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RESIDUES FOR MAPS GENERICALLY TRANSVERSE TO DISTRIBUTIONS

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ABSTRACT. We show a residues formula for maps generically transversal to regular holomorphic distributions.

1. Introduction

Let $f : X \longrightarrow Y$ be a singular holomorphic map between complex manifolds X and Y, with $\dim(X) := n \geq m =: \dim(Y)$, having generic fiber F. Consider the singular set of f defined by

$$S := \mathrm{Sing}(f) = \{p \in X : \mathrm{rank}(df(p)) < m \}.$$

If $Y = C$ is a curve, Iversen in [11] proved the multiplicity formula

$$\chi(X) - \chi(F) \cdot \chi(C) = (-1)^n \sum_{p \in \mathrm{Sing}(f)} \mu_p(f),$$

where $\mu_p(f)$ is the Milnor number of f at p. Izawa and Suwa [14] generalized Iversen's result for the case where X is possibly a singular variety.

A generalization of the multiplicity formula for maps was given by Diop in [7]. In his work he generalized some formulas involving the Chern classes given previously by Iversen [11], Brasselet [3, 4], and Schwartz [17]. More precisely, Diop showed that if S is smooth and $\dim(S) = m - 1$, then

$$\chi(X) - \chi(F)\chi(Y) = (-1)^{n-m+1} \sum_j \mu_j \int_{S_j} c_{q-1}[(f^*TY)|_{S_j} - \mathcal{L}_j],$$

where $S = \bigcup S_j$ is the decomposition of S into irreducible components, $\mu_j = \mu(f|\,\Sigma_j)$ is the Milnor number of the restriction of f to a transversal section Σ_j to S_j at a regular point $p_j \in S_j$, and $\mathcal{L}_j$ is the line bundle over S_j given by the decomposition $f^*df(TX|_{S_j}) \oplus \mathcal{L}_j = f^*(TY)|_{S_j}$.

On the other hand, Brunella in [5] introduced the notion of tangency index of a germ of curve with respect to a germ of holomorphic foliation: given a reduced curve C and a foliation $\mathcal{F}$ (possibly singular) on a complex compact surface, suppose that C is not invariant by $\mathcal{F}$ and that C and $\mathcal{F}$ are given locally by $\{f = 0\}$ and a vector

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field v, respectively. The tangency index $I_p(\mathcal{F}, C)$ of C with respect to $\mathcal{F}$ at p is given by the intersection number

$$I_p(\mathcal{F}, C) = \dim_{\mathbb{C}} \mathcal{O}_2/(f, v(f)).$$

Using this index, Brunella proved the formula

$$c_1(\mathcal{O}(C))^2 - c_1(T_{\mathcal{F}}) \cap c_1(\mathcal{O}(C)) = \sum_{p \in \mathrm{Tang}(\mathcal{F},C)} I_p(\mathcal{F}, C),$$

where $T_{\mathcal{F}}$ is the tangent bundle of $\mathcal{F}$ and $\mathrm{Tang}(\mathcal{F}, C)$ denotes the non-transversality loci of C with respect to $\mathcal{F}$. In [9] and [10], T. Honda also studied Brunella's tangency formula. Distributions and foliations transverse to certain domains in $\mathbb{C}^n$ have been studied by Bracci and Scárdua in [2] and by Ito and Scárdua in [12].

Recently, Izawa [13] generalized certain results due to Diop [7] in the foliated context. More precisely, let $f : X \longrightarrow (Y, \mathcal{F})$ be a holomorphic map such that $\mathcal{F}$ is a regular holomorphic foliation of codimension one in Y. Let $S(f, \mathcal{F})$ be the set of points where f fails to be transverse to $\mathcal{F}$. Suppose $S(f, \mathcal{F})$ is given by isolated points and let $\widetilde{\mathcal{F}} := f^*\mathcal{F}$. Since $\mathcal{F}$ is regular, we may find local coordinates in a neighborhood of $p \in \mathrm{Sing}(f)$ and $f(p)$ in such a way that $f = (f_1, \dots, f_m)$ and $\widetilde{\mathcal{F}}$ is given by $\ker(df_m)$ nearby p. If we pick $g_i := \frac{\partial f_m}{\partial x_i}$ (i.e., $df_m = g_1 dx_1 + \cdots + g_n dx_n$), then

$$\chi(X) - \sum_{i=1}^{r} f_*(c_{n-i}(T_X) \cap [X]) \cap c_1(\mathcal{N}_{\mathcal{F}})^i = (-1)^n \sum_{p \in S(f,\mathcal{F})} \mathrm{Res}_p \left[\begin{array}{c} dg_1 \wedge \cdots \wedge dg_m \\ g_1, \dots, g_m \end{array} \right],$$

where $\mathcal{N}_{\mathcal{F}}$ denotes the normal sheaf of $\mathcal{F}$.

