PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 12, December 2018, Pages 5295–5310
https://doi.org/10.1090/proc/14224
Article electronically published on September 17, 2018

KATO SQUARE ROOT PROBLEM WITH UNBOUNDED LEADING COEFFICIENTS

LUIS ESCAURIAZA AND STEVE HOFMANN

(Communicated by Svitlana Mayboroda)

Abstract. We prove the Kato conjecture for elliptic operators, $L = -\nabla \cdot ((\mathbf{A}+\mathbf{D})\nabla\ )$, with $\mathbf{A}$ a complex measurable bounded coercive matrix and $\mathbf{D}$ a measurable real-valued skew-symmetric matrix in $\mathbb{R}^n$ with entries in $BMO(\mathbb{R}^n)$; i.e., the domain of $\sqrt{L}$ is the Sobolev space $\dot{H}^1(\mathbb{R}^n)$ in any dimension, with the estimate $\|\sqrt{L}\, f\|_2 \lesssim \|\nabla f\|_2$.

1. Introduction

Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix of complex, L^∞ coefficients defined on $\mathbb{R}^n$ and satisfying the ellipticity or accretivity condition

$$\lambda|\xi|^2 \le \Re\langle \mathbf{A}\xi, \xi\rangle \equiv \Re \sum_{i,j} a_{ij}(x)\, \xi_j\, \overline{\xi}_i\,, \quad \|\mathbf{A}\|_\infty \le \lambda^{-1}, \tag{1.1}$$

for $\xi \in \mathbb{C}^n$ and for some $0 < \lambda \le 1$. We consider a divergence form operator

$$Lu \equiv -\nabla \cdot (\mathbf{A}(x)\nabla u)\,. \tag{1.2}$$

The accretivity condition (1.1) enables one to define a square root $\sqrt{L}$ [22], and a fundamental issue was to "solve the square root problem", i.e., to establish the estimate

$$\|\sqrt{L}\, f\|_{L^2(\mathbb{R}^n)} \le N\|\nabla f\|_{L^2(\mathbb{R}^n)}, \tag{1.3}$$

with N depending on n and λ. The latter estimate is connected with the question of the analyticity of the mapping $\mathbf{A} \to \sqrt{L}$, which in turn has applications to the perturbation theory for certain classes of hyperbolic equations [27]. We remark that (1.3) is equivalent to the opposite inequality for the square root of the adjoint operator L^*.

In [21, 22] Kato conjectured that an abstract version of (1.3) might hold for "regularly accretive operators". A counterexample to this abstract conjecture was obtained by McIntosh [26], who then reformulated the conjecture in the following form, bearing in mind that Kato's interest in the problem had been motivated by

Received by the editors December 28, 2017, and, in revised form, April 24, 2018.
2010 *Mathematics Subject Classification.* Primary 35B45, 35J15, 35J25, 42B20, 42B37, 47B44.
Key words and phrases. Kato's conjecture.
The first author was supported by grants MTM2014-53145-P and IT641-13 (GIC12/96).
The second author was supported by NSF grant no. DMS-1664047.
This material is based upon work supported by the National Science Foundation under Grant No. DMS- 1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2017 semester.

the special case of elliptic differential operators:

The estimate (1.3) *holds for* L *defined as in* (1.2)*, for any* L^∞*,* $n \times n$ *matrix* $\mathbf{A}$ *with complex entries for which* (1.1) *holds.*

To establish the validity of this conjecture became known as the Kato Problem or square root problem. In 1982 it was solved in one dimension [9], where it is essentially equivalent to the problem of proving the L^2 boundedness of the Cauchy integral operator on Lipschitz curves [23].

For $n > 1$, a restricted version of the conjecture, also essentially posed by Kato in [22], was proved by P. Auscher, S. Hofmann, J.L. Lewis, and P. Tchamitchian in [4]. The restricted version treated the case that $\mathbf{A}$ is close in the L^∞ norm to a real symmetric matrix of bounded measurable coefficients. It is this version that yields the perturbation results for hyperbolic equations alluded to above [27].

Prior to the latter result, the conjecture was proved in higher dimensions when $\|\mathbf{A} - \mathbf{I}\|_{L^\infty(\mathbb{R}^n)} \le \epsilon(n)$ [8, 13, 14, 20]. [12] gave a different proof using the $T(1)$ theorem. Sharper bounds for the constant $\epsilon(n)$ on the order of $n^{-\frac{1}{2}}$ were obtained in [20]. In [6] it was proved when $\|\mathbf{A} - \mathbf{I}\|_{BMO(\mathbb{R}^n)}$ is small.

Later, the validity of the conjecture was established when the heat kernel of the operator L satisfies the "Gaussian" property, first in 2 dimensions [17] and then in all dimensions [18]. That is, let $G(x, y, t)$ denote the kernel of the operator e^{-tL}. We say that L satisfies the Gaussian property if there are positive constants $0 < \alpha \le 1$ and N such that

$$\begin{aligned} &\text{(i)} \quad |G(x,y,t)| \le N t^{-\frac{n}{2}} e^{-|x-y|^2/Nt}, \\ &\text{(ii)} \quad |G(x+h,y,t) - G(x,y,t)| + |Gx, y+h, t) - G(x,y,t)| \\ &\qquad \le N \left(|h|/\sqrt{t}\right)^{\alpha} t^{-\frac{n}{2}} e^{-|x-y|^2/Nt}, \end{aligned}$$

where the latter holds when $t > 0$ and either $|h| \le t$ or $|h| \le |x - y|/2$.

The Gaussian property holds when $\mathbf{A}$ is real-valued by results of Aronson [2] and in some cases for complex $\mathbf{A}$: in two dimensions from [5] and for perturbations of real operators [3]. Hence, [17], [18] solve the conjecture in the former two-dimensional cases or the latter n-dimensional cases.

Finally, the conjecture was solved for general complex, bounded, and coercive matrices $\mathbf{A}$ satisfying (1.1) in [7].

The purpose of this note is to show that minor modifications of the reasoning in [7] also yield the following extension. Let

$$H^1(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \nabla f \in L^2(\mathbb{R}^n)\}$$

denote the usual Sobolev space, and let $\dot{H}^1(\mathbb{R}^n)$ denote its homogeneous version; i.e., $\dot{H}^1$ is the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the seminorm $\|f\|_{\dot{H}^1(\mathbb{R}^n)} := \|\nabla f\|_{L^2(\mathbb{R}^n)}$.

Theorem 1. *For any operator*

$$L = -\nabla \cdot ((\mathbf{A}(x) + \mathbf{D}(x))\nabla\ ) \tag{1.4}$$

with $\mathbf{A}$ *a bounded complex-valued coercive matrix satisfying* (1.1) *and* $\mathbf{D}$ *a real-valued skew-symmetric matrix in* $\mathbb{R}^n$ *with entries in* $BMO(\mathbb{R}^n)$ *satisfying* (1.5)*, the domain of* $\sqrt{L}$ *contains* $H^1(\mathbb{R}^n)$*, and* (1.3) *holds over* $\dot{H}^1(\mathbb{R}^n)$*, with* $N = N(\lambda, n)$*.*

We remark in passing that the Gaussian property has been shown to hold with $\alpha = \alpha(\lambda, n)$ and $N = N(\lambda, n)$ when

$$Lu \equiv -\nabla \cdot ((\mathbf{A}(x) + \mathbf{D}(x))\nabla u)\,,$$

with $\mathbf{A}$ a real-valued, bounded symmetric and coercive matrix satisfying (1.1) and $\mathbf{D} = (d_{ij}(x))$ a real-valued skew-symmetric $BMO(\mathbb{R}^n)$ matrix [28,31] with

$$\|\mathbf{D}\|_{BMO(\mathbb{R}^n)} \leq \lambda^{-1}. \tag{1.5}$$

The arguments of [18] could be modified to treat this restricted case. On the other hand, as in [7], we do not require the Gaussian property in the proof of Theorem 1 in the present paper.

