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FAITHFULNESS OF BIFREE PRODUCT STATES

CHRISTOPHER RAMSEY

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Abstract. Given a nontrivial family of pairs of faces of unital C*-algebras where each pair has a faithful state, it is proved that if the bifree product state is faithful on the reduced bifree product of this family of pairs of faces, then each pair of faces arises as a minimal tensor product. A partial converse is also obtained.

1. Introduction

The reduced free product was given independently by Avitzour [1] and Voiculescu [7] and it has been foundational in the development of free probability. Dykema proved in [2] that the free product state on the reduced free product of unital C*-algebras with faithful states is faithful. As a consequence of this, if $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ is a free family of unital C*-algebras in the noncommutative C*-probability space $(\mathcal{A},\varphi)$ and if φ is faithful on $C^*(\{\mathcal{A}_i\}_{i\in\mathcal{I}})$, then

$$C^*(\{\mathcal{A}_i\}_{i\in\mathcal{I}}) \simeq *_{i\in\mathcal{I}}(\mathcal{A}_i, \varphi|_{\mathcal{A}_i}),$$

the reduced free product of the $\mathcal{A}_i$'s with respect to the given states. This can be deduced from a paper of Dykema and Rørdam, namely [3, Lemma 1.3].

The present paper is the result of the author's attempt to prove the same result in the new context of bifree probability introduced by Voiculescu [8]. To this end, suppose $(\mathcal{A}_l^{(i)},\mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ is a nontrivial family of pairs of faces in the noncommutative C*-probability space $(\mathcal{A},\varphi)$. If $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)},\mathcal{A}_r^{(i)})}$ is faithful on $C^*(\mathcal{A}_l^{(i)},\mathcal{A}_r^{(i)})$, for all $i\in\mathcal{I}$, then it will be proven that if the bifree product state $**_{i\in\mathcal{I}}\varphi_i$ is faithful on the reduced bifree product $**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)},\mathcal{A}_r^{(i)})$, then $C^*(\mathcal{A}_l^{(i)},\mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)}\otimes_{min}\mathcal{A}_r^{(i)}$, $i\in\mathcal{I}$. A converse is shown with the added asumption that each φ_i is a product state. Moreover, in this case there is a commensurate result to that which follows from Dykema and Rørdam, mentioned above.

It should be mentioned that the failure in general of the faithfulness of the bifree product state has been pointed out in [4] and that this failure has been the cause of the introduction of weaker versions of faithfulness in the bifree context [4, 5].

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2. Bifree independence and the reduced bifree product

We will first take some time to recall the definition of bifree independence from [8] and then define the reduced bifree product of C*-algebras and the bifree product state.

Fix a noncommutative C*-probability space $(\mathcal{A}, \varphi)$, that is a unital C*-algebra and a state. Given a set $\mathcal{I}$, suppose that for each $i \in \mathcal{I}$ there is a pair of unital C*-subalgebras $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ of $\mathcal{A}$, a "left" algebra and a "right" algebra. We call the set $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ a family of pairs of faces in $\mathcal{A}$. Such a family will be called *nontrivial* if $|\mathcal{I}| \geqslant 2$ and $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \neq \mathbb{C}$ for all $i \in \mathcal{I}$. That is, there are at least two pairs of faces and there are no trivial pairs of faces.

Let $(\pi_i, \mathcal{H}_i, \xi_i)$ be the GNS construction for $(C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$ where $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})}$. Voiculescu [8] (and even way back in [7]) observed that there are two natural representations of $B(\mathcal{H}_i)$ on the free product Hilbert space, which we will now introduce. The free product Hilbert space,

$$(\mathcal{H}, \xi) = *_{i\in\mathcal{I}}(\mathcal{H}_i, \xi_i),$$

is given by associating all of the distinguished vectors and then forming a Fock space-like structure. Namely, if $\mathring{\mathcal{H}}_j = \mathcal{H}_j \ominus \mathbb{C}\xi_j$, then

$$\mathcal{H} := \mathbb{C}\xi \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ i_1,\ldots,i_n\in\mathcal{I} \\ i_1\neq\cdots\neq i_n}} \mathring{\mathcal{H}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_n}.$$

To define these representations we need to first build some Hilbert spaces and some unitaries. To this end, define

$$\mathcal{H}(l,i) := \mathbb{C}\xi \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ i_1,\ldots,i_n\in\mathcal{I} \\ i\neq i_1\neq\cdots\neq i_n}} \mathring{\mathcal{H}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_n} \qquad \text{and}$$

$$\mathcal{H}(r,i) := \mathbb{C}\xi \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ i_1,\ldots,i_n\in\mathcal{I} \\ i_1\neq\cdots\neq i_n\neq i}} \mathring{\mathcal{H}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_n}.$$