In this paper we generalize the above results for a regular distribution $\mathcal{F}$ in Y of any codimension with the following residual formula for the non-transversality points of $f(X)$ with respect to $\mathcal{F}$.

In order to state our main result, let us introduce some notions. Let $f : X \longrightarrow (Y, \mathcal{F})$ be a holomorphic map and suppose that X and Y are projective manifolds. We say that the set of points in X where f fails to be transversal to $\mathcal{F}$ is the *ramification locus* of f with respect to $\mathcal{F}$, and denote it by $S(f, \mathcal{F})$. The set $R(f, \mathcal{F}) := f(S(f, \mathcal{F}))$ is called the *branch locus* or the set of *branch points* of f with respect to $\mathcal{F}$. Let $S(f, \mathcal{F}) = \bigcup S_j$ be the decomposition of S into irreducible components. Then we denote by $\mu(f, \mathcal{F}, S_j)$ the multiplicity of S_j and call it the *ramification multiplicity of f along S_j* with respect to $\mathcal{F}$. As usual, we denote by $[W]$ the class in the Chow group of X of the subvariety $W \subset X$. The class $f_*[S_j] =: [R_j]$ is called a *branch class* of f. Observe that $R(f, \mathcal{F})$ is the set of tangency points between $f(X)$ and $\mathcal{F}$ if $\dim(X) \le \dim(Y)$.

Theorem 1.1. *Let $f : X \longrightarrow (Y, \mathcal{F})$ be a holomorphic map of generic rank r and let $\mathcal{F}$ be a non-singular distribution of codimension k on Y. Suppose the ramification locus of f with respect to $\mathcal{F}$ has codimension $n - k + 1$. Then*

$$\begin{aligned} f_*(c_{n-k+1}(T_X) \cap [X]) + \sum_{i=1}^{r} (-1)^i f_*(c_{n-k+1-i}(T_X) \cap [X]) \cap s_i(\mathcal{N}_{\mathcal{F}}^*) \\ = (-1)^{n-k+1} \sum_{R_j \subset R} \mu(f, \mathcal{F}, S_j)[R_j], \end{aligned}$$

where $s_i(\mathcal{N}_{\mathcal{F}}^*)$ is the i-th Segre class of $\mathcal{N}_{\mathcal{F}}^*$.

Some consequences of this result are the following.

Corollary 1.2 (Izawa). *If $k = 1$, then*

$$\chi(X)-\sum_{i=1}^{r} f_*(c_{n-i}(T_X)\cap[X])\cap c_1(\mathcal{N}_{\mathcal{F}})^i = (-1)^n \sum_{p\in S(f,\mathcal{F})} \mathrm{Res}_p \left[\begin{array}{c} dg_1\wedge\cdots\wedge dg_m \\ g_1,\cdots,g_m \end{array} \right].$$

In fact, if $k = 1$ we have $c_n(T_X)\cap[X] = \chi(X)$ by the Chern-Gauss-Bonnet Theorem. Since $\mathcal{N}^*_{\mathcal{F}}$ is a line bundle, then $s_i(\mathcal{N}^*_{\mathcal{F}}) = (-1)^i c_1(\mathcal{N}^*_{\mathcal{F}})^i$ for all i. The above Izawa formula [13, Theorem 4.1] implies the multiplicity formula

$$\chi(X) - \chi(F)\cdot\chi(C) = (-1)^n \sum_{p\in \mathrm{Sing}(f)} \mu_p(f).$$

Corollary 1.3 (Tangency formulae). *Let $X \subset Y$ be a k-dimensional submanifold generically transverse to a non-singular distribution $\mathcal{F}$ on Y of codimension k. Then*

$$[c_1(N_{X|Y}) - c_1(T_{\mathcal{F}})]\cap[X] = \sum_{R_j\subset R} \mu(f,\mathcal{F},S_j)[R_j].$$

In particular, if $\det(T_{\mathcal{F}})|_X - \det(N_{X|Y})$ *is ample, then X is tangent to $\mathcal{F}$.*

If $X = C$ is a curve on a surface Y, we have $[C] = c_1(\mathcal{O}(C)) = c_1(N_{X|Y})$. This yields Brunella's formula

$$c_1(\mathcal{O}(C))^2 - c_1(T_{\mathcal{F}})\cap c_1(\mathcal{O}(C)) = \sum_{p\in \mathrm{Tang}(\mathcal{F},C)} I_p(\mathcal{F},C).$$

Moreover, this formula coincides with Honda's formula [10] in case $\mathcal{F}$ is a one-dimensional foliation and X is a curve.

In Section 3, we prove Theorem 1.1 and Corollary 1.3.

2. Holomorphic distributions

Let X be a complex manifold of dimension n.