We recall that a function $\beta : \mathbb{R}^n \longrightarrow \mathbb{C}$ is in $BMO(\mathbb{R}^n)$ or has bounded mean oscillation [19] when it is locally integrable and

$$\|\beta\|_{BMO(\mathbb{R}^n)} = \sup_{Q\subset\mathbb{R}^n} \fint_Q |\beta - m_Q(\beta)|\, dx < +\infty,$$

where Q ranges over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axis and

$$m_Q(\beta) = \fint_Q \beta\, dx.$$

If one defines other norms by

$$\|\beta\|_{BMO(\mathbb{R}^n)_p} = \sup_{Q\subset\mathbb{R}^n} \left(\fint_Q |\beta - m_Q(\beta)|^p\, dx\right)^{\frac{1}{p}},\ 1 < p < \infty,$$

the John-Nirenberg inequality [19] implies that all the $BMO(\mathbb{R}^n)_p$-norms are equivalent when $1 \leq p < \infty$. Finally, $BMO(\mathbb{R}^n)$ is the dual of $H^1_{at}(\mathbb{R}^n)$, the real Hardy space in $\mathbb{R}^n$, where f in $L^1(\mathbb{R}^n)$ is in $H^1_{at}(\mathbb{R}^n)$ when

$$\sup_{\epsilon>0} |\theta_\epsilon * f|$$

is in $L^1(\mathbb{R}^n)$, where $\theta_\epsilon = \epsilon^{-n}\theta(x/\epsilon)$ and θ is any smooth non-negative compactly supported mollifier with integral equal to 1 [15]. In particular, when β is in $BMO(\mathbb{R}^n)$ and f is in $H^1_{at}(\mathbb{R}^n)$, the principal value of the integral of βf is well defined [24] and

$$\left|\int_{\mathbb{R}^n} \beta f\, dx\right| \leq N(n)\|\beta\|_{BMO(\mathbb{R}^n)}\|f\|_{H^1_{at}(\mathbb{R}^n)},$$

with

$$\|f\|_{H^1_{at}(\mathbb{R}^n)} = \|\sup_{\epsilon>0} |\theta_\epsilon * f|\|_{L^1(\mathbb{R}^n)}.$$

Following [28], when

$$L = -\nabla \cdot (\mathbf{A}\nabla\ ) - \mathbf{b} \cdot \nabla, \tag{1.6}$$

with $\mathbf{A}$ a complex-valued bounded matrix verifying (1.1) and $\mathbf{b}$ a real-valued divergence-free vector field with

$$\sup_{x\in\mathbb{R}^n,\, r>0} r \fint_{B_r(x)} |\mathbf{b}|\, dx < \infty, \tag{1.7}$$

the matrix

$$\mathbf{D} = \Delta^{-1}(\nabla\mathbf{b} - \nabla\mathbf{b}^\top)$$

is a skew-symmetric and real-valued matrix with

$$\|\mathbf{D}\|_{BMO(\mathbb{R}^n)} \le N(n) \sup_{x\in\mathbb{R}^n,\, r>0} r \fint_{B_r(x)} |\mathbf{b}|\, dx,$$

and L can be written in the form (1.4). Thus, according to Theorem 1, the domain of the square root of the accretive operator (1.6) contains $H^1(\mathbb{R}^n)$ when $\mathbf{A}$ is as above, $\mathbf{b}$ is real-valued, and (1.7) holds.

The proof of Theorem 1 requires simple modifications to the original reasoning in [7] but most importantly the following two compensated compactness-type results.

Proposition 1. *When $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ are in $H^1(\mathbb{R}^n)$ and $i, j \in \{1, \dots, n\}$, the functions $\partial_i f \partial_j g - \partial_j f \partial_i g$ and $f\partial_i f$ are in $H^1_{at}(\mathbb{R}^n)$ and there is $N = N(n)$ such that*

$$\|\partial_i f \partial_j g - \partial_j f \partial_i g\|_{H^1_{at}(\mathbb{R}^n)} \le N\|\nabla f\|_{L^2(\mathbb{R}^n)}\|\nabla g\|_{L^2(\mathbb{R}^n)} \tag{1.8}$$

and

$$\|f\partial_i f\|_{H^1_{at}(\mathbb{R}^n)} \le N\|f\|_{L^2(\mathbb{R}^n)}\|\nabla f\|_{L^2(\mathbb{R}^n)}. \tag{1.9}$$

The reader can find the proofs of (1.8) and (1.9) in [10] and [31] respectively.

In the next section we explain the minor modifications one must make to the reasoning in the proof of Conjecture 1.4 in [7] to derive its extension in Theorem 1. Throughout the next pages N denotes a constant which depends at most on λ and n, B_r is an open ball in $\mathbb{R}^n$ of radius $r > 0$, Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axis, x_Q is its center, and $\delta(Q)$ is its side length.

2. Proof of Theorem 1

Setting

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g}(x)\, dx, \text{ for } f, g \in L^2(\mathbb{R}^n),$$

it follows from Hölder's inequality, (1.1), (1.5), the identity

$$\int_{\mathbb{R}^n} \mathbf{D}(x)\nabla u \cdot \nabla\overline{v}\, dx = \frac{1}{2}\int_{\mathbb{R}^n} d_{ij}(x)\left(\partial_i u\partial_j\overline{v} - \partial_j u\partial_i\overline{v}\right) dx, \tag{2.1}$$

(1.8), and the fact that

$$\Re\mathbf{D}(x)\xi\cdot\overline{\xi} = 0, \text{ when } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{C}^n,$$

that the sesquilinear form

$$\mathcal{B}(u, v) = \int_{\mathbb{R}^n} \left(\mathbf{A}(x) + \mathbf{D}(x)\right)\nabla u \cdot \nabla\overline{v}\, dx,$$

associated to the unbounded operator L in (1.4), with domain

$$\mathcal{D}(L) = \{u \in H^1(\mathbb{R}^n) : Lu \in L^2(\mathbb{R}^n)\},$$

and by the relation

$$\langle Lu, v\rangle = \mathcal{B}(u, v), \text{ when } u \in \mathcal{D}(L) \text{ and } v \in H^1(\mathbb{R}^n),$$

is bounded and coercive on $H^1(\mathbb{R}^n)$ with

$$|\mathcal{B}(u, v)| \le N\|\nabla u\|_{L^2(\mathbb{R}^n)}\|\nabla v\|_{L^2(\mathbb{R}^n)}$$

and

$$\Re\mathcal{B}(u, u) \ge \lambda\int_{\mathbb{R}^n} |\nabla u|^2\, dx, \text{ when } u \in H^1(\mathbb{R}^n),$$

when the matrices $\mathbf{A}$ and $\mathbf{D}$ satisfy the conditions in Theorem 1. L is an accretive unbounded operator,

$$\Re\langle Lu, u\rangle = \Re\mathcal{B}(u,u) \geq 0, \text{ when } u \in \mathcal{D}(L);$$

L is also m-accretive [21, p. 279], and the operators

$$(2.2)\quad (1+t^2L)^{-1}, \quad t\nabla(1+t^2L)^{-1}, \quad (1+t^2L)^{-1}t\,\nabla\cdot, \quad \text{and} \quad t^2\nabla(1+t^2L)^{-1}\nabla\cdot$$

are uniformly $L^2(\mathbb{R}^n)$-bounded with bounds depending on n and λ for all $t > 0$, where

$$u = \left(1+t^2L\right)^{-1} f \quad \text{and} \quad w = \left(1+t^2L\right)^{-1} t\nabla\cdot\mathbf{f}$$

are the unique Lax-Milgram weak solutions in $H^1(\mathbb{R}^n)$ satisfying respectively

$$\int_{\mathbb{R}^n} u\overline{v} + t^2\left(\mathbf{A}+\mathbf{D}\right)\nabla u\cdot\overline{\nabla v}\,dx = \int_{\mathbb{R}^n} f\,\overline{v}\,dx \tag{2.3}$$

and

$$\int_{\mathbb{R}^n} w\overline{v} + t^2\left(\mathbf{A}+\mathbf{D}\right)\nabla w\cdot\overline{\nabla v}\,dx = -t\int_{\mathbb{R}^n} \mathbf{f}\cdot\nabla\overline{v}\,dx, \tag{2.4}$$

for all v in $C_0^\infty(\mathbb{R}^n)$, when f and $\mathbf{f}$ are in $L^2(\mathbb{R}^n)$. Similar bounds hold when L is replaced above by the adjoint of L,

$$L^* = -\nabla\cdot\left(\left(\mathbf{A}^* - \mathbf{D}\right)\nabla\right),$$

where $\mathbf{A}^*$ denotes the transpose conjugate matrix of $\mathbf{A}$.