Then there are unitaries $V_i : \mathcal{H}_i \otimes \mathcal{H}(l,i) \to \mathcal{H}$ and $W_i : \mathcal{H}(r,i) \otimes \mathcal{H}_i$ given by concatenation (with appropriate handling of ξ_i and ξ). Finally, the two natural representations are the left representation $\lambda_i : B(\mathcal{H}_i) \to B(\mathcal{H})$ which is defined as

$$\lambda_i(T) = V_i(T \otimes I_{\mathcal{H}(l,i)})V_i^*$$

and the right representation $\rho_i : B(\mathcal{H}_i) \to B(\mathcal{H})$ which is defined as

$$\rho_i(T) = W_i(I_{\mathcal{H}(r,i)} \otimes T)W_i^*.$$

With all of this groundwork established we can finally define bifree independence. Note that $\circledast$ below refers to the full (or universal) free product of C*-algebras.

Definition 2.1 (Voiculescu [8])**.** The family of pairs of faces $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ in the noncommutative probability space $(\mathcal{A}, \varphi)$ is said to be *bifreely independent* with respect to φ if the following diagram commutes:

$$\begin{array}{ccccc}
\check{*}_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}\check{*}\mathcal{A}_r^{(i)}) & \xrightarrow{\iota} & \mathcal{A} & \xrightarrow{\varphi} & \mathbb{C} \\
{\scriptstyle *_{i\in\mathcal{I}}(\pi_i*\pi_i)}\downarrow & & & & \| \\
\check{*}_{i\in\mathcal{I}}(B(\mathcal{H}_i)\check{*}B(\mathcal{H}_i)) & \xrightarrow{*_{i\in I}(\lambda_i*\rho_i)} & B(\mathcal{H}) & \xrightarrow{\langle\cdot\xi,\xi\rangle} & \mathbb{C}
\end{array}$$

where ι is the unique $*$-homomorphism extending the identity on each $\mathcal{A}_\chi^{(i)}$, for all $\chi \in \{l, r\}$ and $i \in \mathcal{I}$.

From this we can now define the main objects of this paper.

Definition 2.2. Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ be a family of pairs of faces in the noncommutative C*-probability space $(\mathcal{A}, \varphi)$. As before, denote φ_i to be the restriction of φ to $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ and let $(\pi_i, \mathcal{H}_i, \xi_i)$ be the GNS construction of $(C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$.

The *reduced bifree product* of $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ with respect to the states φ_i is

$$(**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), **_{i\in\mathcal{I}}\varphi_i) = **_{i\in\mathcal{I}}((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$$

which is made up of the unital C*-subalgebra of $B(\mathcal{H})$, called the *reduced bifree product of C*-algebras*,

$$**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) := C^*((\lambda_i \circ \pi_i(\mathcal{A}_l^{(i)}), \rho_i \circ \pi_i(\mathcal{A}_r^{(i)}))_{i\in\mathcal{I}}) \subset B(\mathcal{H}),$$

and the *bifree product state*

$$**_{i\in\mathcal{I}}\varphi_i(\cdot) := \langle\cdot\xi, \xi\rangle.$$

It is an immediate fact that the family of pairs of faces $(\lambda_i \circ \pi_i(\mathcal{A}_l^{(i)}), \rho_i \circ \pi_i(\mathcal{A}_r^{(i)}))_{i\in\mathcal{I}}$ is bifreely independent with respect to the bifree product state.

It should be noted that we are working within the framework of the original noncommutative C*-probability space $(\mathcal{A}, \varphi)$. This means that the reduced bifree product is taking into account the behaviour of φ not just on the left and right faces but on the C*-algebra they generate, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$. Since bifree independence is a statement about the behaviour in the original C*-probability space, this definition makes sense.

That being said, one can create the reduced bifree product as an external product. Start with pairs of faces in different C*-probability spaces and simply create a new C*-probability space by taking the full free product of the C*-algebras and their associated states and then proceed with the above reduced bifree product construction.

3. Faithfulness of bifree product states

We first establish what happens when the bifree product state is faithful.

Theorem 3.1. *Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ be a nontrivial family of pairs of faces in the noncommutative C*-probability space $(\mathcal{A}, \varphi)$ such that $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})}$ is faithful on $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ for each $i \in \mathcal{I}$. If $**_{i\in\mathcal{I}}\varphi_i$ is faithful on the reduced bifree product $**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ then*

$$C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}.$$

Proof. First we will establish that $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute in $\mathcal{A}$, then we will show that they induce a C*-norm on the algebraic tensor product $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$, and finally that this is in fact the minimal tensor norm.

