An Introduction to Teaching Mathematics at the College Level

Suzanne Kelton
Assumption College

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Introduction

This guide is intended to assist a mathematician who has little or no teaching experience at the college level, but who will be teaching courses as a graduate teaching assistant or as a newly-hired professor.

The author is a newly-tenured associate professor, and therefore recently been through the process of establishing and evaluating her philosophies and course structures. The lessons learned in these early years of teaching are highlighted for your consideration. The aim is to generate ideas and provide examples of core goals for your classroom and this guide is intended to be useful prior to your first class, as well as for courses you teach a few years from now.

Chapters 1 and 2 relate to general teaching ideas and may be used for any course at hand. The focus of these chapters is to encourage behavior that will lead to a positive and successful classroom environment and to stimulate thought on the structure you will want for your class. This is not intended to be a handbook for how to run your course, but rather a springboard for your own ideas and ambitions.

Chapter 3 covers a variety of courses. Hopefully, if you have no teaching experience at all, you will have been assigned to teach a lower-level course. The sections for these courses discuss general instructional approaches important for this level as well as specific comments on material. You may find it helpful to read all of the lower-level courses, rather than just the course of interest, since there is overlap of pedagogical issues for all of these courses. The sections discussing the higher-level material presume that some teaching experience is likely if you have been asked to teach one of these courses.

These sections focus more on the specifics of the material and offer less in the way of general teaching suggestions.

Chapter 4 proposes ways to continually evaluate and improve your teaching while simultaneously preparing you to advance in your career.

Finally, the appendix contains some handouts for a few calculus topics. The handout on continuity is intended for use as students are just learning the topic, while the handouts for graphing skills and integration techniques are intended as reviews and summaries after students have been working with the material for some time.

Chapter 1: The Basics of the Classroom

Encouragement & Rapport

When teaching lower-level material, be aware that you can expect a wide variety of ability levels. In these courses, you will most likely encounter a number of students who struggle with math. If you are teaching mathematics at the college level, then repeated failure in math is something you have not likely experienced. If you have done any mathematical research, you may understand some level of frustration, but you are fortunate to have had a background that has given you confidence and a reasonable *hope* that you will eventually find the solution you seek. If some of your students have faced repeated struggles and failure in math, they may have been stripped of that very valuable sense of hope. Give your students the respect and sensitivity that they deserve. Diffuse the frustrations some students may feel towards the course by separating their performances from their dedication to do well. There are several avenues by which you may achieve this.

Primarily, ***be approachable***. Inherent in being able to effectively encourage your students is that they feel a connection with you. To be of use to your students, they must feel that they can come to you with questions or seek your support when they feel discouraged. Being friendly and supportive of questions in class will encourage your students to seek help both in the class and in office hours. A bright smile for the wary or timid student at your office door may help dispel the notion that he or she is bothering you. Some students miss the point that office-hour time is specifically designated for them, so try to make that clear. In your efforts to be friendly, remain professional, of course. Students should get the message that you care, but will still require them to work hard, to learn, and to perform well on graded papers.

Let your students see a distinction between you and the course you teach, so that they can view you as someone who is helping them through the material. For instance, avoid acting defensively if a student complains that a skill you are teaching is useless or that something was confusing in class. In the first case, it may help to briefly explain how a topic may be used in higher mathematics. Explain that for students going farther in math, this skill will be necessary. Even students not pursuing math generally respect your need to properly instruct those who are. In the latter case, ask where the student became confused in the lecture. Ask questions that help the two of you pinpoint where the connection was lost. You can gain useful insight to improve your instruction on a topic from listening to students in office hours explaining where they got lost in your lecture.

Reward hard work. When you know students are working hard by attending office hours and study sessions, tell them that. It helps students to know that even if their efforts are not resulting in the grade they desire in your course, at least you are aware and appreciative of the hard work they are putting into the subject. Often, one of the struggling student's fears is that you will misinterpret a poor performance as an indication of apathy and laziness.