Following Kato [21, p. 281], L has a unique m-accretive square root $\sqrt{L}$ given by

$$\sqrt{L}f = \frac{1}{\pi}\int_0^{+\infty} \lambda^{-\frac{1}{2}}(\lambda+L)^{-1}Lf\,d\lambda, \quad \text{when } f\in\mathcal{D}(L).$$

The identities

$$(\lambda+L)^{-k-1} = -\frac{1}{k}\frac{d}{d\lambda}(\lambda+L)^{-k}, \quad \lambda > 0,$$

for $k = 1, 2$, integration by parts, and the change of variables $\lambda = 1/t^2$, show that

$$\sqrt{L}f = \frac{8}{\pi}\int_0^{+\infty}(1+t^2L)^{-3}t^3L^2f\,\frac{dt}{t}, \quad \text{when } f\in\mathcal{D}(L^2) = (1+L)^{-1}\mathcal{D}(L),$$

and as in [7], we use the latter resolution formula for $\sqrt{L}$ to prove Theorem 1.

As in [7], Theorem 1 follows once (1.3) is derived for f in a dense subspace of $H^1(\mathbb{R}^n)$, as $\mathcal{D}(L^2)$ (here, $L^2 = L\circ L$), because then (2.7), (2.12), and the closedness of $\sqrt{L}$ as an unbounded operator over $L^2(\mathbb{R}^n)$ show that $H^1(\mathbb{R}^n)$ is contained in the domain of $\sqrt{L}$ and (1.3) holds for f in $H^1(\mathbb{R}^n)$. Finally, $H^1(\mathbb{R}^n)$ is dense in $\dot{H}^1(\mathbb{R}^n)$ and $\sqrt{L}$ can be uniquely extended by density to $\dot{H}^1(\mathbb{R}^n)$.

We have

$$\begin{aligned}
|\langle\sqrt{L}f, g\rangle| &= \left|\int_0^{+\infty}\langle\left(1+t^2L\right)^{-1}tLf, \left(1+t^2L^*\right)^{-2}t^2L^*g\rangle\frac{dt}{t}\right| \\
&\leq \left(\int_0^{+\infty}\|\left(1+t^2L\right)^{-1}tLf\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t}\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_0^{+\infty}\|\left(1+t^2L^*\right)^{-2}t^2L^*g\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t}\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\left(\int_0^{+\infty} \| \left(1+t^2L^*\right)^{-2} t^2L^*g\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t}\right)^{\frac12} \le N\|g\|_{L^2(\mathbb{R}^n)}. \tag{2.5}$$

To verify the latter inequality, define

$$\mathcal{S}_t = \left(1+t^2L\right)^{-2} t^2L = \left(1+t^2L\right)^{-1} - \left(1+t^2L\right)^{-2}.$$

By duality

$$\int_0^{+\infty} \| \left(1+t^2L^*\right)^{-2} t^2L^*g\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} = \langle \int_0^{+\infty} \mathcal{S}_t\mathcal{S}_t^* g\frac{dt}{t}, g\rangle$$
$$\le \| \int_0^{+\infty} \mathcal{S}_t\mathcal{S}_t^* g\frac{dt}{t}\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)},$$

and because the operators $\mathcal{S}_t$ are uniformly bounded in $\mathcal{B}(L^2(\mathbb{R}^n))$ and

$$\|\mathcal{S}_t^*\mathcal{S}_s\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le N \min\{t/s, s/t\}, \text{ when } s,t>0, \tag{2.6}$$

Cotlar's Lemma for integrals [11, 29] and (2.6) imply that

$$\| \int_0^{+\infty} \mathcal{S}_t\mathcal{S}_t^* g\frac{dt}{t}\|_{L^2(\mathbb{R}^n)} \le N\|g\|_{L^2(\mathbb{R}^n)},$$

which gives (2.5). Thus,

$$\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \le N\left(\int_0^{+\infty} \| \left(1+t^2L\right)^{-1} tLf\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t}\right)^{\frac12}, \tag{2.7}$$

provided that (2.6) holds. Write then for $t, s > 0$,

$$\begin{aligned}
&\langle \mathcal{S}_t^*\mathcal{S}_s f, g\rangle \\
&= \langle \left(1+s^2L\right)^{-2} s^2Lf, \left(1+t^2L\right)^{-2} t^2Lg\rangle \\
&= \langle s^2L\left(1+s^2L\right)^{-2} f, \left(1+t^2L\right)^{-1} g - \left(1+t^2L\right)^{-2} g\rangle \\
&= -\frac{s}{t}\ \langle (\mathbf{A}+\mathbf{D})\, s\nabla\left(1+s^2L\right)^{-2} f, t\nabla\left(1+t^2L\right)^{-1} g - t\nabla\cdot\left(1+t^2L\right)^{-2} g\rangle \\
&= -\frac{t}{s}\ \langle s\nabla\left(1+s^2L\right)^{-1} f - s\nabla\cdot\left(1+s^2L\right)^{-2} f, (\mathbf{A}+\mathbf{D})\, t\nabla\left(1+t^2L\right)^{-2} g\rangle
\end{aligned}$$

and use Hölder's inequality, (2.1), (2.2), and (1.8) to derive (2.6) from the previous identities.

The next goal is to show that the operator

$$\mathcal{T}_t = \left(1+t^2L\right)^{-1} t^2L = I - \left(1+t^2L\right)^{-1} \tag{2.8}$$

has Gaffney bounds and a well defined action over $L^\infty(\mathbb{R}^n)$ and the space of Lipschitz functions over $\mathbb{R}^n$. To show it we prove first the following lemma.

Lemma 1. *There are $\theta = \theta(\lambda, n)$ and N such that the inequalities*

$$\|e^{x\cdot\xi/t}\left(1+t^2L\right)^{-1} f\|_{L^2(\mathbb{R}^n)} + \|e^{x\cdot\xi/t} t\nabla\left(1+t^2L\right)^{-1} f\|_{L^2(\mathbb{R}^n)} \le N\|e^{x\cdot\xi/t}f\|_{L^2(\mathbb{R}^n)},$$
$$\|e^{x\cdot\xi/t}\left(1+t^2L\right)^{-1} t\nabla\cdot\mathbf{f}\|_{L^2(\mathbb{R}^n)} + \|e^{x\cdot\xi/t} t^2\nabla\left(1+t^2L\right)^{-1} \nabla\cdot\mathbf{f}\|_{L^2(\mathbb{R}^n)}$$
$$\le N\|e^{x\cdot\xi/t}\mathbf{f}\|_{L^2(\mathbb{R}^n)}$$

hold when ξ is in $\mathbb{R}^n$ and $|\xi| \le \theta$.

