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NOTE ON THE ABSENCE OF REMAINDERS IN THE WIENER-IKEHARA THEOREM

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Abstract. We show that it is impossible to get a better remainder than the classical one in the Wiener-Ikehara theorem even if one assumes analytic continuation of the Mellin transform after subtraction of the pole to a half-plane. We also prove a similar result for the Ingham-Karamata theorem.

1. Introduction

The Wiener-Ikehara theorem is a landmark in 20th-century analysis. It states:

Theorem 1.1. *Let S be a non-decreasing function and suppose that*

$$G(s) := \int_1^\infty S(x)x^{-s-1}\mathrm{d}x \quad \text{converges for } \Re e\ s > 1 \tag{1}$$

and that there exists a such that $G(s) - a/(s-1)$ admits a continuous extension to $\Re e\ s \geq 1$. Then

$$S(x) = ax + o(x). \tag{2}$$

This result is well-known in number theory as it leads to one of the quickest proofs of the prime number theorem. However, it also has important applications in other fields such as operator theory (see e.g. [1]). Over the last century the Wiener-Ikehara theorem has been extensively studied and generalized in many ways (e.g., [6,8,9,11,13,16,18,20]). We refer the interested reader to [12, Chap. III] for more information about the Wiener-Ikehara theorem.

If one wishes to attain a stronger remainder in (2) (compared to $o(x)$), it is natural to strengthen the assumptions on the Mellin transform (1). We investigate here whether one can obtain remainders if the Mellin transform after subtraction of the pole at $s = 1$ admits an analytic extension to a half-plane $\Re e\ s > \alpha$ where $0 < \alpha < 1$. It is well-known that one can get reasonable error terms in the asymptotic formula for S if bounds are known on the analytic function G. The question of obtaining remainders if one does not have such bounds was recently raised by Müger

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[15], who actually conjectured the error term $O(x^{(\alpha+2)/3+\varepsilon})$ could be obtained for each $\varepsilon > 0$.

We show in this article that this is false. In fact, we shall prove in Section 3 the more general result that no reasonably good remainder can be expected in the Wiener-Ikehara theorem with solely the classical Tauberian condition (of S being non-decreasing) and the analyticity of $G(s) - A/(s-1)$ on $\Re e\, s > \alpha$ for $0 < \alpha < 1$. To show this result we will adapt an attractive functional analysis argument given by Ganelius[1] [10, Thm. 3.2.2]. Interestingly, the nature of our problem requires us to consider a suitable Fréchet space of functions instead of working with a Banach space.

In Section 4 we shall apply our result on the Wiener-Ikehara theorem to study another cornerstone in complex Tauberian theory, namely, the Ingham-Karamata theorem for Laplace transforms [12, Chap. III] (see [7, 8] for sharp versions of it). Notably, a very particular case of this theorem captured special attention when Newman found an elementary contour integration proof that leads to a simple deduction of the prime number theorem; in fact, this proof is nowadays a chapter in various popular expository textbooks in analysis [5, 14]. We will show that, just as for the Wiener-Ikehara theorem, no reasonable error term can be obtained in the Ingham-Karamata theorem under just an analytic continuation hypothesis on the Laplace transform. On the other hand, the situation is then pretty much the same as for the Wiener-Ikehara theorem: error terms can be achieved if the Laplace transform satisfies suitable growth assumptions. We point out that the problem of determining such growth conditions on the Laplace transform has been extensively studied in recent times [2, 4, 17] and such results have numerous applications in operator theory and in the study of the asymptotic behavior of solutions to various evolution equations.

2. Some lemmas

We start with some preparatory lemmas that play a role in our constructions. The first one is a variant of the so-called smooth variation theorem from the theory of regularly varying functions [3, Thm. 1.8.2, p. 45].

Lemma 2.1. *Let ℓ be a positive non-increasing function on $[0,\infty)$ such that $\ell(x) = o(1)$ (as $x \to \infty$). Then, there is a positive smooth function L such that*

$$\ell(x) \ll L(x) = o(1),$$

and, for some positive C, A, and B,

$$\left|L^{(n)}(x)\right| \leq CA^n n! x^{-n}, \qquad \text{for all } x \geq B \text{ and } n \in \mathbb{N}. \tag{3}$$

Proof. We consider the Poisson kernel of the real line

$$P(x,y) = \frac{y}{\pi(y^2+x^2)} = \frac{i}{2\pi}\left(\frac{1}{x+iy} - \frac{1}{x-iy}\right).$$

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[1] According to him [10, p. 3], the use of functional analysis arguments to avoid cumbersome constructions of counterexamples in Tauberian theory was suggested by L. Hörmander.