Distinguish mathematical achievement from intelligence. Assuring your student that you are confident that he or she is a talented and dedicated pupil goes a long way in establishing rapport with an individual who may have come to loathe the very subject you are teaching. Sometimes students will assume that since this material is quite natural to you, you will think that it should come easily to them. It usually provides much relief when students realize that you think that they are neither stupid nor lazy, and that you respect their intelligence and appreciate their feelings. Be cautious during lectures to avoid describing problems as “easy” or “obvious.” Momentary confusion on a student’s part after such a description could lead to ill will.

Relate an experience to them. When a student has come to office hours to discuss frustration over poor course performance, it may help to explain that you understand such dissatisfaction by explaining how you struggled in some endeavor, whether it was academic or not. Challenges found in other subjects, sports, musical pursuits, and artistic endeavors may provide the common ground you seek.

Essentially, you should be respectful and considerate. This will improve your class’ ability to learn from you and soothe the tensions for the students who struggle.

Pace

There are two paces that you will need to set for yourself. One is the pace at which the course moves and the other is the pace of your individual lectures. Naturally, these are strongly intertwined. Do not let your desired pace for the course rush a lecture that needs special attention, and in return, do not belabor points that do not require it, as you will unnecessarily slow your course. Attempt to maintain a consistent course pace with varying lecture paces, as needed.

You may find pace to be a challenging part of any new course you teach. If you have your course schedule provided to you by your department, you may have a very structured guideline to follow. This is helpful when teaching a new course, but also restrictive when you are faced with a class that needs more or less review than your average group of students. If the course outline is left entirely up to you, the latitude to stop every time your students' comprehension and performance are below your expectations may hinder your ability to cover sufficient material. It is a good idea to plan out your entire course schedule before the semester begins, but remain open to adjusting it constantly, especially the first time through a course.

An important point to realize in setting the pace of your course is that you will almost undoubtedly move faster than some students would like. While it is important to assess whether the bulk of your students have the general gist of what you are doing in both an individual lecture and the course as a whole, you must also be sure to recognize that most mastery comes with practice and not a belabored presentation on your part. When important topics arise and relatively little lecture time is needed, alert your class to the fact that these skills are essential and will continue throughout this semester (and possibly beyond). Urge them to weed out any confusion immediately. An example would be a differentiation rule, such as the product or quotient rule. The rules take little time to put on the board and one lecture is generally ample time to review a variety of examples of low to moderate difficulty. It is usually necessary to allow sufficient practice time outside of class as well as ample time to review questions in the next class before moving on to the more complicated examples, especially those requiring multiple rules.

Break up your presentations with problems for the class to try on their own. It will help moderate the pace of your lecture as well as give the students a chance to find where they might have questions before they leave the class and begin the homework. In most cases, you will likely not have time for students to work more than a few problems each class, depending on the ability level of your class and the available time for lecture. Whatever practice time you can give them will be beneficial and the students will appreciate. It is easy for students to have the misconception that the problems are easy when they witness you solving them with no difficulty. Giving the students a chance to see where they might stumble will help them formulate questions and motivate them to give the homework proper attention. It may be best to only give the students one or two problems at a time to solve on their own. That is generally sufficient for the students to ascertain whether they are having significant difficulty. After you have demonstrated the skill, give the class a basic problem or two to try on their own. Then move on to demonstrating harder examples and allow the students to try one or two of those. Because time is often tight in the college classroom, keep a watchful eye on productivity and focus if you choose to have students break off into groups.

Allow your students time to absorb the material. It is essential to consider the amount of time outside of class that your students will have to work on the material. If you are faced with a topic that does not require excessive lecture time, but *does* need more than the usual amount of attention outside of class, attempt to place this lecture at a time when you feel your students will have sufficient time to work on the material and seek help. You may want to consider when students may have access to tutors at your institution and when they will have the most access to you through office hours. An alternative is to break the material into two shorter lectures, providing a longer period for the students to absorb it. To utilize the rest of the available class time, you could use the opportunity to allow the class to do group work or in-class worksheets on the material. If neither above option is viable, stress the fact that this material will need extra attention outside of class. Sample topics for which students require more practice than average are implicit differentiation and the formal definition of continuity. In the case of the latter, I would recommend an additional handout, since many texts may not offer sufficient practice in the problem set. (Note: If possible, do not place such material prior to a vacation break, as this will not only mean that many students may spend *less* than the usual amount of time studying prior to the next class, but also that your attendance may be down for the lecture. Even if your students diligently attend the lecture and work on the assignment, there will be a delay in the time before they are able to ask you questions.)