Proof. We first prove the lemma when the domain of L is replaced by

$$\mathcal{D}(L) = \{f \in H_0^1(\Omega) : Lf \in L^2(\Omega)\},$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $L^2(\mathbb{R}^n)$ is replaced by $L^2(\Omega)$ in Lemma 1. In that case, when f is in $L^2(\Omega)$, $u = \left(1 + t^2 L\right)^{-1} f$ is the unique Lax-Milgram weak solution in $H_0^1(\Omega)$, which satisfies

$$\int_\Omega u\overline{v} + t^2 \left(\mathbf{A} + \mathbf{D}\right) \nabla u \cdot \overline{\nabla v}\, dx = \int_\Omega f\, \overline{v}\, dx, \tag{2.9}$$

for all v in $H_0^1(\Omega)$. Then take $v = e^{2x\cdot\xi/t} u$ in (2.9) to find that

$$\int_\Omega e^{2x\cdot\xi/t} \left[|u|^2 + t^2 \left(\mathbf{A} + \mathbf{D}\right) \nabla u \cdot \left(\nabla\overline{u} + 2(\xi/t)\overline{u}\right)\right]\, dx = \int_\Omega e^{2x\cdot\xi/t} f\overline{u}\, dx.$$

Taking real parts, together with the fact that $\Re\mathbf{D}\eta \cdot \overline{\eta} = 0$, for all $\eta \in \mathbb{C}^n$, we get

$$\begin{aligned}\|e^{x\cdot\xi/t}u\|_{L^2(\Omega)}^2 + \lambda\|e^{x\cdot\xi/t}t\nabla u\|_{L^2(\Omega)}^2 &\le \|e^{x\cdot\xi/t}f\|_{L^2(\Omega)}\|e^{x\cdot\xi/t}u\|_{L^2(\Omega)}\\ &+ \lambda^{-1}|\xi|\|e^{x\cdot\xi/t}t\nabla u\|_{L^2(\Omega)}\|e^{x\cdot\xi/t}u\|_{L^2(\Omega)} - t\int_\Omega \mathbf{D}\nabla\left(|e^{x\cdot\xi/t}u|^2\right)\cdot\xi\, dx.\end{aligned}$$

According to (1.9) and because $e^{x\cdot\xi/t}u$ is in $H_0^1(\Omega) \subset H^1(\mathbb{R}^n)$, the absolute value of the last integral above is bounded by a multiple of

$$\lambda^{-1}|\xi|\|e^{x\cdot\xi/t}u\|_{L^2(\Omega)}\|e^{x\cdot\xi/t}t\nabla u\|_{L^2(\Omega)} + |\xi|^2\|e^{x\cdot\xi/t}u\|_{L^2(\Omega)}^2,$$

and the inequality

$$\|e^{x\cdot\xi/t}u\|_{L^2(\Omega)} + \|e^{x\cdot\xi/t}t\nabla u\|_{L^2(\Omega)} \le N\|e^{x\cdot\xi/t}f\|_{L^2(\Omega)} \tag{2.10}$$

follows when $|\xi| \le \theta$, ξ is in $\mathbb{R}^n$, and θ is sufficiently small.

For f in $C_0^\infty(\mathbb{R}^n)$ with the support of f contained in B_R, let u_R denote the Lax-Milgram weak solution to (2.9) when $\Omega = B_R$. Because $\mathbf{D}\nabla\varphi$ is in $L^2(\mathbb{R}^n)$, when φ is in $C_0^\infty(\mathbb{R}^n)$ and the bounds that we have for u_R and ∇u_R are independent of $R \ge 1$, we can derive that u_R converges to u in $L^2_{loc}(\mathbb{R}^n)$ and ∇u_R converges weakly to ∇u in $L^2_{loc}(\mathbb{R}^n)$, where now $u = \left(1 + t^2 L\right)^{-1} f$ is as in Lemma 1. The first part of the lemma now follows from (2.10) and the local weak convergences of $e^{x\cdot\xi/t}u_R$ and $e^{x\cdot\xi/t}\nabla u_R$ to $e^{x\cdot\xi/t}u$ and $e^{x\cdot\xi/t}\nabla u$ when R tends to infinity.

The second part of the lemma follows after replacing (2.9) by (2.4) and taking $v = e^{2x\cdot\xi/t}w$. □

Lemma 2. *There is N such that the following inequalities hold for all cubes Q in $\mathbb{R}^n$ with side length $\delta(Q)$ and $t > 0$:*

$$\|\left(1 + t^2 L\right)^{-1} f\|_{L^2(Q)} + \|t\nabla\left(1 + t^2 L\right)^{-1} f\|_{L^2(Q)} \le Ne^{-2^k\delta(Q)/Nt}\|f\|_{L^2(\mathbb{R}^n)},$$

$$\|\left(1 + t^2 L\right)^{-1} t\nabla\cdot\mathbf{f}\|_{L^2(Q)} + \|t^2\nabla\left(1 + t^2 L\right)^{-1}\nabla\cdot\mathbf{f}\|_{L^2(Q)} \le Ne^{-2^k\delta(Q)/Nt}\|\mathbf{f}\|_{L^2(\mathbb{R}^n)},$$

when the supports of f and $\mathbf{f}$ are contained in $2^{k+1}Q \setminus 2^kQ$ and $k \ge 1$.

Proof. Without loss of generality we may assume that the cube Q is centered at the origin and $2^kQ = \{x \in \mathbb{R}^n : \|x\|_\infty \le 2^{k-1}\delta(Q)\}$, when $k \ge 1$. Assume then that f is supported in $2^{k+1}Q \setminus 2^kQ$ and write $\mathbb{R}^n \setminus \{0\}$ as the union of the sets, $\bigcup_{i=1}^n A_i \cup B_i$, where

$$A_i = \{x \in \mathbb{R}^n : \|x\|_\infty = x_i\} \quad \text{and} \quad B_i = \{x \in \mathbb{R}^n : \|x\|_\infty = -x_i\}, \quad i = 1, \dots, n.$$

Then,

$$\left(1+t^2L\right)^{-1}f=\sum_{i=1}^{n}\left(1+t^2L\right)^{-1}(f\chi_{A_i})+\left(1+t^2L\right)^{-1}(f\chi_{B_i}),$$

and we show that the first inequality in the lemma holds for each of these parts of $\left(1+t^2L\right)^{-1}f$. To get it for $\left(1+t^2L\right)^{-1}(f\chi_{A_1})$, apply Lemma 1 to $f\chi_{A_1}$ with $\xi=-\theta e_1$, $e_1=(1,0,\dots,0)$, and observe that $e^{-\theta x_1/t}\geq e^{-\theta\delta(Q)/2t}$ inside Q and $e^{-\theta x_1/t}\leq e^{-\theta 2^k\delta(Q)/2t}$ inside $A_1\cap 2^{k+1}Q\setminus 2^kQ$.

The other inequalities in Lemma 2 follow in the same way from Lemma 1. □

For f in $L^\infty(\mathbb{R}^n)$, define

$$\left(1+t^2L\right)^{-1}f=\lim_{R\to+\infty}\left(1+t^2L\right)^{-1}(f\chi_{B_R(x_0)}),$$

where x_0 is any point in $\mathbb{R}^n$ and the limit is taken in the $L^2_{loc}(\mathbb{R}^n)$-sense. The limit is well defined due to the Gaffney bounds in Lemma 2, for if x_1 is any other point in $\mathbb{R}^n$, the symmetric difference between $B_R(x_0)$ and $B_R(x_1)$ is contained in an annulus $B_{2R}\setminus B_{\frac{R}{2}}$ for R sufficiently large and

$$\begin{aligned}\|\left(1+t^2L\right)^{-1}(f\chi_{B_R(x_0)})-\left(1+t^2L\right)^{-1}(f\chi_{B_R(x_1)})\|_{L^2(B_{\frac{R}{4}})}&\\ \leq NR^{\frac{n}{2}}e^{-R/Nt}\|f\|_{L^\infty(\mathbb{R}^n)}.&\end{aligned}$$

Also, after writing

$$\chi_{B_{R_2}\setminus B_{R_1}}=\chi_{B_{R_2}\setminus B_{2^{l+1}R_1}}+\sum_{i=0}^{l}\chi_{B_{2^{i+1}R_1}\setminus B_{2^iR_1}}$$

when $2R<R_1<2^{l+1}R_1<R_2\leq 2^{l+2}R_1$, it follows from Lemma 2 that

$$\|\left(1+t^2L\right)^{-1}(f\chi_{B_{R_2}})-\left(1+t^2L\right)^{-1}(f\chi_{B_{R_1}})\|_{L^2(B_R)}\leq Nt^{\frac{n}{2}+1}R_1^{-1}\|f\|_{L^\infty(\mathbb{R}^n)},$$

which shows that $\left(1+t^2L\right)^{-1}(f\chi_{B_R})$ is a Cauchy sequence in $L^2_{loc}(\mathbb{R}^n)$ when f is in $L^\infty(\mathbb{R}^n)$. Also, the Gaffney control that we have in Lemma 2 over the operator

$$t\nabla\left(1+t^2L\right)^{-1}$$

shows with similar reasoning that for f in $L^\infty(\mathbb{R}^n)$, $u=\left(1+t^2L\right)^{-1}f$ is a weak $H^1_{loc}(\mathbb{R}^n)$ solution over $\mathbb{R}^n$ to $u+t^2Lu=f$.