Differentiating the last expression with respect to y, it is clear that we find

$$\begin{aligned}\left|\frac{\partial^n P}{\partial y}(x,y)\right| &\leq \frac{2^{n+1}n!y^{n+1}}{\pi(y^2+x^2)^{1+n}} \max_{0\leq j\leq n+1} |x/y|^j \\ &< \frac{2^{n+1}n!}{\pi(y^2+x^2)^{(1+n)/2}}, \quad \text{for all } n\geq 1.\end{aligned}$$

We set

$$L(y)=\int_0^\infty \ell(yx)P(x,1)\mathrm{d}x=\int_0^\infty \ell(x)P(x,y)\mathrm{d}x.$$

By the dominated convergence theorem, we have $L(y)=o(1)$. Since ℓ is non-increasing and $P(x,1)$ is positive, it follows that

$$L(y)\geq \int_0^1 \ell(y)P(x,1)\mathrm{d}x=\frac{\ell(y)}{4}\,.$$

For the derivatives we have $\left|L^{(n)}(y)\right|\leq \ell(0)2^n n! y^{-n}$ for all $n\in\mathbb{N}$ and $y>0$. ☐

We also need to study the analytic continuation of the Laplace transform of functions satisfying the regularity assumption (3).

Lemma 2.2. *Suppose that $L\in L^1_{loc}[0,\infty)$ satisfies the regularity assumption* (3) *for some $A,B,C>0$, and set $\theta=\arccos(1/(1+A))$. Then its Laplace transform $\mathcal{L}\{L;s\}=\int_0^\infty e^{-sx}L(x)\mathrm{d}x$ converges for $\Re e\, s>0$ and admits analytic continuation to the sector $-\pi+\theta<\arg s<\pi-\theta$.*

Proof. It is clear that $F(s)=\int_0^\infty e^{-sx}L(x)\mathrm{d}x$ converges for $\Re e\ s>0$. Since the Laplace transform of a compactly supported function is entire, we may suppose that L is supported on $[B,\infty)$. Since we can write

$$F(s)=e^{-sB}\int_0^\infty e^{-sx}L(x+B)\mathrm{d}x,$$

we may w.l.o.g. assume $B=0$ and replace x^{-n} in the estimates for $L^{(n)}(x)$ by $(1+x)^{-n}$. We consider the kth derivative of F, namely $(-1)^k\int_0^\infty x^k e^{-sx}L(x)\mathrm{d}x$. We use integration by parts $k+2$ times to find

$$F^{(k)}(s)=(-1)^k\frac{k!L(0)}{s^{k+1}}+(-1)^k\frac{(k+1)!L'(0)}{s^{k+2}}+\frac{(-1)^k}{s^{k+2}}\int_0^\infty (L(x)x^k)^{(k+2)}e^{-sx}\mathrm{d}x.$$

Because of the regularity assumption (3) the latter integral absolutely converges, and hence F admits a C^∞-extension on the imaginary axis except possibly at the origin. The bounds (3) actually give for arbitrary $\varepsilon > 0$:

$$\begin{aligned}\left|F^{(k)}(it)\right| &\le \frac{|L(0)|}{|t|}\frac{k!}{|t|^{k}} + \frac{|L'(0)|}{|t|^{2}}\frac{(k+1)!}{|t|^{k}} \\ &\quad + \frac{1}{|t|^{k+2}}\int_0^\infty \sum_{j=2}^{k+2}\binom{k+2}{j}\left|L^{(j)}(x)\right|\frac{k!}{(j-2)!}x^{j-2}\mathrm{d}x \\ &\le C'\frac{(1+|t|)(k+1)!}{|t|^{k+2}} + \frac{1}{|t|^{k+2}}\int_0^\infty \sum_{j=2}^{k+2} CA^{j}\frac{(k+2)!k!}{(k+2-j)!(j-2)!}(1+x)^{-2}\mathrm{d}x \\ &\le C'\frac{(1+|t|)(k+1)!}{|t|^{k+2}} + \frac{A^2C\pi(k+2)!}{2\,|t|^{k+2}}\sum_{j=0}^{k}A^{j}\binom{k}{j} \\ &\le C_\varepsilon\frac{(1+|t|)}{|t|^2}\frac{k!(1+A+\varepsilon)^k}{|t|^k},\end{aligned}$$