Definition 2.1. A codimension k distribution $\mathcal{F}$ on X is given by an exact sequence

$$\mathcal{F}:\ 0 \longrightarrow \mathcal{N}^*_{\mathcal{F}} \longrightarrow \Omega^1_X \longrightarrow \Omega_{\mathcal{F}} \longrightarrow 0, \tag{1}$$

where $\mathcal{N}^*_{\mathcal{F}}$ is a coherent sheaf of rank $k \le \dim(X)-1$ and $\Omega_{\mathcal{F}}$ is a torsion free sheaf. We say that $\mathcal{F}$ is a foliation if at the level of local sections we have $d(\mathcal{N}^*_{\mathcal{F}}) \subset \mathcal{N}^*_{\mathcal{F}}\wedge\Omega^1_X$. The singular set of the distribution $\mathcal{F}$ is defined by $\mathrm{Sing}(\mathcal{F}) := \mathrm{Sing}(\Omega_{\mathcal{F}})$. We say that $\mathcal{F}$ is regular if $\mathrm{Sing}(\mathcal{F}) = \emptyset$.

Taking determinants of the map $\mathcal{N}^*_{\mathcal{F}} \longrightarrow \Omega^1_X$, we obtain a map

$$\det(\mathcal{N}^*_{\mathcal{F}}) \longrightarrow \Omega^k_X,$$

which induces a twisted holomorphic k-form $\omega \in H^0(X, \Omega^k_X\otimes\det(\mathcal{N}^*_{\mathcal{F}})^*)$. Therefore, a distribution can be induced by a twisted holomorphic k-form

$$H^0(X, \Omega^k_X \otimes \det(\mathcal{N}^*_{\mathcal{F}})^*),$$

which is locally decomposable outside the singular set of $\mathcal{F}$. That is, for each point $p \in X \setminus \mathrm{Sing}(\mathcal{F})$ there exists a neighborhood U and holomorphic 1-forms $\omega_1,\ldots,\omega_k \in H^0(U,\Omega^1_U)$ such that

$$\omega|_U = \omega_1\wedge\cdots\wedge\omega_k.$$

Moreover, if $\mathcal{F}$ is a foliation, then by Definition 2.1 we have

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0$$

for all $i = 1, \ldots, k$. The tangent sheaf of $\mathcal{F}$ is the coherent sheaf of rank $(n-k)$ given by

$$T_{\mathcal{F}} = \{v \in T_X;\ i_v\omega = 0\}.$$

The normal sheaf of $\mathcal{F}$ is defined by $\mathcal{N}_{\mathcal{F}} = T_X/T_{\mathcal{F}}$. It is worth noting that $\mathcal{N}_{\mathcal{F}} \neq (\mathcal{N}^*_{\mathcal{F}})^*$ whenever $\mathrm{Sing}(\mathcal{F}) \neq \emptyset$. Dualizing the sequence (1) one obtains the exact sequence

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_X \longrightarrow (\mathcal{N}^*_{\mathcal{F}})^* \longrightarrow \mathrm{Ext}^1(\Omega_{\mathcal{F}}, \mathcal{O}_X) \longrightarrow 0,$$

so that there is an exact sequence

$$0 \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow (\mathcal{N}^*_{\mathcal{F}})^* \longrightarrow \mathrm{Ext}^1(\Omega_{\mathcal{F}}, \mathcal{O}_X) \longrightarrow 0.$$

Definition 2.2. Let $V \subset X$ be an analytic subset. We say that V is tangent to $\mathcal{F}$ if $T_pV \subset (T_{\mathcal{F}})_p$, for all $p \in V \setminus \mathrm{Sing}(V)$.

3. Proof of the main results

We begin by proving the main theorem.

Proof of Theorem 1.1. Consider a map $f : X \longrightarrow Y$ and let (U, x) and (V, y) be local systems of coordinates for X and Y such that $f(U) \subset V$. Since $\mathcal{F}$ is a regular distribution, we may suppose that it is induced on U by the k-form $\omega_1 \wedge \cdots \wedge \omega_k$. Therefore, the ramification locus of f with respect to $\mathcal{F}$ on U is given by

$$S(f, \mathcal{F})|_U = \{f^*(\omega_1 \wedge \cdots \wedge \omega_k) = f^*(\omega_1) \wedge \cdots \wedge f^*(\omega_k) = 0\}.$$

In other words, the ramification locus $S(f, \mathcal{F})$ coincides with $\mathrm{Sing}(f^*(\mathcal{F}))$.