In particular, $\left(1+t^2L\right)^{-1}1=1$ and $\nabla\left(1+t^2L\right)^{-1}1=0$ in the above sense, because if $\eta_R(x)=\eta(x/R)$, with η in $C_0^\infty(\mathbb{R}^n)$, $\eta=1$ in B_1, and $\eta=0$ outside B_2, then $u_R=\left(1+t^2L\right)^{-1}(\eta_R)$ satisfies

$$u_R-\eta_R+t^2L(u_R-\eta_R)=-t^2L\eta_R.$$

At the same time, the skew-symmetry of $\mathbf{D}$ implies that in the sense of distributions

$$-t^2L\eta_R=t^2\nabla\cdot\left(\left(\mathbf{A}+\mathbf{D}-m_{B_{2R}}(\mathbf{D})\right)\nabla\eta_R\right),\quad m_{B_{2R}}(\mathbf{D})=\fint_{B_{2R}}\mathbf{D}\,dx.$$

Then, the second inequality in Lemma 2 gives

$$\begin{aligned}&\|u_R-1\|_{L^2(B_{\frac{R}{2}})}+\|t\nabla u_R\|_{L^2(B_{\frac{R}{2}})}\\&\le Nte^{-R/Nt}\|\mathbf{A}\nabla\eta_R+(\mathbf{D}-m_{B_{2R}}(\mathbf{D}))\nabla\eta_R\|_{L^2(B_{2R})}\\&\le NtR^{\frac{n}{2}-1}e^{-R/Nt},\end{aligned}$$

which tends to zero as R tends to $+\infty$. The latter shows that the $L^2(\mathbb{R}^n)$-uniformly bounded operators $\mathcal{T}_t$ defined by (2.8) verify Gaffney bounds, map $L^\infty(\mathbb{R}^n)$ into $L^2_{loc}(\mathbb{R}^n)$, and $\mathcal{T}_t(1)=0$, for $t>0$.

For a Lipschitz function f in $\mathbb{R}^n$, define in a similar manner

$$\mathcal{T}_t(f)=\lim_{R\to+\infty}\mathcal{T}_t\left((f-f(x_0))\,\chi_{B_R(x_1)}\right),$$

where x_0 and x_1 are any points in $\mathbb{R}^n$. The limit is measured in the $L^2_{loc}(\mathbb{R}^n)$-sense, and the definition is again independent of the choices of x_0 and x_1. Clearly, for f Lipschitz, $\mathcal{T}_t(f)$ is a weak $H^1_{loc}(\mathbb{R}^n)$ solution over $\mathbb{R}^n$ to

$$\mathcal{T}_t(f)+t^2L\mathcal{T}_t(f)=t^2Lf.$$

This follows from the Gaffney bounds verified by the operators $\mathcal{T}_t$ and the following lemma.

Lemma 3. *Let f be a Lipschitz function in $\mathbb{R}^n$ and let Q be a cube in $\mathbb{R}^n$ with $0<t\le\delta(Q)$. Then,*

$$\|\mathcal{T}_t(f)\|_{L^2(Q)}\le Nt|Q|^{\frac{1}{2}}\|\nabla f\|_{L^\infty(\mathbb{R}^n)}$$

and

$$\|\nabla\mathcal{T}_t(f)\|_{L^2(Q)}\le N|Q|^{\frac{1}{2}}\|\nabla f\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. Let x_Q denote the center of the cube Q. Write

$$\begin{aligned}\mathcal{T}_t(f)&=\lim_{R\to+\infty}\mathcal{T}_t\left((f-f(x_Q))\,\chi_{B_R(x_Q)}\right)\\&=\mathcal{T}_t\big((f-f(x_Q))\,\eta_0\big)+\sum_{k=0}^{+\infty}\mathcal{T}_t\big((f-f(x_Q))\,(\eta_{k+1}-\eta_k)\big),\end{aligned}$$

where $\eta\in C_0^\infty(\mathbb{R}^n)$ is equal to 1 in $2Q-x_Q$, 0 outside $3Q-x_Q$, and $\eta_k(x)=\eta(x-x_Q/2^k)$, $k\ge0$. Then, the Gaffney bounds in Lemma 2 show that

(2.11)
$$\begin{aligned}\|\mathcal{T}_t((f-f(x_Q))\,(\eta_{k+1}-\eta_k))\|_{L^2(Q)}+\|t\nabla\mathcal{T}_t((f-f(x_Q))\,(\eta_{k+1}-\eta_k))\|_{L^2(Q)}\\\le N2^{-k}\|\nabla f\|_{L^\infty(\mathbb{R}^n)}t|Q|^{1/2},\end{aligned}$$

when $k\ge0$. Next, $u=\mathcal{T}_t\left((f-f(x_Q))\,\eta_0\right)$ is a weak $H^1(\mathbb{R}^n)$ solution to

$$u+t^2Lu=-t^2\nabla\cdot(\mathbf{A}+\mathbf{D})\,\nabla\left((f-f(x_Q))\,\eta_0\right),$$

and recalling that the distribution

$$\nabla\cdot\left(\mathbf{D}\nabla\left((f-f(x_Q))\,\eta_0\right)\right)$$

is the same as

$$\nabla\cdot\left((\mathbf{D}-m_Q(\mathbf{D}))\,\nabla\left((f-f(x_Q))\,\eta_0\right)\right),$$

we find that

$$u=-\left(1+t^2L\right)^{-1}t^2\nabla\cdot\left[(\mathbf{A}+(\mathbf{D}-m_Q(\mathbf{D}))\,\nabla\left((f-f(x_Q))\,\eta_0\right))\right].$$

Then, the uniform boundedness of the last two operators in (2.2) gives

$$\|u\|_{L^2(Q)} + \|t\nabla u\|_{L^2(Q)} \le Nt|Q|^{\frac12}\left(1+\|\mathbf{D}\|_{BMO}\right)\|\nabla\left(\left(f-f(x_Q)\right)\eta_0\right)\|_{L^\infty(4Q)}$$
$$\le Nt|Q|^{\frac12}\left(1+\|\mathbf{D}\|_{BMO}\right)\|\nabla f\|_{L^\infty(\mathbb{R}^n)},$$

and the lemma follows after adding up (2.11) and the last inequality. □

Next, we recall the following result in [1, Lemma 3.9].

Lemma 4. *Let $\{\mathcal{T}_t : t>0\}$ be a family of bounded operators on $L^2(\mathbb{R}^n)$ satisfying for some $N>0$:*

(1) $\sup_{t>0}\|\mathcal{T}_t\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le N$.

(2) *$\mathcal{T}_t$ verifies Gaffney bounds; i.e., when Q is a cube in $\mathbb{R}^n$ and $k\ge 1$,*

$$\|\mathcal{T}_t\left(f\chi_{2^{k+1}Q\setminus 2^kQ}\right)\|_{L^2(Q)} \le Ne^{-2^k\delta(Q)/Nt}\|f\chi_{2^{k+1}Q\setminus 2^kQ}\|_{L^2(\mathbb{R}^n)}.$$

(3) *$\mathcal{T}_t(1)\equiv 0$ in $L^2_{loc}(\mathbb{R}^n)$.*

Then,

$$\left(\int_{\mathbb{R}^{n+1}_+}|\frac1t\mathcal{T}_t(f)|^2\frac{dxdt}{t}\right)^{\frac12} \le N\left[1+\|\frac1t\mathcal{T}_t(\Phi)\|_C\right]\|\nabla f\|_{L^2(\mathbb{R}^n)},$$

for all f in $H^1(\mathbb{R}^n)$, where

$$\|\frac1t\mathcal{T}_t(\Phi)\|_C = \sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_{R_Q}|\frac1t\mathcal{T}_t(\Phi)|^2\frac{dxdt}{t}\right)^{\frac12},$$

Φ is the identity map of $\mathbb{R}^n$, and R_Q is the Carleson box $Q\times(0,\delta(Q))$.