where C_ε depends only on ε and L. Therefore, F admits an analytic extension to the disk around it with radius $|t|/(1+A)$. The union of all such disks and the half-plane $\Re e\ s > 0$ is precisely the sector in the statement of the lemma. ☐

3. Absence of remainders in the Wiener-Ikehara theorem

We are ready to show our main theorem, which basically tells us that no remainder of the form $O(x\rho(x))$ with $\rho(x)$ a function tending arbitrarily slowly to 0 could be expected in the Wiener-Ikehara theorem from just the hypothesis of analytic continuation of $G(s) - a/(s-1)$ to a half-plane containing $\Re e\ s \ge 1$. As customary, the Ω below stands for the Hardy-Littlewood symbol, namely, the negation of Landau's o symbol. Our general reference for functional analysis is the textbook [19].

Theorem 3.1. *Let ρ be a positive function, let $a > 0$, and let $0 < \alpha < 1$. Suppose that every non-decreasing function S on $[1,\infty)$, whose Mellin transform $G(s)$ is such that $G(s) - a/(s-1)$ admits an analytic extension to $\Re e\ s > \alpha$, satisfies*

$$S(x) = ax + O(x\rho(x)).$$

Then, one must necessarily have

$$\rho(x) = \Omega(1).$$

Proof. Since $a > 0$, we may actually assume that the "Tauberian theorem" hypothesis holds for every possible constant $a > 0$. Assume that $\rho(x) \to 0$. Then, one can choose a positive non-increasing function $\ell(x) \to 0$ such that $\ell(\log x)/\rho(x) \to \infty$. We now apply Lemma 2.1 to ℓ to get a smooth function L with $\ell(x) \ll L(x) \to 0$ and the estimates (3) on its derivatives. We set $x\rho(x) = 1/\delta(x)$. If we manage to show that $\delta(x) = O(1/xL(\log x))$, then one obtains a contradiction with $\ell(\log x)/\rho(x) \to \infty$ and hence $\rho(x) \not\to 0$. We thus proceed to show that $\delta(x) = O(1/xL(\log x))$. Obviously, we may additionally assume that L satisfies

$$L(x) \gg x^{-1/2}. \tag{4}$$

We are going to define two Fréchet spaces. The first one consists of all Lipschitz continuous functions on $[1,\infty)$ such that their Mellin transforms can be analytically continued to $\Re e\, s > \alpha$ and continuously extended to the closed half-plane $\Re e\, s \geq \alpha$. We topologize it via the countable family of complete norms

$$\|T\|_{n,1} = \operatorname*{ess\,sup}_x |T'(x)| + \sup_{\Re e\ s \geq \alpha, |\Im m\ s| \leq n} |G_T(s)|\,,$$

where G_T stands for (the analytic continuation of) the Mellin transform of T. The second Fréchet space is defined via the norms

$$\|T\|_{n,2} = \sup_x |T(x)\delta(x)| + \|T\|_{n,1}\,.$$

The hypothesis in the theorem ensures that the two spaces have the same elements. The verification of completeness with respect to these two families of norms is standard and we therefore omit it. Obviously the inclusion mapping from the second space into the first one is continuous. Hence, by the open mapping theorem, the inclusion mapping from the first space into the second one is also continuous. Therefore, there exist sufficiently large N and C such that

$$\sup_x |T(x)\delta(x)| \leq C\, \|T\|_{N,1} \tag{5}$$

for all T in our Fréchet space. This inequality extends to the completion of the Fréchet space with regard to the norm $\|\cdot\|_{N,1}$. We note that any function T for which $T'(x) = o(1)$, $T(1) = 0$, and whose Mellin transform has analytic continuation in a neighborhood of $\{s : \Re e\, s \geq \alpha, |\Im m\ s| \leq N\}$ is in that completion. Indeed, let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\varphi(0) = 1$ and its Fourier transform has compact support. Then $\tilde{T}_\lambda(x) := \int_1^x T'(u)\varphi(\lambda \log u)\mathrm{d}u$ converges to T as $\lambda \to 0^+$ in the norm $\|\cdot\|_{N,1}$. We now consider