We will be using the notation from Section 2. To simplify things a little bit, because the φ_i are assumed to be faithful, consider $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ as already a subalgebra of $B(\mathcal{H}_i)$, and so $\varphi_i(\cdot) = \langle \cdot \xi_i, \xi_i \rangle$. That is, we are suppressing the π_i notation from the GNS construction. Moreover, we will be using the convention that $\lambda_i(x), \rho_i(x), \lambda_i * \rho_i(x)$ all are living in $B(\mathcal{H})$.

Suppose $a_\chi \in \mathcal{A}_\chi^{(i)}$ such that $\varphi_i(a_\chi) = 0$, $\chi \in \{l, r\}$, and $0 \neq b \in \mathcal{A}_l^{(j)} \cup \mathcal{A}_r^{(j)}$ for $j \neq i$ such that $\varphi_j(b) = 0$. Such a b exists by the nontriviality of the family of pairs of faces. This gives that $\langle b^* \xi_j, \xi_j \rangle = \varphi_j(b) = 0$, and so $b^* \xi_j \in \mathring{\mathcal{H}}_j$ while $\langle b(b^* \xi_j), \xi_j \rangle = \varphi_j(bb^*) \neq 0$ by the faithfulness of φ_j.

Now, [8, Section 1.5] establishes that $[\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)})](\mathcal{H} \ominus \mathcal{H}_i) = 0$, which gives that

$$(\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b))\xi = 0$$

since $b\xi \in \mathring{\mathcal{H}}_j \subset \mathcal{H}$. The faithfulness of $**_{i \in \mathcal{I}} \varphi_i$ implies that ξ is a separating vector for the reduced bifree product, and thus

$$\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b) = 0,$$

which gives that

$$\begin{aligned} 0 &= P_{\mathcal{H}_i}(\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b))b^*\xi_j \\ &= (\lambda_i(a_l)\rho_i(a_r) - \rho_i(a_r)\lambda_i(a_l))\langle bb^*\xi_j, \xi_j \rangle \xi \\ &= \langle bb^*\xi_j, \xi_j \rangle (a_l a_r - a_r a_l)\xi_i. \end{aligned}$$

Since ξ_i is separating for $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ this implies that a_l and a_r commute. Thus, $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute in $\mathcal{A}$ for every $i \in \mathcal{I}$.

Claim. The canonical map from $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ to $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ is injective.

Since $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute, the universal property of $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ gives that there exists a $*$-homomorphism

$$\sum_{k=1}^m a_{k,l} \odot a_{k,r} \mapsto \sum_{k=1}^m a_{k,l} a_{k,r}.$$

We need to establish its injectivity. To this end, consider $h \in \mathring{\mathcal{H}}_j$, $\|h\| = 1$, where $j \neq i$, and the isometric map

$$V_h : \mathcal{H}_i \otimes \mathcal{H}_i \to \mathcal{H}_i \otimes h \otimes \mathcal{H}_i$$

defined by $V_h(h_l \otimes h_r) = h_l \otimes h \otimes h_r$ for $h_l, h_r \in \mathcal{H}_i$. This map is inspired by Dykema's proof of the faithfulness of the free product state [2, Theorem 1.1]. Note that in $\mathcal{H}$ we really have that

$$\mathcal{H}_i \otimes h \otimes \mathcal{H}_i = \mathbb{C}h \oplus (\mathring{\mathcal{H}}_i \otimes h) \oplus (h \otimes \mathring{\mathcal{H}}_i) \oplus (\mathring{\mathcal{H}}_i \otimes h \otimes \mathring{\mathcal{H}}_i),$$

but hopefully the reader will pardon the simplified notation.

Now $\mathcal{H}_i \otimes h \otimes \mathcal{H}_i$ is a reducing subspace of $C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))$ since for all $a \in \mathcal{A}_l^{(i)}, b \in \mathcal{A}_r^{(i)}$, and $\eta_1, \eta_2 \in \mathcal{H}_i$ we have that

$$\begin{aligned} V_h^* \lambda_i(a)\rho_i(b)V_h(\eta_1 \otimes \eta_2) &= V_h^* \lambda_i(a)\rho_i(b)(\eta_1 \otimes h \otimes \eta_2) \\ &= a\eta_1 \otimes b\eta_2. \end{aligned}$$

Thus, compressing to $\mathcal{H}_i \otimes h \otimes \mathcal{H}_i$ gives

$$V_h^* C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))V_h = \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}.$$