Hence, as in [7], Lemma 4 implies that

$$\left(\int_0^{+\infty}\|\left(1+t^2L\right)^{-1}tLf\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t}\right)^{\frac12} \le N\|\nabla f\|_{L^2(\mathbb{R}^n)} \tag{2.12}$$

after one shows with $\mathcal{T}_t$ as in (2.8) that the measure

$$|\frac1t\mathcal{T}_t(\Phi)|^2\frac{dxdt}{t}$$

is a Carleson measure with

$$\|\frac1t\mathcal{T}_t(\Phi)\|_C \le N, \tag{2.13}$$

and (1.3) for f in $\mathcal{D}(L^2)$ follows from (2.7) and (2.12).

To obtain (2.13), it suffices to adapt the construction of [18] to verify a variant of the $T(b)$ theorem for square roots [6]: for a fixed cube Q in $\mathbb{R}^n$, $0<\epsilon<1$, and ξ a unit vector in $\mathbb{C}^n$, define the scalar-valued function

$$f^\epsilon_{Q,\xi} = \Phi_Q\cdot\xi - \mathcal{T}_t\left(\Phi_Q\cdot\xi\right), \tag{2.14}$$

with $\Phi_Q(x) = x - x_Q$ and $t=\epsilon\delta(Q)$. Then, if follows from Lemma 3 with Q replaced by $10Q$, $t=\epsilon\delta(Q)$, and $f=\Phi_Q\cdot\xi$ that

$$\left(⨍_{10Q}|f^\epsilon_{Q,\xi}-\Phi_Q\cdot\xi|^2\,dx\right)^{\frac12} \le N\epsilon\delta(Q), \tag{2.15}$$

$$\left(\fint_{10Q} |\nabla f^{\epsilon}_{Q,\xi} - \xi|^2 \, dx\right)^{\frac{1}{2}} \leq N. \tag{2.16}$$

Also $f^{\epsilon}_{Q,\xi}$ is a weak $H^1_{loc}(\mathbb{R}^n)$ solution to $f^{\epsilon}_{Q,\xi} + t^2 L f^{\epsilon}_{Q,\xi} = \Phi_Q \cdot \xi$ over $\mathbb{R}^n$, with $t = \epsilon\delta(Q)$ and

$$\left(\fint_{10Q} |L f^{\epsilon}_{Q,\xi}|^2 \, dx\right)^{\frac{1}{2}} \leq N / \left(\epsilon\delta(Q)\right). \tag{2.17}$$

The reasoning in [7, Lemma 5.4] shows that given functions $f^{\epsilon}_{Q,\xi}$ in $H^1_{loc}(\mathbb{R}^n)$ verifying (2.15) and (2.16) for some $N > 0$, there are $0 < \epsilon \leq 1$, $\epsilon = \epsilon(N, n)$, and a finite set W of unit vectors in $\mathbb{C}^n$, whose cardinality depends only on ϵ and n, such that the inequality

$$\|\Psi\|_C \leq N \sum_{\xi \in W} \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_{R_Q} |\Psi \cdot S^Q_t(\nabla f^{\epsilon}_{Q,\xi})|^2 \frac{dxdt}{t}\right)^{\frac{1}{2}} \tag{2.18}$$

holds for all measurable functions $\Psi : \mathbb{R}^{n+1} \longrightarrow \mathbb{C}^n$ in $L^2_{loc}(\mathbb{R}^{n+1}_+)$, where for each Q cube in $\mathbb{R}^n$, S^Q_t denotes the dyadic averaging operator associated to the dyadic mesh generated by Q; i.e.,

$$S^Q_t(h)(x) = \fint_{Q'} h(y) \, dy,$$

for x in the dyadic cube Q' with $\frac{1}{2}\delta(Q') < t \leq \delta(Q')$. In fact, the proof of (2.18) in [7, Lemma 5.4] uses the compactness of the unit sphere in $\mathbb{C}^n$, properties of the distance function in $\mathbb{C}^n$, Hölder's inequality, the boundedness of the Hardy-Littlewood maximal function in $L^2(\mathbb{R}^n)$, a suitable stopping time argument independent of Ψ, and the interpolation inequality in [7, Lemma 5.15]. Thus, its proof is independent of the choice of Ψ, and (2.18) holds with $\Psi = \frac{1}{t}\mathcal{T}_t(\Phi)$ when L is as in Theorem 1 and for the choice of functions $f^{\epsilon}_{Q,\xi}$ defined in (2.14).

Then, (2.13) follows from (2.18) with $\Psi = \frac{1}{t}\mathcal{T}_t(\Phi)$ and Lemma 5 below, which adapts [7, Lemma 5.5] to the more general hypothesis on the coefficients matrix of L in Theorem 1.

Lemma 5. *Let $\epsilon = \epsilon(N, n)$ be the choice of ϵ in* (2.18)*. Then, there is $N > 0$ such that*

$$\left(\frac{1}{|Q|} \int_{R_Q} |\frac{1}{t}\mathcal{T}_t(\Phi) \cdot S^Q_t(\nabla f^{\epsilon}_{Q,\xi})|^2 \frac{dxdt}{t}\right)^{\frac{1}{2}} \leq N,$$

for all cubes Q in $\mathbb{R}^n$ and ξ a unit vector in $\mathbb{C}^n$.

Proof. Fix Q, ξ in $\mathbb{C}^n$ with $|\xi| = 1$ and make $\epsilon = \epsilon(N, n)$. Let χ be in $C^\infty_0(4Q)$ with $\chi = 1$ in $2Q$, $\chi = 0$ outside $3Q$, and

$$\|\chi\|_\infty + \delta(Q)\|\nabla X\|_\infty \leq N.$$

To simplify the notation set $f = f^{\epsilon}_{Q,\xi}$ and $S_t = S^Q_t$. Then,

$$\|\frac{1}{t}\mathcal{T}_t(\Phi) \cdot S_t(\nabla f)\|_{L^2(R_Q, dxdt/t)} = \|\frac{1}{t}\mathcal{T}_t(\Phi) \cdot S_t(\nabla\,(\chi f))\|_{L^2(R_Q, dxdt/t)}$$

because $\nabla(\chi f) = \nabla f$ over $2Q$ and $S_t(\nabla f)$ only reads information about ∇f inside Q to calculate its values at points (x, t) in R_Q. Next, let P_t denote the convolution with an even smooth mollifier, $\theta_t(x) = t^{-n}\theta(x/t)$, θ with integral 1 and supported in B_1. We have

$$\begin{aligned}
&\|\frac{1}{t}\mathcal{T}_t(\Phi)\cdot S_t(\nabla(\chi f))\|_{L^2(R_Q,dxdt/t)}\\
&\leq \|\frac{1}{t}\mathcal{T}_t(\Phi)\cdot(S_t-P_t^2)\,(\nabla(\chi f))\,\|_{L^2(\mathbb{R}^{n+1}_+,dxdt/t)}\\
&+\|\frac{1}{t}\left(\mathcal{T}_t(\Phi)\cdot\nabla P_t^2-\mathcal{T}_t\right)(\chi f)\,\|_{L^2(\mathbb{R}^{n+1}_+,dxdt/t)}+\|\frac{1}{t}\mathcal{T}_t(\chi f)\|_{L^2(R_Q,dxdt/t)}\\
&= I+II+III.
\end{aligned}$$