Let us denote $\widetilde{\mathcal{F}} := f^*(\mathcal{F})$. Let $\{U_\alpha\}$ be a covering of Y such that the distribution $\mathcal{F}$ is induced on U_α by the holomorphic 1-forms $\omega_1^\alpha, \ldots, \omega_k^\alpha$. Hence, on $U_\alpha \cap U_\beta \neq \emptyset$ we have $(\omega_1^\alpha \wedge \cdots \wedge \omega_k^\alpha) = g_{\alpha\beta}(\omega_1^\beta \wedge \cdots \wedge \omega_k^\beta)$, where $\{g_{\alpha\beta}\}$ is a cocycle generating the line bundle $\det(\mathcal{N}^*_{\mathcal{F}})^*$. Then the distribution $\widetilde{\mathcal{F}}$ is induced locally by $f^*(\omega_1^\alpha), \ldots, f^*(\omega_k^\alpha)$. This shows that $\mathcal{N}^*_{\widetilde{\mathcal{F}}}$ is locally free. Therefore the singular set of $\widetilde{\mathcal{F}}$ is the loci of degeneracy of the induced map

$$\mathcal{N}^*_{\widetilde{\mathcal{F}}} \longrightarrow \Omega^1_X.$$

By hypothesis, the ramification locus of f with respect to $\mathcal{F}$, which is given by $\mathrm{Sing}(\widetilde{\mathcal{F}})$, has codimension $n-k+1$. Then it follows from the Thom-Porteous formula [8] that

$$c_{n-k+1}(\Omega^1_X - \mathcal{N}^*_{\widetilde{\mathcal{F}}}) \cap [X] = \sum_j \mu_j [S_j],$$

where μ_j is the multiplicity of the irreducible component S_i. It follows from $c(\Omega^1_X - \mathcal{N}^*_{\widetilde{\mathcal{F}}}) = c(\Omega^1_X) \cdot s(\mathcal{N}^*_{\widetilde{\mathcal{F}}})$ that

$$c_{n-k+1}(\Omega^1_X - \mathcal{N}^*_{\widetilde{\mathcal{F}}}) = \sum_{i=0}^{n-k+1} c_{n-k+1-i}(\Omega^1_X) \cap s_i(\mathcal{N}^*_{\widetilde{\mathcal{F}}}),$$

where $s_i(\mathcal{N}^*_{\widetilde{\mathcal{F}}})$ is the i-th Segre classe of $\mathcal{N}^*_{\widetilde{\mathcal{F}}}$. Since $X_0 := X - \mathrm{Sing}(\widetilde{\mathcal{F}})$ is a dense and open subset of X, by taking the cap product we have

$$c_{n-k+1}(\Omega^1_X - \mathcal{N}^*_{\widetilde{\mathcal{F}}}) \cap [X] = c_{n-k+1}(\Omega^1_X - \mathcal{N}^*_{\widetilde{\mathcal{F}}}) \cap [X_0]$$
$$= \sum_{i=0}^{n-k+1} (c_{n-k+1-i}(\Omega^1_X)) \cap [X_0] \cap s_i(f^*\mathcal{N}^*_{\mathcal{F}}).$$

It follows from the projection formula that

$$f_*(c_{n-k+1}(\Omega^1_X - \mathcal{N}^*_{\widetilde{\mathcal{F}}}))\cap[X]) = \sum_{i=0}^{n-k+1} f_*(c_{n-k+1-i}(\Omega^1_X)\cap[X])\cap s_i(\mathcal{N}^*_{\mathcal{F}}) = \sum_j \mu_j f_*[S_j].$$

□

Now, we prove our tangency formulae as a consequence of the main theorem.

Proof of Corollary 1.3. Let $i : X \hookrightarrow Y$ be the inclusion map. It follows from Theorem 1.1 that

$$i_*(c_1(T_X) \cap [X]) - i_*([X]) \cap s_1(\mathcal{N}^*_{\mathcal{F}}) = -\sum_{R_j \subset R} \mu(f, \mathcal{F}, S_j)[R_j].$$

On the one hand, we have $c_1(T_Y|_X) = c_1(N_{X|Y}) + c_1(T_X)$, and on the other hand, we have $c_1(T_Y|_X) = c_1(T_{\mathcal{F}}|_X) + c_1(\mathcal{N}_{\mathcal{F}}|_X)$. Since $s_1(\mathcal{N}^*_{\mathcal{F}}) = -c_1(\mathcal{N}^*_{\mathcal{F}}) = c_1(\mathcal{N}_{\mathcal{F}})$, we obtain

$$[c_1(N_{X|Y}) - c_1(T_{\mathcal{F}})] \cap [X] = \sum_{R_j \subset R} \mu(f, \mathcal{F}, S_j)[R_j].$$

Now notice that, by construction, the cycle

$$Z = \sum_{R_j \subset R} \mu(f, \mathcal{F}, S_j)[R_j]$$

is an effective divisor on X, since $\mu(f, \mathcal{F}, S_j) \geq 0$. If the line bundle $\det(T_{\mathcal{F}})|_X - \det(N_{X|Y}) = -[\det(N_{X|Y}) - \det(T_{\mathcal{F}})|_X]$ is ample, we obtain

$$0 < -[\det(N_{X|Y}) - \det(T_{\mathcal{F}})|_X] \cdot C = -Z \cdot C,$$

for all irreducible curves $C \subset X$. If X is not invariant by $\mathcal{F}$ and $\det(T_{\mathcal{F}})|_X - \det(N_{X|Y})$ is ample, we obtain an absurdity. In fact, in this case $Z \cdot C < 0$, contradicting the fact that Z is effective. □

4. Examples

4.1. **Integrable example.** This example is inspired by an example due to Izawa [13].