So, if $\sum_{k=1}^m a_{k,l} \odot a_{k,r} \neq 0 \in \mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$, then $\sum_{k=1}^m a_{k,l} \otimes a_{k,r} \neq 0 \in \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$, which implies that

$$\begin{aligned} 0 &\neq \sum_{k=1}^m a_{k,l} \otimes a_{k,r}(\xi_i \otimes \xi_i) \\ &= V_h^* \sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})V_h(\xi_i \otimes \xi_i) \\ &= \sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})h \end{aligned}$$

since the state $\langle \cdot \xi_i \otimes \xi_i, \xi_i \otimes \xi_i \rangle$ is faithful on the min tensor product. But then $\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r}) \neq 0 \in C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))$, which by the faithfulness of $**_{i\in\mathcal{I}}\varphi_i$ gives that $\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi \neq 0$. Finally,

$$\begin{aligned} 0 &\neq \left\langle \sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi, \sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi \right\rangle \\ &= \left\langle \left(\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\right)^* \left(\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\right) \xi, \xi \right\rangle \\ &= \varphi_i \left(\left(\sum_{k=1}^m a_{k,l}a_{k,r}\right)^* \left(\sum_{k=1}^m a_{k,l}a_{k,r}\right)\right), \end{aligned}$$

which gives by the faithfulness of φ_i that $\sum_{k=1}^m a_{k,l}a_{k,r} \neq 0$. Therefore, the claim is verified.

Now, this implies that $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_\alpha \mathcal{A}_r^{(i)}$ where $\| \cdot \|_\alpha$ is a C*-norm on $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$. So by Takesaki's Theorem [6] we have that there exists a surjective $*$-homomorphism

$$q : C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \to \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}.$$

To finish the proof all we need to do is show that q is injective.

To this end, let $a \in C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ such that $q(a) = 0$. Again as in the first part of this proof, find $0 \neq b \in \mathcal{A}_l^{(j)} \cup \mathcal{A}_r^{(j)}$ for $j \neq i$ such that $\varphi_j(b) = 0$ and $h \in \mathring{\mathcal{H}}_j$ such that $\langle bh, \xi_j \rangle \neq 0$. Additionally, assume that $\|b\xi_j\| = 1$.

In the second part of this proof we saw that compressing to $\mathcal{H}_i \otimes b\xi_j \otimes \mathcal{H}_i$ is tantamount to this quotient homomorphism q. Namely, suppose

$$\iota_i : \mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)} \to C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \ \ (\subseteq B(\mathcal{H}_i) \text{ by assumption})$$

is the unique $*$-homomorphism extending the identity in each component. There then exists $\tilde{a} \in \mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)}$ such that $\iota_i(\tilde{a}) = a$. An important fact to record is that, by uniqueness,

$$\lambda_i * \rho_i(\cdot)|_{\mathcal{H}_i} = \iota_i(\cdot),$$

remembering that we have that $\lambda_i * \rho_i(\cdot) \in B(\mathcal{H})$. Thus,

$$V_{b\xi_j}^* \lambda_i * \rho_i(\tilde{a}) V_{b\xi_j} = q(a) = 0,$$

which implies, by the fact that $V_{b\xi_j}(\mathcal{H}_i \otimes \mathcal{H}_i)$ is reducing for $\lambda_i * \rho_i(\mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)})$, that

$$\begin{aligned} 0 &= \lambda_i * \rho_i(\tilde{a}) V_{b\xi_j}(\xi_i \otimes \xi_i) \\ &= \lambda_i * \rho_i(\tilde{a})(b\xi_j) \\ &= \lambda_i * \rho_i(\tilde{a}) \lambda_j * \rho_j(b)\xi. \end{aligned}$$

By the faithfulness of the bifree product state $\lambda_i * \rho_i(\tilde{a})\lambda_j * \rho_j(b) = 0$, and so

$$\begin{aligned} 0 &= P_{\mathcal{H}_i} \lambda_i * \rho_i(\tilde{a}) \lambda_j * \rho_j(b) h \\ &= \lambda_i * \rho_i(\tilde{a}) \langle bh, \xi_j \rangle \xi \\ &= \langle bh, \xi_j \rangle \iota_i(\tilde{a}) \xi \\ &= \langle bh, \xi_j \rangle a \xi_i. \end{aligned}$$

Hence, by the faithfulness of φ_i we have that $a = 0$. Therefore, for all $i \in \mathcal{I}$, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$. □

We turn now to a partial converse of the previous theorem. This is probably known among the experts in bifree probability, but we could not find a published proof. The following proof may be a tad clunky but we find it the clearest from a nonexpert perspective.