$$T_b(x) := \int_1^x L(\log u)\cos(b \log u)\mathrm{d}u.$$

Obviously the best Lipschitz constant for T_b is bounded by the supremum of L. Its Mellin transform is

$$G_b(s) = \frac{1}{2s}\left(\mathcal{L}\{L; s-1+ib\} + \mathcal{L}\{L; s-1-ib\}\right).$$

Because of Lemma 2.2 it follows that G_b is analytic in $\{s : \Re e\ s \geq \alpha, |\Im m\ s| \leq N\}$ for all sufficiently large b; let us say for every $b > M$. Hence, the norm $\|T_b\|_{N,1}$ is uniformly bounded in b for $b \in [M, M+1]$. A quick calculation shows for $b \in [M, M+1]$ that

$$T_b(x) := \frac{xL(\log x)}{b^2+1}\left(\cos(b\log x) + b\sin(b\log x)\right) + O\left(\frac{x}{\log x}\right),$$

where the O-constant is independent of b. For each y large enough there is $b \in [M, M+1]$ such that $\sin(b \log y) = 1$. Therefore, for y sufficiently large, taking also (4) into account, we have

$$\sup_{b \in [M,M+1]} T_b(y) \geq \inf_{b \in [M,M+1]} \frac{byL(\log y)}{b^2+1} + O\left(\frac{y}{\log y}\right) \geq C_M y L(\log y),$$

with C_M a positive constant. Consequently, for all sufficiently large y, the inequality (5) yields

$$\begin{aligned}\delta(y) &\leq \sup_{b\in[M,M+1]} \frac{T_b(y)\delta(y)}{C_M y L(\log y)} \leq \sup_{b\in[M,M+1]} \sup_{x} \frac{|T_b(x)\delta(x)|}{C_M y L(\log y)} \\ &\leq \frac{C}{C_M y L(\log y)} \sup_{b\in[M,M+1]} \|T_b\|_{N,1} = O\left(\frac{1}{yL(\log y)}\right).\end{aligned}$$

□

4. The Ingham-Karamata theorem

We start by stating the Ingham-Karamata theorem. A real-valued function τ is called *slowly decreasing* [12] if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

Theorem 4.1. *Let $\tau \in L^1_{loc}[0,\infty)$ be slowly decreasing and have convergent Laplace transform*

$$\mathcal{L}\{\tau; s\} = \int_0^\infty \tau(x)e^{-sx}\mathrm{d}x \quad \text{for } \Re e\, s > 0.$$

Suppose that $\mathcal{L}\{\tau; s\}$ has a continuous extension to the imaginary axis. Then,

$$\tau(x) = o(1).$$

We also have the ensuing result on the absence of remainders in the Ingham-Karamata theorem.

Theorem 4.2. *Let η be a positive function and let $-1 < \alpha < 0$. Suppose that every slowly decreasing function $\tau \in L^1_{loc}[0,\infty)$, whose Laplace transform converges on $\Re e\, s > 0$ and has an analytic continuation to the half-plane $\Re e\, s > \alpha$, satisfies*

$$\tau(x) = O(\eta(x)).$$

Then, we necessarily have

$$\eta(x) = \Omega(1).$$

Proof. We reduce the problem to Theorem 3.1. So set $\rho(x) = \eta(\log x)$, and we will show that $\rho(x) = \Omega(1)$. Suppose that S is non-decreasing on $[1,\infty)$ such that its Mellin transform $G(s)$ converges on $\Re e\, s > 1$ and

$$G(s) - \frac{1}{s-1}$$

analytically extends to $\Re e\, s > 1 + \alpha$. By the Wiener-Ikehara theorem $\tau(x) = e^{-x}S(e^x) - 1 = o(1)$, and in particular it is slowly decreasing. Its Laplace transform

$$\mathcal{L}\{\tau; s\} = G(s+1) - \frac{1}{s}$$

is analytic on $\Re e\, s > \alpha$, and thus $\tau(x) = O(\eta(x))$, or, equivalently, $S(x) = x + O(x\rho(x))$. Since S was arbitrary, Theorem 3.1 gives at once $\rho(x) = \Omega(1)$. The proof is complete. □