Consider $Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the subvariety $X = F^{-1}(0) \cap g^{-1}(0)$ given by the homogenous equations

$$F(x, y, z) = \sum_{i=0}^{3} x_i^{\ell}, \qquad G(x, y, z) = \sum_{i=0}^{1} x_i y_i,$$

where $([x], [y], [z]) = ((x_0 : x_1 : x_2 : x_3), (y_0 : y_1), (z_0 : z_1)) \in Y$ are homogeneous coordinates. By a straightforward calculation one may verify that X is smooth. In Y we consider the foliation $\mathcal{F}$ given by the fibers of the map $\pi : \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and let $f : X \to Y$ be the inclusion map. We will analyze the branch points of the f with respect to $\mathcal{F}$.

A simple but exhaustive calculation shows that there is no branch point in the hypersurface $x_0 = 0$; thus we concentrate on the Zariski open set $x_0 \neq 0$.

The affine charts for $y_0 \neq 0$. In the affine charts for $x_0 \neq 0$ and $y_0 \neq 0$ the equations defining X assume the form

$$\begin{aligned} 1 + x^\ell + y^\ell + z^\ell &= 0, \\ 1 + ux &= 0, \end{aligned}$$

where $(1 : x : y : z) = (1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \frac{x_3}{x_0})$ and $(1 : v) = (1 : \frac{y_1}{x_0})$. This yields the parametrization of X given by

$$\begin{aligned} x &= (-1)^{\frac{1}{\ell}}(y^\ell + z^\ell + 1)^{\frac{1}{\ell}}, \\ v &= (-1)^{\frac{1+\ell}{\ell}}(y^\ell + z^\ell + 1)^{-\frac{1}{\ell}}. \end{aligned}$$

Now, recall that the leaves of $\mathcal{F}$ are given by $\{\text{const}\} \times \mathbb{C}$; hence the tangency points between X and $\mathcal{F}$ are the solutions to the equation $du = u_y dy + u_z dz = 0$. Thus the set of tangency points coincides with the solutions of the system of equations

$$\begin{aligned} 0 = \frac{\partial v}{\partial y} &= (-1)^{-\frac{1}{\ell}} y^{\ell-1}(y^\ell + z^\ell + 1)^{-\frac{\ell+1}{\ell}}, \\ 0 = \frac{\partial v}{\partial z} &= (-1)^{-\frac{1}{\ell}} z^{\ell-1}(y^\ell + z^\ell + 1)^{-\frac{\ell+1}{\ell}} \end{aligned}$$

or, in other words,

$$(2) \qquad \begin{cases} x = (-1)^{\frac{1}{\ell}}, \\ y^{\ell-1} = 0, \\ z^{\ell-1} = 0, \\ v = -(-1)^{-\frac{1}{\ell}}. \end{cases}$$

The solutions to this system of equations are given in terms of homogeneous coordinates by

$$S_k^{0,0} = \{(1 : \alpha_k : 0 : 0)\} \times \{(1 : -1/\alpha_k)\} \times \mathbb{P}^1,$$

where $\alpha_k = \exp(\frac{(2k+1)\pi i}{\ell})$, $k = 0, \ldots, \ell - 1$. Note that $S_k^{0,0}$ is a solution with multiplicity $(\ell - 1)^2$ and that these solutions are contained in the codimension 2 variety given by $x_2 = x_3 = 0$.

The affine chart for $y_1 \neq 0$. On the other hand, in the affine charts for $x_0 \neq 0$ and $y_1 \neq 0$ the equations defining X assume the form

$$\begin{aligned} 1 + x^\ell + y^\ell + z^\ell &= 0, \\ u + x &= 0, \end{aligned}$$

where $(1 : x : y : z) = (1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \frac{x_3}{x_0})$ and $(u : 1) = (\frac{y_0}{y1} : 1)$. This leads to the parametrization of X given by

$$\begin{aligned} x &= (-1)^{\frac{1}{\ell}}(y^\ell + z^\ell + 1)^{\frac{1}{\ell}}, \\ u &= (-1)^{\frac{1+\ell}{\ell}}(y^\ell + z^\ell + 1)^{\frac{1}{\ell}}. \end{aligned}$$