Theorem 3.2. *Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ be a family of pairs of faces in the noncommutative C^*-probability space $(\mathcal{A}, \varphi)$. If $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$ and $\varphi_i = \varphi_i|_{\mathcal{A}_l^{(i)}} \otimes \varphi_i|_{\mathcal{A}_r^{(i)}}$ is a faithful product state on $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$, for all $i \in \mathcal{I}$, then $**_{i \in \mathcal{I}} \varphi_i$ is faithful on the reduced bifree product and*

$$**_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}} \ \simeq \ *_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \otimes_{min} *_{i \in \mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi).$$

Proof. As before, we will be using the notation of Section 2.

For each $i \in \mathcal{I}$, since $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$ and φ_i is a product state we can a priori choose $\mathcal{H}_i = \mathcal{H}_{i,l} \otimes \mathcal{H}_{i,r}$, unit vectors $\xi_{i,l} \in \mathcal{H}_{i,l}, \xi_{i,r} \in \mathcal{H}_{i,r}$ such that $\xi_i = \xi_{i,l} \otimes \xi_{i,r}$, and $*$-homomorphisms $\pi_{i,\chi} : \mathcal{A}_\chi^{(i)} \to B(\mathcal{H}_{i,\chi})$ such that $\pi_i = \pi_{i,l} \otimes \pi_{i,r}$. This will give for $a_\chi \in \mathcal{A}_\chi^{(i)}$, $\chi \in \{l, r\}$, that

$$\begin{aligned} \varphi_i(a_l a_r) &= \langle \pi_i(a_l a_r)\xi_i, \xi_i \rangle \\ &= \langle \pi_{i,l}(a_l)\xi_{i,l}, \xi_{i,l} \rangle \langle \pi_{i,r}(a_r)\xi_{i,r}, \xi_{i,r} \rangle. \end{aligned}$$

Along with the free product Hilbert space

$$(\mathcal{H}, \xi) = *_{i \in \mathcal{I}}(\mathcal{H}_i, \xi_i)$$

we need to also define, for $\chi \in \{l, r\}$, the free product Hilbert spaces

$$(\mathcal{H}_\chi, \xi_\chi) = *_{i \in \mathcal{I}}(\mathcal{H}_{i,\chi}, \xi_{i,\chi}).$$

Since there are multiple free product Hilbert spaces we will use subscripts to denote the different left and right representations, namely,

$$\lambda_{\mathcal{H}_i} : B(\mathcal{H}_i) \to B(\mathcal{H}) \quad \text{and} \quad \lambda_{\mathcal{H}_{i,l}} : B(\mathcal{H}_{i,l}) \to B(\mathcal{H}_l)$$

for the left representations and

$$\rho_{\mathcal{H}_i} : B(\mathcal{H}_i) \to B(\mathcal{H}) \quad \text{and} \quad \rho_{\mathcal{H}_{i,r}} : B(\mathcal{H}_{i,r}) \to B(\mathcal{H}_r)$$

for the right representations.

Dykema's original result [2] proves that $\langle \cdot \xi_\chi, \xi_\chi \rangle$ is faithful on $\ast_{i\in\mathcal{I}}(\mathcal{A}_\chi^{(i)}, \varphi)$ for $\chi \in \{l, r\}$, and it is a folklore result that the minimal tensor product of faithful states is faithful. Thus, $\langle \cdot\, \xi_l \otimes \xi_r, \xi_l \otimes \xi_r \rangle$ is faithful on $\ast_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \otimes_{min} \ast_{i\in\mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi)$.

Fix $k \geqslant 1$ and $j_1, \dots, j_k \in \mathcal{I}$ such that $j_i \neq j_{i+1}, 1 \leqslant i \leqslant k-1$. Now fix a unit vector

$$\begin{aligned} h &= (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r}) \\ &\in (\xi_{j_1,l} \otimes \mathring{\mathcal{H}}_{j_1,r}) \otimes \mathring{\mathcal{H}}_{j_2} \otimes \cdots \otimes \mathring{\mathcal{H}}_{j_{k-1}} \otimes (\mathring{\mathcal{H}}_{j_k,l} \otimes \xi_{j_k,r}). \end{aligned}$$

If $k = 1$ the only possible h is $\xi = \xi_{j_1} = \xi_{j_1,l} \otimes \xi_{j_1,r}$. Call the collection of such h, as k and the indices vary, $\mathcal{S} \subset \mathcal{H}$.

As will be shown below, this set of unit vectors $\mathcal{S}$ plays an important role in decomposing simple tensors in $\mathcal{H}$. In particular, for every simple tensor $\eta \in \mathcal{H}$ that is also a simple tensor in each component, there exists a unique $h \in \mathcal{S}$ such that $\eta \in \mathcal{H}_l \otimes h \otimes \mathcal{H}_r$. By abuse of tensor notation this is not very hard to see in one's mind, but the reality of proving this carefully needs plenty of indices.