REFERENCES

[1] J. Aramaki, *An extension of the Ikehara Tauberian theorem and its application*, Acta Math. Hungar. **71** (1996), no. 4, 297–326, DOI 10.1007/BF00114420. MR1397559

[2] Charles J. K. Batty, Alexander Borichev, and Yuri Tomilov, *L^p-tauberian theorems and L^p-rates for energy decay*, J. Funct. Anal. **270** (2016), no. 3, 1153–1201, DOI 10.1016/j.jfa.2015.12.003. MR3438332

[3] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987. MR898871

[4] Ralph Chill and David Seifert, *Quantified versions of Ingham's theorem*, Bull. Lond. Math. Soc. **48** (2016), no. 3, 519–532, DOI 10.1112/blms/bdw024. MR3509911

[5] D. Choimet and H. Queffélec, *Twelve landmarks of twentieth-century analysis*, illustrated by Michaël Monerau, translated from the 2009 French original by Danièle Gibbons and Greg Gibbons, with a foreword by Gilles Godefroy, Cambridge University Press, New York, 2015. MR3445361

[6] Gregory Debruyne and Jasson Vindas, *Generalization of the Wiener-Ikehara theorem*, Illinois J. Math. **60** (2016), no. 2, 613–624. MR3680551

[7] Gregory Debruyne and Jasson Vindas, *Optimal Tauberian constant in Ingham's theorem for Laplace transforms,* Israel J. Math., in press, DOI 10.1007/s11856-018-1758-1.

[8] Gregory Debruyne and Jasson Vindas, *Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior,* J. Anal. Math., to appear, preprint, arXiv:1604.05069.

[9] Hubert Delange, *Généralisation du théorème de Ikehara* (French), Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 213–242. MR0068667

[10] Tord H. Ganelius, *Tauberian remainder theorems*, Lecture Notes in Mathematics, Vol. 232, Springer-Verlag, Berlin-New York, 1971. MR0499898

[11] S. W. Graham and Jeffrey D. Vaaler, *A class of extremal functions for the Fourier transform*, Trans. Amer. Math. Soc. **265** (1981), no. 1, 283–302, DOI 10.2307/1998495. MR607121

[12] Jacob Korevaar, *Tauberian theory,* A century of developments, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 329, Springer-Verlag, Berlin, 2004. MR2073637

[13] Jacob Korevaar, *Distributional Wiener-Ikehara theorem and twin primes*, Indag. Math. (N.S.) **16** (2005), no. 1, 37–49, DOI 10.1016/S0019-3577(05)80013-8. MR2138049

[14] Peter D. Lax and Lawrence Zalcman, *Complex proofs of real theorems*, University Lecture Series, vol. 58, American Mathematical Society, Providence, RI, 2012. MR2827550

[15] Michael Müger, *On Ikehara type Tauberian theorems with $O(x^\gamma)$ remainders*, Abh. Math. Semin. Univ. Hambg. **88** (2018), no. 1, 209–216, DOI 10.1007/s12188-017-0187-0. MR3785794

[16] Szilárd Gy. Révész and Anne de Roton, *Generalization of the effective Wiener-Ikehara theorem*, Int. J. Number Theory **9** (2013), no. 8, 2091–2128, DOI 10.1142/S1793042113500760. MR3145162

[17] David Seifert, *A quantified Tauberian theorem for sequences*, Studia Math. **227** (2015), no. 2, 183–192, DOI 10.4064/sm227-2-7. MR3397278

[18] Gérald Tenenbaum, *Introduction to analytic and probabilistic number theory*, 3rd ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015. Translated from the 2008 French edition by Patrick D. F. Ion. MR3363366

[19] François Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967. MR0225131

[20] Wen-Bin Zhang, *Wiener-Ikehara theorems and the Beurling generalized primes*, Monatsh. Math. **174** (2014), no. 4, 627–652, DOI 10.1007/s00605-013-0597-8. MR3233115

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