Since the leaves of $\mathcal{F}$ are given by $\{\text{const}\} \times \mathbb{C}$, the tangency points between X and $\mathcal{F}$ are the solutions to the equation $du = u_y dy + u_z dz = 0$. Therefore the set of tangency points coincides with the solution to the system of equations

$$0 = \frac{\partial u}{\partial y} = (-1)^{\frac{\ell+1}{\ell}} y^{\ell-1}(y^\ell + z^\ell + 1)^{\frac{1-\ell}{\ell}},$$
$$0 = \frac{\partial u}{\partial z} = (-1)^{\frac{\ell+1}{\ell}} z^{\ell-1}(y^\ell + z^\ell + 1)^{\frac{1-\ell}{\ell}}$$

or, in other words, with the solutions to the system of equations

$$\text{(3)} \qquad \begin{cases} x = (-1)^{\frac{1}{\ell}}, \\ y^{\ell-1} = 0, \\ z^{\ell-1} = 0, \\ u = -(-1)^{\frac{1}{\ell}}. \end{cases}$$

In homogeneous coordinates the solutions to this system of equations are given by

$$S_k^{0,1} = \{(1 : \alpha_k : 0 : 0)\} \times \{(-\alpha_k : 1)\} \times \mathbb{P}^1,$$

where $\alpha_k = \exp(\frac{(2k+1)\pi i}{\ell})$, $k = 0, \dots, \ell - 1$. Note that $S_k^{0,1}$ is a solution with multiplicity $(\ell-1)^2$ and that this solution is contained in the codimension 2 variety $x_2 = x_3 = 0$. Notice also that $S_k^{0,1} = S_k^{0,0}$ for all $k = 0, \dots, \ell - 1$.

The residual formula. Consider the projections $\pi_1 : Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $\pi_2 : Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, $\pi_3 : Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, and $\rho : Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. As usual, we denote a line bundle on Y by $\mathcal{O}(a, b, c) := \pi_1^*\mathcal{O}_{\mathbb{P}^3}(a) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(b) \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(c)$, with $a, b, c \in \mathbb{Z}$. Now denote $h_3 = c_1(\mathcal{O}(1, 0, 0))$, $h_{1,1} = c_1(\mathcal{O}(0, 1, 0))$, and $h_{1,2} = c_1(\mathcal{O}(0, 0, 1))$.

Summing up, the set of non-transversal points is given by the following cycle:

$$S = \sum_{k=0}^{\ell-1} (\ell - 1)^2 S_k^{0,0}.$$

Since $[S_k^{0,0}] = h_3^3 \cdot h_{1,1}$, we conclude that

$$\begin{aligned} [S] &= (\ell - 1)^2 \sum_{k=0}^{\ell-1} [S_k^{0,0}] \\ &= \ell(\ell - 1)^2 h_3^3 \cdot h_{1,1}. \end{aligned}$$

Recall that $n = 3$, $k = 2$, and $r = 2$; thus the left side of the formula stated in Theorem 1.1 assumes the form

$$\begin{aligned} &f_*(c_{n-k+1}(T_X) \cap [X]) + \sum_{i=1}^{r} (-1)^i f_*(c_{n-k+1-i}(T_X) \cap [X]) \cap s_i(\mathcal{N}_{\mathcal{F}}^*) \\ &= c_2(T_X) \cap [X] - c_1(T_X) \cap [X] \cap s_1(\mathcal{N}_{\mathcal{F}}^*) + c_0(T_X) \cap [X] \cap s_2(\mathcal{N}_{\mathcal{F}}^*) \\ &= \{ c_2(T_X) - c_1(T_X) \cap s_1(\mathcal{N}_{\mathcal{F}}^*) + s_2(\mathcal{N}_{\mathcal{F}}^*) \} \cap [X]. \end{aligned}$$

Since the associated line bundles of $V(x_0^\ell + x_1^\ell + x_2^\ell + x_3^\ell)$ and $V(x_0 y_0 + x_1 y_1)$ are $\mathcal{O}(\ell, 0, 0)$ and $\mathcal{O}(1, 1, 0)$, respectively, we have the short exact sequence

$$0 \longrightarrow T_X \longrightarrow T_Y|_X \longrightarrow \mathcal{O}(\ell, 0, 0) \oplus \mathcal{O}(1, 1, 0)|_X \longrightarrow 0.$$