To this end, for $m \geqslant 1$ suppose $s_1, \dots, s_m \in \mathcal{I}$ such that $s_t \neq s_{t+1}$ for $1 \leqslant t \leqslant m-1$, and $\eta_{t,l} \in \mathcal{H}_{s_t,l}$, $\eta_{t,r} \in \mathcal{H}_{s_t,r}$ such that $\eta_{t,l} \otimes \eta_{t,r} \in \mathring{\mathcal{H}}_{s_t}$ for $1 \leqslant t \leqslant m$. This last condition implies that $\eta_{t,\chi} = \|\eta_{t,\chi}\|\xi_{t,\chi}$ cannot hold for both $\chi = l$ and $\chi = r$. In summary,

$$\eta := (\eta_{1,l} \otimes \eta_{1,r}) \otimes \cdots \otimes (\eta_{m,l} \otimes \eta_{m,r}) \in \mathring{\mathcal{H}}_{s_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{s_m}.$$

Note that the conditions imposed on the $\eta_{t,\chi}$ in the above paragraph imply that the form of η above is as reduced as it can be.

As mentioned above, it will be established that there exists $h \in \mathcal{S}$ such that

$$\eta \in \mathcal{H}_l \otimes h \otimes \mathcal{H}_r.$$

To prove the required decomposition, let

$$v = \max\{0 \leqslant t \leqslant m \;:\; \eta_{j,r} = \|\eta_{j,r}\|\xi_{s_j,r},\ 1 \leqslant j \leqslant t\}$$

and

$$w = \min\{1 \leqslant t \leqslant m+1 \;:\; \eta_{j,l} = \|\eta_{j,l}\|\xi_{s_j,l},\ t \leqslant j \leqslant m\}.$$

This gives that v is the number of terms in a row from the left with trivial right tensor components and that $m+1-w$ is the number of terms in a row from the right with trivial left tensor components.

By the fact that $\eta_{t,l} \otimes \eta_{t,r} \in \mathring{\mathcal{H}}_{s_t}$, that is, $\eta_{t,\chi} = \|\eta_{t,\chi}\|\xi_{t,\chi}$ cannot hold for both $\chi = l$ and $\chi = r$, we have that $v < w$. If $v = m$, then $w = m+1$ and $\eta \in \mathcal{H}_l$, and if $w = 1$, then $v = 0$ and $\eta \in \mathcal{H}_r$. Otherwise, when $0 \leqslant v \leqslant m-1$ and $2 \leqslant w \leqslant m$, define

$$\begin{aligned} \eta_l &= (\eta_{1,l} \otimes \|\eta_{1,r}\|\xi_{s_1,r}) \otimes \cdots \otimes (\eta_{v,l} \otimes \|\eta_{v,r}\|\xi_{s_v,r}) \otimes (\eta_{v+1,l} \otimes \xi_{s_{v+1},r}), \\ \eta_{\mathcal{S}} &= (\xi_{v+1,l} \otimes \eta_{v+1,r}) \otimes \cdots \otimes (\eta_{w-1,l} \otimes \xi_{w-1,r}), \\ \eta_r &= (\xi_{w-1,l} \otimes \eta_{w-1,r}) \otimes (\|\eta_{w,l}\|\xi_{w,l} \otimes \eta_{w,r}) \otimes \cdots \otimes (\|\eta_{m,l}\|\xi_{m,l} \otimes \eta_{m,r}) \end{aligned}$$

with $\eta_{\mathcal{S}} = \xi$ if $v + 1 = w$. Hence, by the usual slight abuse of the tensor notation, $\eta = \eta_l \otimes \eta_{\mathcal{S}} \otimes \eta_r \in \mathcal{H}_l \otimes \eta_{\mathcal{S}} \otimes \mathcal{H}_r$ with $\frac{1}{\|\eta_{\mathcal{S}}\|}\eta_{\mathcal{S}} \in \mathcal{S}$. Therefore,

$$\overline{\operatorname{span}}\{\mathcal{H}_l \otimes h \otimes \mathcal{H}_r : h \in \mathcal{S}\} = \mathcal{H}.$$