Then, I and II in the right hand side above are handled exactly as their analogues in [7, Lemma 5.5]. In particular, the only information about f that one needs to bound I and II by $N\sqrt{|Q|}$ is that (2.15) and (2.16) imply the bound

$$\left(\fint_{5Q}|\nabla(\chi f)|^2\,dx\right)^{\frac{1}{2}}\leq N, \tag{2.19}$$

and it suffices to apply the same harmonic analysis techniques, which allow us to handle the operators $\frac{1}{t}\mathcal{T}_t(\Phi)\cdot(S_t^Q-P_t^2)$ and $\frac{1}{t}\left(\mathcal{T}_t(\Phi)\cdot\nabla P_t^2-\mathcal{T}_t\right)$ in [7, Lemma 5.5]. In particular, for I use that S_t is a projection operator; i.e., $S_t^2=S_t$,

$$\begin{aligned}
&\|\frac{1}{t}\mathcal{T}_t(\Phi)\cdot(S_t-P_t^2)\,(\nabla(\chi f))\,\|_{L^2(\mathbb{R}^{n+1}_+,dxdt/t)}\\
&\qquad\qquad=\|\frac{1}{t}\mathcal{T}_t(\Phi)\cdot(S_t+P_t)(S_t-P_t)\,(\nabla(\chi f))\,\|_{L^2(\mathbb{R}^{n+1}_+,dxdt/t)}
\end{aligned}$$

and that $\frac{1}{t}\mathcal{T}_t(\Phi)\cdot(S_t+P_t)$ is a bounded operator on $L^2(\mathbb{R}^n)$ because the pointwise bounds of the kernel of S_t+P_t and duality show that

$$\|\frac{1}{t}\mathcal{T}_t(\Phi)(S_t+P_t)\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\leq N\|P_{Nt}\left(|\frac{1}{t}\mathcal{T}_t(\Phi)|^2P_{Nt}\right)\|^{\frac{1}{2}}_{\mathcal{B}(L^2(\mathbb{R}^n))}.$$

The first inequality in Lemma 3 implies that

$$\left(\fint_{B_{2Nt}(x)}|\frac{1}{t}\mathcal{T}_t(\Phi)|^2\,dx\right)^{\frac{1}{2}}\leq N,$$

which shows that the kernel of $P_{Nt}\left(|\frac{1}{t}\mathcal{T}_t(\Phi)|^2P_{Nt}\right)$ is bounded by $Nt^{-n}\chi_{|x-y|\leq 4Nt}$. Finally, the proof of the inequality

$$\left(\int_{\mathbb{R}^{n+1}_+}|(S_t-P_t)(h)|^2\frac{dxdt}{t}\right)\leq N\|h\|_{L^2(\mathbb{R}^n)},\quad \text{for } h\in L^2(\mathbb{R}^n),$$

is explained in [20] or [6, pp. 168 and 172-173]. The bound for II follows from (2.19), and Lemma 4 applied to the family of operators $\mathcal{T}_t(\Phi)\cdot\nabla P_t^2-\mathcal{T}_t$, which are uniformly bounded in $L^2(\mathbb{R}^n)$, verify Gaffney bounds and map 1 and Φ to zero.

In order to bound III, the presence of the $BMO(\mathbb{R}^n)$ matrix $\mathbf{D}$ obliges us to use some additional information about the gradient of $f=f^\epsilon_{Q.\xi}$, in particular, local higher integrability; i.e. there is $p=p(\lambda,n)>0$ independent of $\delta(Q)$ and ξ such that

$$\left(\fint_{5Q}|\nabla f^\epsilon_{Q,\xi}|^p\,dx\right)^{\frac{1}{p}}\leq N. \tag{2.20}$$

Once the latter is known, the skew-symmetry of $\mathbf{D}$ implies that as a distribution

$$L(\chi f) = \chi L f - \nabla \cdot (f(\mathbf{A}+\mathbf{D}-m_{4Q}(\mathbf{D}))\nabla\chi) - (\mathbf{A}+\mathbf{D}-m_{4Q}(\mathbf{D}))\nabla f\cdot\nabla\chi$$

and

(2.21)

$$\begin{aligned}&\frac{1}{t}\mathcal{T}_t(\chi f) = \left(1+t^2L\right)^{-1} tL(\chi f)\\ &= t\left(1+t^2L\right)^{-1}(\chi L f)\\ &- \left(1+t^2L\right)^{-1} t\left[\nabla\cdot(f(\mathbf{A}+\mathbf{D}-m_{4Q}(\mathbf{D}))\nabla\chi) + (\mathbf{A}+\mathbf{D}-m_{4Q}(\mathbf{D}))\nabla f\cdot\nabla\chi\right].\end{aligned}$$

Then, (2.2) and (2.17) give

$$\|t\left(1+t^2L\right)^{-1}(\chi L f)\|_{L^2(Q)} \le Nt\delta(Q)^{-1}\sqrt{|Q|}, \tag{2.22}$$

while the Gaffney bounds in Lemma 2, (1.1), (1.5), (2.20), (2.15), (2.16), Hölder's inequality, and the Poincaré-Sobolev inequality over $4Q$ imply that for $0<t\le\delta(Q)$,

(2.23)

$$\begin{aligned}&N^{-1}\|\left(1+t^2L\right)^{-1} t\nabla\cdot(f(\mathbf{A}+\mathbf{D}-m_{4Q}(\mathbf{D}))\nabla\chi)\|_{L^2(Q)}\\ &+N^{-1}\|\left(1+t^2L\right)^{-1} t\left[(\mathbf{A}+\mathbf{D}-m_{4Q}(\mathbf{D}))\nabla f\cdot\nabla\chi\right]\|_{L^2(Q)}\\ &\le e^{-\delta(Q)/Nt}\delta(Q)^{-1}\left[\|(\mathbf{A}+\mathbf{D}-m_{4Q}(\mathbf{D}))(f-m_{4Q}(f))\|_{L^2(4Q)} + |m_{4Q}(f)|\sqrt{|Q|}\right]\\ &+e^{-\delta(Q)/Nt}t\delta(Q)^{-1}\|(\mathbf{A}+\mathbf{D}-m_{4Q}(\mathbf{D}))\nabla f\|_{L^2(4Q)} \le Ne^{-\delta(Q)/2Nt}\sqrt{|Q|}.\end{aligned}$$

Finally, (2.21), (2.22), and (2.23) show that III is also bounded by $N\sqrt{|Q|}$, which proves Lemma 5.

It only remains to show that (2.20) holds, but this follows from (2.15) and (2.16) and standard higher integrability methods [16, 28, 30] because $f = f^{\epsilon}_{Q,\xi}$ is a weak $H^1_{loc}(\mathbb{R}^n)$ solution to $f + t^2Lf = \Phi_Q\cdot\xi$ over $\mathbb{R}^n$, with $t = \epsilon\delta(Q)$. We include the details for the reader's convenience.

When B_{2r} is any ball, multiply the equation

$$-t^2\nabla\cdot((\mathbf{A}+\mathbf{D})\nabla f) = \Phi_Q\cdot\xi - f$$

by $\left(\overline{f} - m_{B_{2r}}(\overline{f})\right)\eta^2$, with $\eta = 1$ over B_r, η in $C_0^\infty(B_{2r})$. This yields

$$\begin{aligned}&\int t^2\mathbf{A}\nabla f\cdot\nabla\overline{f}\eta^2 + 2t^2(\mathbf{A}+\mathbf{D})\nabla f\cdot\nabla\eta\left(\overline{f}-m_{B_{2r}}(\overline{f})\right)\eta\,dx\\ &\qquad\qquad = \int(\Phi_Q\cdot\xi - f)\left(\overline{f}-m_{B_{2r}}(\overline{f})\right)\eta^2\,dx.\end{aligned}$$

Taking real parts, dividing by t^2, and using the cancellations provided by the skew-symmetry of the matrix $m_{B_{2r}}(\mathbf{D})$, one gets

$$\begin{aligned}&\int|\nabla f|^2\eta^2\,dx \le Nr^{-2}\int_{B_{2r}}\left(1+|\mathbf{D}-m_{B_{2r}}(\mathbf{D})|^2\right)|f-m_{B_{2r}}(f)|^2\,dx\\ &\qquad\qquad + \left(\int_{B_{2r}}|(\Phi_Q\cdot\xi - f)t^{-2}|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}}.\end{aligned}$$

