to the original one on Z and X. Assume that $v \in \mathfrak{su}(n+1)$ is the infinitesimal generator of the linear flow and let $c_0 < \dots < c_r$ be the critical values of the function $f([z]) := i\langle v(z), z\rangle / |z|^2$, that is the Hamiltonian of the flow on $\mathbb{P}^n(\mathbb{C})$. Denote by C_i the critical manifold corresponding to c_i and let $W^u_i(\mathbb{P}^n(\mathbb{C}))$ be its unstable manifold. Then $\mathbb{P}^n(\mathbb{C}) = \bigsqcup_{i=0}^r W^u_i$. Moreover, for each j the set $\bigsqcup_{i \leq j} W^u_i$ is equal to a linear subspace $L_j \subset \mathbb{P}^n(\mathbb{C})$. This is an elementary computation; see, e.g., [6, Lemma 7.4]. Since $\mu^\beta_{\mathfrak{a}} = f|_X$ the critical points of $\mu^\beta_{\mathfrak{a}}$ on X are given by $\bigcup_i (C_i \cap X)$. If $\max_X \mu^\beta_{\mathfrak{a}} = c_j$, then $X \subset L_j$, X is not contained in L_{j-1}, and $W^\beta_{\max}(X) = W^u_j(\mathbb{P}^n(\mathbb{C})) \cap X = X - L_{j-1}$. Assume by contradiction that this set is not dense in X. Then $X \cap L_{j-1}$ contains an open subset of X and $A := (X \cap L_{j-1})^0$ is not empty. On the other hand $A \neq X$, since X is not contained in L_{j-1}. Hence there is some point $x \in \partial A = \overline{A} - A$. Fix a real analytic chart $\varphi : U \to U'$ with $x \in U$ and U' an open ball in $\mathbb{R}^k$. Locally around x we have $L_{j-1} = \{h_1 = \dots = h_p = 0\}$ for some local holomorphic functions $h_1, \dots, h_p$. Therefore the set

$$U'' := \{y \in U' : h_1\varphi^{-1}(y) = \dots = h_p\varphi^{-1}(y) = 0\}$$

contains the open set $\varphi(A \cap U)$. Therefore $U'' = U'$, $U \subset X \cap L_{j-1}$, and $x \in A$, a contradiction. $\square$

Thanks to the previous lemma we can apply Proposition 3.1 and we get the following result.

Theorem 3.4. *Assume that $Z = \mathbb{P}^n(\mathbb{C})$ with the Fubini study metric. Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a compact connected A-invariant real analytic submanifold. Then:*

(a) *$P = \mu_{\mathfrak{a}}(X)$ is a convex polytope with vertices in $\mu(X^A)$;*
(b) *the set $\{x \in X : \mu_{\mathfrak{a}}(\overline{A \cdot x}) = \mu_{\mathfrak{a}}(X)\}$ is open and dense;*
(c) *if $\sigma \subset \mu_{\mathfrak{a}}(X)$ is a face of P there exists an A-orbit Y such that $\mu_{\mathfrak{a}}(\overline{Y}) = \sigma$.*

We remark that by [4] the image of the gradient map is independent of the Kähler metric within a fixed Kähler class.

4. Action on the space of measures

Let Z, X, G, and K be as in §1.2. Denote by $\mathscr{P}(X)$ the set of Borel probability measures on X, which is a compact Hausdorff space when endowed with the weak topology; see [6, 11] for more details and [5, 7–10] for background and motivation.

Assume that $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subalgebra. Let Ψ^A be the Kempf–Ness function for $(X, A, \{e\})$ as in Proposition 1.1. Since A acts on X, we have an action on the probability measures on X as follows:

$$A \times \mathscr{P}(X) \to \mathscr{P}(X), \quad (g, \nu) \mapsto g_* \nu.$$

In [11] it is proven that this action is continuous with respect to the weak topology and that the function

$$\Psi^{\mathscr{P}} : \mathscr{P}(M) \times A \to \mathbb{R}, \quad \Psi^{\mathscr{P}}(\nu, g) := \int_M \Psi^A(x, g) d\nu(x), \tag{4.1}$$

is a Kempf–Ness function for $(A, \mathscr{P}(M), \{e\})$ in the sense of Definition 1.1. Moreover, the gradient map is given by the formula

$$\mathfrak{F} : \mathscr{P}(M) \to \mathfrak{a}, \quad \mathfrak{F}(\nu) := \int_M \mu_{\mathfrak{a}}(x) d\nu(x). \tag{4.2}$$

Since $\mathscr{P}(X)$ is compact, Theorem 2.1 gives a short proof of the following result proved in [12].

Theorem 4.1. *Let $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subalgebra. If $\nu \in \mathscr{P}(M)$, then:*

(a) *$\mathfrak{F}(A \cdot \nu)$ is a convex set.*
(b) *$\mathfrak{F}(\overline{A \cdot \nu})$ coincides with the convex hull of $\mathfrak{F}\big(\mathscr{P}(M)^A \cap \overline{A \cdot \nu}\big)$, where $\mathscr{P}(M)^A = \{\tilde{\nu} \in \mathscr{P}(M) : A \cdot \tilde{\nu} = \tilde{\nu}\}$.*

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