For any $h \in \mathcal{S}$, which is a unit vector, there is a natural isometric map $S_h : \mathcal{H}_l \otimes \mathcal{H}_r \to \mathcal{H}$ given by the concatenation $\mathcal{H}_l \otimes \mathcal{H}_r \mapsto \mathcal{H}_l \otimes h \otimes \mathcal{H}_r$ with the appropriate simplification of tensors when needed. In particular, there exist $k \geqslant 1$ and $j_1, \ldots, j_k \in \mathcal{I}$ such that $j_i \neq j_{i+1}, 1 \leqslant i \leqslant k-1$, and then

$$\begin{aligned} h &= (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r}) \\ &\in (\xi_{j_1,l} \otimes \mathring{\mathcal{H}}_{j_1,r}) \otimes \mathring{\mathcal{H}}_{j_2} \otimes \cdots \otimes \mathring{\mathcal{H}}_{j_{k-1}} \otimes (\mathring{\mathcal{H}}_{j_k,l} \otimes \xi_{j_k,r}). \end{aligned}$$

We can now carefully specify that the isometric map is given by

$$\xi_l \otimes \xi_r \mapsto h,$$

$$\xi_l \otimes (\mathring{\mathcal{H}}_{i_1,r} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,r})$$
$$\to \begin{cases} h \otimes (\xi_{i_1,l} \otimes \mathring{\mathcal{H}}_{i_1,r}) \otimes \cdots \otimes (\xi_{i_m,l} \otimes \mathring{\mathcal{H}}_{i_m,r}), & i_1 \neq j_k, \\ (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \mathring{\mathcal{H}}_{i_1,r}) & \\ \qquad \otimes \cdots \otimes (\xi_{i_m,l} \otimes \mathring{\mathcal{H}}_{i_m,r}), & i_1 = j_k, \end{cases}$$

$$(\mathring{\mathcal{H}}_{i_1,l} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,l}) \otimes \xi_r$$
$$\to \begin{cases} (\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes \xi_{i_m,r}) \otimes h, & i_m \neq j_1 \\ (\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes h_{j_1,r}) \otimes h_{j_2} & \\ \qquad \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r}), & i_m = j_1 \end{cases}$$

and

$$(\mathring{\mathcal{H}}_{i_1,l} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,l}) \otimes (\mathring{\mathcal{H}}_{t_1,r} \otimes \cdots \otimes \mathring{\mathcal{H}}_{t_s,r})$$
$$\to (\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes \xi_{i_m,r}) \otimes h \otimes (\xi_{t_1,l} \otimes \mathring{\mathcal{H}}_{t_1,r}) \otimes \cdots \otimes (\xi_{t_s,l} \otimes \mathring{\mathcal{H}}_{t_s,r})$$

if $i_m \neq j_1$ and $j_k \neq t_1$, with similar statements as the cases above when $i_m = j_1$ or $j_k = t_1$ or both happen. Perhaps the most natural case of S_h is when $h = \xi$. It certainly minimizes, but doesn't remove, the need for all of the cases above.

A careful examination of the S_h isometric map implies that for $a \in \mathcal{A}_l^{(i_1)}$, $b \in \mathcal{A}_r^{(i_2)}$, and $\eta_\chi \in \mathcal{H}_\chi$ for $\chi \in \{l, r\}$ we have that, by abuse of the tensor notation,

$$\begin{aligned} &\lambda_{\mathcal{H}_{i_1}}(\pi_{i_1}(a))\rho_{\mathcal{H}_{i_2}}(\pi_{i_2}(b))S_h(\eta_l \otimes \eta_r) \\ &\qquad = \lambda_{\mathcal{H}_{i_1}}(\pi_{i_1}(a))\rho_{\mathcal{H}_{i_2}}(\pi_{i_2}(b))(\eta_l \otimes h \otimes \eta_r) \\ &\qquad = \lambda_{\mathcal{H}_{i_1,l}}(\pi_{i_1,l}(a))\eta_l \otimes h \otimes \rho_{\mathcal{H}_{i_2,r}}(\pi_{i_2,r}(b))\eta_r \\ &\qquad = S_h(\lambda_{\mathcal{H}_{i_1,l}}(\pi_{i_1,l}(a))\eta_l \otimes \rho_{\mathcal{H}_{i_2,r}}(\pi_{i_2,r}(b))\eta_r). \end{aligned}$$

Hence, $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$ is a reducing subspace of the reduced bifree product. Moreover,

$$S_h^* \lambda_{\mathcal{H}_i} \circ \pi_i(\cdot) S_h = (\lambda_{\mathcal{H}_{i,l}} \circ \pi_{i,l}(\cdot)) \otimes I_{\mathcal{H}_r} \qquad \text{on } \mathcal{A}_l^{(i)}$$

and

$$S_h^* \rho_{\mathcal{H}_i} \circ \pi_i(\cdot) S_h = I_{\mathcal{H}_l} \otimes (\rho_{\mathcal{H}_{i,r}} \circ \pi_{i,r}(\cdot)) \qquad \text{on } \mathcal{A}_r^{(i)}.$$

Therefore, for any $h \in \mathcal{S}$,

$$S_h^*(\ast\!\ast_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}))S_h = \ast_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \ \otimes_{min} \ \ast_{i\in\mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi),$$

and furthermore, by the identities involving S_h, λ, and ρ, $S_h^* a S_h = S_\xi^*(a) S_\xi$ for all $h \in \mathcal{S}$ and $a \in \ast\ast_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$.