Next, by Hölder's inequality and a Sobolev-Poincaré inequality

$$\int_{B_{2r}} |\mathbf{D} - m_{B_{2r}}(\mathbf{D})|^2 |f - m_{B_{2r}}(f)|^2\,dx$$
$$\leq \left(\int_{B_{2r}} |\mathbf{D} - m_{B_{2r}}(\mathbf{D})|^{2n}\,dx\right)^{\frac{1}{n}} \left(\int_{B_{2r}} |f - m_{B_{2r}}(f)|^{\frac{2n}{n-1}}\,dx\right)^{\frac{n-1}{n}}$$
$$\leq Nr \left(\int_{B_{2r}} |\nabla f|^{\frac{2n}{n+1}}\,dx\right)^{\frac{n+1}{n}}.$$

Hence, recalling that $t = \epsilon\delta(Q)$, one gets

$$\left(\fint_{B_r} |\nabla f|^2\,dx\right)^{\frac{1}{2}} \leq N \left(\fint_{B_{2r}} |\nabla f|^{\frac{2n}{n+1}}\,dx\right)^{\frac{n+1}{2n}} + N \left(\fint_{B_{2r}} |\left(\Phi_Q \cdot \xi - f\right) \delta(Q)^{-1}|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{2n}}, \tag{2.24}$$

when B_{2r} is any ball contained in $10Q$. From [30] and (2.24), there is some $p = p(\lambda, n) > 2$ such that

$$\left(\fint_{5Q} |\nabla f|^p\,dx\right)^{\frac{1}{p}} \leq N \left(\fint_{10Q} |\nabla f|^2\,dx\right)^{\frac{1}{2}} + N\delta(Q)^{-1} \left(\fint_{10Q} |\Phi_Q \cdot \xi - f|^p\,dx\right)^{\frac{1}{p}}. \tag{2.25}$$

Finally, we may assume that $2 < p < \frac{2n}{n-2}$, and the interpolation of (2.15) and (2.16) shows that the second term in the right hand of (2.25) is bounded by N, while (2.16) implies that the same holds with the first term. □

References

[1] M. Angeles Alfonseca, Pascal Auscher, Andreas Axelsson, Steve Hofmann, and Seick Kim, *Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients*, Adv. Math. **226** (2011), no. 5, 4533–4606, DOI 10.1016/j.aim.2010.12.014. MR2770458

[2] D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc. **73** (1967), 890–896, DOI 10.1090/S0002-9904-1967-11830-5. MR0217444

[3] Pascal Auscher, *Regularity theorems and heat kernel for elliptic operators*, J. London Math. Soc. (2) **54** (1996), no. 2, 284–296, DOI 10.1112/jlms/54.2.284. MR1405056

[4] Pascal Auscher, Steve Hofmann, John L. Lewis, and Philippe Tchamitchian, *Extrapolation of Carleson measures and the analyticity of Kato's square-root operators*, Acta Math. **187** (2001), no. 2, 161–190, DOI 10.1007/BF02392615. MR1879847

[5] Pascal Auscher, Alan McIntosh, and Philippe Tchamitchian, *Heat kernels of second order complex elliptic operators and applications*, J. Funct. Anal. **152** (1998), no. 1, 22–73, DOI 10.1006/jfan.1997.3156. MR1600066

[6] Pascal Auscher and Philippe Tchamitchian, *Square root problem for divergence operators and related topics* (English, with English and French summaries), Astérisque **249** (1998), viii+172. MR1651262

[7] Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh, and Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$*, Ann. of Math. (2) **156** (2002), no. 2, 633–654, DOI 10.2307/3597201. MR1933726

[8] R. R. Coifman, D. G. Deng, and Y. Meyer, *Domaine de la racine carrée de certains opérateurs différentiels accrétifs* (French, with English summary), Ann. Inst. Fourier (Grenoble) **33** (1983), no. 2, x, 123–134. MR699490

[9] R. R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes* (French), Ann. of Math. (2) **116** (1982), no. 2, 361–387, DOI 10.2307/2007065. MR672839

[10] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces* (English, with English and French summaries), J. Math. Pures Appl. (9) **72** (1993), no. 3, 247–286. MR1225511

[11] M. Cotlar, *A combinatorial inequality and its applications to L^2-spaces*, Rev. Mat. Cuyana **1** (1955), 41–55 (1956). MR0080263

[12] Guy David and Jean-Lin Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. (2) **120** (1984), no. 2, 371–397, DOI 10.2307/2006946. MR763911

[13] Eugene B. Fabes, David S. Jerison, and Carlos E. Kenig, *Multilinear square functions and partial differential equations*, Amer. J. Math. **107** (1985), no. 6, 1325–1368 (1986), DOI 10.2307/2374409. MR815765

[14] Eugene B. Fabes, David S. Jerison, and Carlos E. Kenig, *Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure*, Ann. of Math. (2) **119** (1984), no. 1, 121–141, DOI 10.2307/2006966. MR736563

[15] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193, DOI 10.1007/BF02392215. MR0447953

[16] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277, DOI 10.1007/BF02392268. MR0402038

[17] Steve Hofmann and Alan McIntosh, *The solution of the Kato problem in two dimensions*, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), Publ. Mat. **Vol. Extra** (2002), 143–160, DOI 10.5565/PUBLMAT_Esco02_06. MR1964818

[18] Steve Hofmann, Michael Lacey, and Alan McIntosh, *The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds*, Ann. of Math. (2) **156** (2002), no. 2, 623–631, DOI 10.2307/3597200. MR1933725

[19] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426, DOI 10.1002/cpa.3160140317. MR0131498

[20] Jean-Lin Journé, *Remarks on Kato's square-root problem*, Conference on Mathematical Analysis (El Escorial, 1989), Publ. Mat. **35** (1991), no. 1, 299–321, DOI 10.5565/PUBLMAT_35191_16. MR1103623

[21] Tosio Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag New York, Inc., New York, 1966. MR0203473

[22] Tosio Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan **13** (1961), 246–274, DOI 10.2969/jmsj/01330246. MR0138005

[23] C. Kenig and Y. Meyer, *Kato's square roots of accretive operators and Cauchy kernels on Lipschitz curves are the same*, Recent progress in Fourier analysis (El Escorial, 1983), North-Holland Math. Stud., vol. 111, North-Holland, Amsterdam, 1985, pp. 123–143, DOI 10.1016/S0304-0208(08)70282-2. MR848144

[24] Robert H. Latter, *A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms*, Studia Math. **62** (1978), no. 1, 93–101, DOI 10.4064/sm-62-1-93-101. MR0482111

[25] J.-L. Lions, *Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs* (French), J. Math. Soc. Japan **14** (1962), 233–241, DOI 10.2969/jmsj/01420233. MR0152878

[26] Alan McIntosh, *On the comparability of $A^{1/2}$ and $A^{*1/2}$*, Proc. Amer. Math. Soc. **32** (1972), 430–434, DOI 10.2307/2037834. MR0290169

[27] Alan McIntosh, *Square roots of operators and applications to hyperbolic PDEs*, Miniconference on operator theory and partial differential equations (Canberra, 1983), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 5, Austral. Nat. Univ., Canberra, 1984, pp. 124–136. MR757577

[28] Gregory Seregin, Luis Silvestre, Vladimír Šverák, and Andrej Zlatoš, *On divergence-free drifts*, J. Differential Equations **252** (2012), no. 1, 505–540, DOI 10.1016/j.jde.2011.08.039. MR2852216

[29] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, with the assistance of Timothy S. Murphy, Princeton Mathematical Series, vol. 43,

Monographs in Harmonic Analysis, III, Princeton University Press, Princeton, NJ, 1993. MR1232192

[30] E. W. Stredulinsky, *Higher integrability from reverse Hölder inequalities*, Indiana Univ. Math. J. **29** (1980), no. 3, 407–413, DOI 10.1512/iumj.1980.29.29029. MR570689

[31] Q. Zhongmin and X. Guangyu, *Parabolic equations with singular divergence-free drift vector fields*, arXiv:1612.07727.

UPV/EHU, Departamento de Matemáticas, Barrio Sarriena s/n, 48940 Leioa, Spain
Email address: luis.escauriaza@ehu.eus

Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65211
Email address: hofmann@math.missouri.edu