Now let $h_3 = c_1(\mathcal{O}(1, 0, 0))$, $h_{1,1} = c_1(\mathcal{O}(0, 1, 0))$, and $h_{1,2} = c_1(\mathcal{O}(0, 0, 1))$. Then by the Euler sequence for multiprojective spaces [6], we conclude that

$$c(T_Y) = (1 + h_3)^4 (1 + h_{1,1})^2 (1 + h_{1,2})^2,$$

with relations $(h_3)^4 = (h_{1,1})^2 = (h_{1,2})^2 = 0$. Since $c(\mathcal{O}(\ell,0,0) \oplus \mathcal{O}(1,1,0)) = (1+\ell h_3)(1+h_3+h_{1,1})$ and

$$c(T_Y)|_X = c(T_X) \cdot c(\mathcal{O}(\ell,0,0) \oplus \mathcal{O}(1,1,0)|_X)$$

it follows that

$$\begin{aligned} c_1(T_X) &= (3-\ell)h_3 + h_{1,1} + 2h_{1,2}, \\ c_2(T_X) &= (4-\ell)h_3h_{1,1} + (6-2\ell)h_3h_{1,2} + (3-3\ell+\ell^2)h_3^2 + 2h_{1,1}h_{1,2}. \end{aligned}$$

We calculate the Segre classes $s_i(\mathcal{N}^*_{\mathcal{F}})$ for $i = 1,\dots,r$. Since in our example $r = 2$, it is enough to calculate $s_i(\mathcal{N}^*_{\mathcal{F}})$, $i = 1,2$. The foliation $\mathcal{F}$ is the restriction of $\rho : Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ to X. Then the normal bundle of $\mathcal{F}$ is

$$N_{\mathcal{F}} = \rho^*(T_{\mathbb{P}^1} \oplus T_{\mathbb{P}^1})|_X = (\mathcal{O}(0,2,0) \oplus \mathcal{O}(0,0,2))|_X.$$

Thus $N^*_{\mathcal{F}} = (\mathcal{O}(0,-2,0) \oplus \mathcal{O}(0,0,-2))|_X$. Since $(h_{1,1})^2 = (h_{1,2})^2 = 0$ we get

$$s_1(N^*_{\mathcal{F}}) = 2(h_{1,1}+h_{1,2}), \qquad s_2(N^*_{\mathcal{F}}) = 4h_{1,1}h_{1,2}.$$

Observe that

$$\begin{aligned} c_1(T_X) \cap s_1(\mathcal{N}^*_{\mathcal{F}}) &= ((3-\ell)h_3 + h_{1,1} + 2h_{1,2}) \cdot (2(h_{1,1}+h_{1,2})) \\ &= (6-2\ell)h_3h_{1,1} + (6-2\ell)h_3h_{1,2} + 6h_{1,1}h_{1,2}. \end{aligned}$$

Thus

$$\begin{aligned} & c_2(T_X) - c_1(T_X) \cap s_1(\mathcal{N}^*_{\mathcal{F}}) + s_2(\mathcal{N}^*_{\mathcal{F}}) \\ &= (4-\ell)h_3h_{1,1} + (6-2\ell)h_3h_{1,2} + (3-3\ell+\ell^2)h_3^2 + 2h_{1,1}h_{1,2} \\ &- ((6-2\ell)h_3h_{1,1} + (6-2\ell)h_3h_{1,2} + 6h_{1,1}h_{1,2}) + 4h_{1,1}h_{1,2} \\ &= (\ell-2)h_3h_{1,1} + (3-3\ell+\ell^2)h_3^2. \end{aligned}$$

Moreover, we have

$$[X] = [V(x_0^\ell + x_1^\ell + x_2^\ell + x_3^\ell)] \cap [V(x_0y_0 + x_1y_1)] = \ell h_3(h_3 + h_{1,1}) = \ell h_3^2 + \ell h_3 h_{1,1}.$$

Thus

$$\begin{aligned} & \{c_2(T_X) - c_1(T_X) \cap s_1(\mathcal{N}^*_{\mathcal{F}}) + s_2(\mathcal{N}^*_{\mathcal{F}})\} \cap [X] \\ &\quad = [(\ell-2)h_3h_{1,1} + (3-3\ell+\ell^2)h_3^2] \cdot [\ell h_3^2 + \ell h_3 h_{1,1}]. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \{c_2(T_X) - c_1(T_X) \cap s_1(\mathcal{N}^*_{\mathcal{F}}) + s_2(\mathcal{N}^*_{\mathcal{F}})\} \cap [X] &= \ell(\ell - 2 + 3 - 3\ell + \ell^2)]h_3^3h_{1,1} \\ &= \ell(\ell-1)^2 h_3^3 h_{1,1} = [S]. \end{aligned}$$

4.2. **Non-integrable example.** Let X be a complex-projective manifold of dimension $\dim(X) = 2n+1$. A contact structure on X is a regular distribution $\mathcal{F}$ induced by a twisted 1-form

$$\omega \in H^0(X, \Omega^1_X \otimes L),$$

such that $\omega \wedge (d\omega)^n \neq 0$ and L is a holomorphic line bundle. Suppose that the second Betti number of X is $b_2(X) = 1$ and that X is not isomorphic to the projective space $\mathbb{P}^{2n+1}$. Then it follows from [15] that there exists a compact irreducible component $H \subset \mathrm{RatCurves}^n(X)$ of the space of rational curves on X such that the intersection of L with the curves associated with H is 1. Moreover, if $C \subset X$ is a generic element of H, then C is smooth, tangent to $\mathcal{F}$, and

$$TX|_C = \mathcal{O}_C(2) \oplus \mathcal{O}_C(1)^{n-1} \oplus \mathcal{O}_C^{n+1},$$
$$T_{\mathcal{F}}|_C = \mathcal{O}_C(2) \oplus \mathcal{O}_C(1)^{n-1} \oplus \mathcal{O}_C^{n-1} \oplus \mathcal{O}_C(-1).$$