Finally, we want to show that compression to $S_\xi(\mathcal{H}_l \otimes \mathcal{H}_r)$ is a $*$-isomorphism. Note that this is the same as compression to $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$ being injective for any $h \in \mathcal{S}$. This gives us a way forward. Suppose that $a \in \ast\ast_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ such that $\ast\ast_{i\in\mathcal{I}}\varphi_i(a^*a) = 0$. This implies that

$$\begin{aligned} 0 &= a\xi \\ &= S_\xi^* a S_\xi(\xi_l \otimes \xi_r). \end{aligned}$$

By the faithfulness of $\langle \cdot\, \xi_l \otimes \xi_r, \xi_l \otimes \xi_r \rangle$ this gives that $S_\xi^* a S_\xi = 0$ or rather a is 0 on the reducing subspace $S_\xi(\mathcal{H}_l \otimes \mathcal{H}_r)$. But then for all $h \in \mathcal{S}$ we have that

$$S_h^* a S_h = S_\xi^* a S_\xi = 0$$

and a is 0 on the reducing subspace $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$. By what we proved about the set $\mathcal{S}$, we have that a is 0 on

$$\overline{\operatorname{span}}\{S_h(\mathcal{H}_l \otimes \mathcal{H}_r) : h \in \mathcal{S}\} = \overline{\operatorname{span}}\{\mathcal{H}_l \otimes h \otimes \mathcal{H}_r : h \in \mathcal{S}\} = \mathcal{H}.$$

Therefore, $a = 0$, and thus $\ast\ast_{i\in\mathcal{I}}\varphi_i$ is faithful. ☐

There may exist a full converse to Theorem 3.1 but the previous proof highly depends on the state φ_i arising as a tensor product of states. In general, φ_i need not be of this form. We should note here that if $\varphi_i|_{\mathcal{A}_l^{(i)}}$ or $\varphi_i|_{\mathcal{A}_l^{(i)}}$ is a pure state, then φ_i will be a tensor product of states.

To end this paper, we summarize with the following corollary.

Corollary 3.3. *Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ be a nontrivial family of pairs of faces in the noncommutative C^*-probability space $(\mathcal{A}, \varphi)$. If φ is faithful on $C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}})$, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$, $\varphi_i = \varphi_i|_{\mathcal{A}_l^{(i)}} \otimes \varphi_i|_{\mathcal{A}_r^{(i)}}$, and $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ is bifreely independent with respect to φ, then*

$$\begin{aligned} C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}) &\simeq \ast\ast_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}} \\ &\simeq \ast_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \otimes_{min} \ast_{i\in\mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi). \end{aligned}$$

Proof. Recall, that by bifree independence we know that the following diagram commutes:

$$\begin{array}{ccccc} \check{\ast}_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)} \check{\ast} \mathcal{A}_r^{(i)}) & \xrightarrow{\iota} & C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}) & \xrightarrow{\varphi} & \mathbb{C} \\ {\scriptstyle \ast_{i\in\mathcal{I}}(\pi_i \ast \pi_i)}\big\downarrow & & & & \big\| \\ \check{\ast}_{i\in\mathcal{I}}(B(\mathcal{H}_i) \check{\ast} B(\mathcal{H}_i)) & \xrightarrow{\ast_{i\in I}(\lambda_i \ast \rho_i)} & B(\mathcal{H}) & \xrightarrow{\langle \cdot \xi, \xi \rangle} & \mathbb{C} \end{array}$$

Because both of the states are faithful on their algebras, then for any $a^*a \in \check{\ast}_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)} \check{\ast} \mathcal{A}_r^{(i)})$, a^*a is in the kernel of ι if and only if a^*a is in the kernel of $\ast_{i\in I}(\lambda_i \ast \rho_i) \circ \ast_{i\in\mathcal{I}}(\pi_i \ast \pi_i)$. Therefore, both quotients are $*$-isomorphic and Theorem 3.2 gives the final $*$-isomorphism. ☐

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Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada

Current address: Department of Mathematics and Statistics, MacEwan University, Edmonton, Alberta, Canada

Email address: `Ramseyc5@macewan.ca`