See [16, Facts 2.2 and 2.3]. In particular, we obtain that $N_{C|X} = \mathcal{O}_C(1)^{n-1} \oplus \mathcal{O}_C^{n+1}$, since $T_C = \mathcal{O}_C(2)$. Then

$$\det(T_{\mathcal{F}})|_C - \det(N_{C|X}) = \mathcal{O}_C(1)$$

is ample. Examples of such manifolds are given by homogeneous Fano contact manifolds; cf. [1]. This example satisfies the conditions of Corollary 1.3.

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References

[1] Arnaud Beauville, *Fano contact manifolds and nilpotent orbits*, Comment. Math. Helv. **73** (1998), no. 4, 566–583, DOI 10.1007/s000140050069. MR1639888

[2] Filippo Bracci and Bruno Scárdua, *Holomorphic vector fields transverse to polydiscs*, J. Lond. Math. Soc. (2) **75** (2007), no. 1, 99–115, DOI 10.1112/jlms/jdl005. MR2302732

[3] Jean-Paul Brasselet, *Une généralisation de la formule de Riemann-Hurwitz* (French), Journées de géométrie analytique (Univ. Poitiers, Poitiers, 1972), Bull. Soc. Math. France Suppl. Mém., No. 38, Soc. Math. France, Paris, 1974, pp. 99–106. MR0361176

[4] Jean-Paul Brasselet, *Sur une formule de M. H. Schwartz relative aux revêtements ramifiés* (French, with English summary), C. R. Acad. Sci. Paris Sér. A-B. **283** (1976), no. 2, Ai, A41–A44. MR0419818

[5] Marco Brunella, *Feuilletages holomorphes sur les surfaces complexes compactes* (French, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 5, 569–594, DOI 10.1016/S0012-9593(97)89932-6. MR1474805

[6] Maurício Corrêa Jr. and Márcio G. Soares, *A Poincaré type inequality for one-dimensional multiprojective foliations*, Bull. Braz. Math. Soc. (N.S.) **42** (2011), no. 3, 485–503, DOI 10.1007/s00574-011-0026-3. MR2833814

[7] El Hadji Cheikh Mbacké Diop, *Résidus d'applications holomorphes entre variétés* (French, with English summary), Hokkaido Math. J. **29** (2000), no. 1, 171–200, DOI 10.14492/hokmj/1350912963. MR1745509

[8] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984. MR732620

[9] Tomoaki Honda, *A localization lemma and its applications*, Singularities and complex analytic geometry (Japanese) (Kyoto, 1997), Sūrikaisekikenkyūsho Kōkyūroku **1033** (1998), 110–118. MR1660634

[10] Tomoaki Honda, *Tangential index of foliations with curves on surfaces*, Hokkaido Math. J. **33** (2004), no. 2, 255–273, DOI 10.14492/hokmj/1285766165. MR2072998

[11] Birger Iversen, *Critical points of an algebraic function*, Invent. Math. **12** (1971), 210–224, DOI 10.1007/BF01418781. MR0342512

[12] Toshikazu Ito and Bruno Scárdua, *On the classification of non-integrable complex distributions*, Indag. Math. (N.S.) **17** (2006), no. 3, 397–406, DOI 10.1016/S0019-3577(06)80040-6. MR2321108

[13] Takeshi Izawa, *Residues for non-transversality of a holomorphic map to a codimension one holomorphic foliation*, J. Math. Soc. Japan **59** (2007), no. 3, 899–910. MR2344833

[14] Takeshi Izawa and Tatsuo Suwa, *Multiplicity of functions of singular varieties*, Internat. J. Math. **14** (2003), no. 5, 541–558, DOI 10.1142/S0129167X03001910. MR1993796

[15] Stefan Kebekus, *Lines on contact manifolds*, J. Reine Angew. Math. **539** (2001), 167–177, DOI 10.1515/crll.2001.072. MR1863858

[16] Stefan Kebekus, *Lines on complex contact manifolds. II*, Compos. Math. **141** (2005), no. 1, 227–252, DOI 10.1112/S0010437X04000880. MR2099777

[17] M.-H. Schwartz, *Champs de repères tangents à une variété presque complexe* (French), Bull. Soc. Math. Belg. **19** (1967), 389–420. MR0243449

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