You may find that moving slower results in a similar comprehension level by your students as when you moved at a brisker pace, but your students learned fewer topics. Obviously, the goal is to cover the most appropriate material while preserving the maximum comprehension and maintaining a reasonable workload for your students. If you err too far on either side of the proper pace for your course, your class' mastery of the material will suffer. If you move extremely slowly, undoubtedly in an attempt to make sure that *everyone* understands *everything*, in reality your class may learn very little. It is impossible to achieve across-the-board learning, unless your class is equally skilled. This is not a common reality, especially in lower-level courses. Moving too slowly will result in everyone learning only a few tools. The students will likely be bored and unprepared for the next course. If you move too quickly, then many students may learn the material on a surface level, but most have not had precious practice time to achieve any level of mastery. They, too, will be unprepared to move to the next course, and will likely have developed a frustration with the subject. You may find that you adjust your pace a bit each time you teach a course until ultimately finding the right pace for your typical audience. Since each group of students is unique, keep a watchful eye on each class to ensure that your typical pace is suited to that group.

A final note on lecture pace: watch your students. Where are the bulk of the students looking? If they are still a board behind you, then let them catch up before you continue. Does anyone seem to be listening to what you are saying or is everyone madly writing? You probably will not be able to please everyone. You may have one or two students who appear to be very slow note-takers and are always behind. Some students have difficulty adjusting to the college classroom in which there may not be time to listen to everything without simultaneously writing notes down. Also be aware that some students do not write complete notes (some will not take any), so do not assume that if a few students appear to be done writing then you have allotted sufficient time for the class to take down your notes.

Chapter 2: Course Policies, Philosophies & Syllabi

Course Policies

Before you write out your course syllabus, give some serious thought to the policies you want to implement. A few common issues you will likely have to deal with follow.

Grades: How will the grades be broken down? Decide the number of exams, frequency of quizzes and homework assignments, etc. and how much each will be worth. If attendance or participation counts towards the grade, be as specific as possible as to how you will assess them and how they will factor into the grade. Will you allow extra credit? You may also want to indicate how an honor violation will affect the grade. Check the policies of your institution to make sure your syllabus is in agreement.

You may want to give yourself a little latitude in the number of quizzes and/or homework assignments. If you want the quizzes to count for 20% of the course grade, you may find it easier to make the quiz *average* worth 20% of the grade rather than planning 10 quizzes, each 2% of the final grade. This way, you can avoid headaches if cancelled classes or other delays interfere with your plan to have 10 quizzes.

Dates: You may want at least approximate dates for the exams, but give yourself latitude where possible, indicating that the dates are tentative and you will announce any changes in class. If you plan to give quizzes each Wednesday, you may want to say "*typically* we will have a quiz each Wednesday." This gives your students a sense of the routine, but allows you room to alter the schedule occasionally, if necessary.

Make-ups: Decide what your policy will be for students who miss exams/quizzes or fail to turn assignments in on time. Will the grade be a zero? Will you allow a make-up? Will you drop some number of the lowest quiz or homework grades instead of allowing missed papers to be made up? Do your best to provide an across-the-board policy to avoid having to handle everything on a case-by-case basis. You may also want to be aware of any policies at your institution.

Calculator policy: Will you allow or require graphing calculators? Be specific as to the features the students will need on their calculators, such as powers, log functions, trigonometric functions, exponentials, etc. Otherwise, your students may bring calculators that only do arithmetic and do not preserve order of operation. If you allow graphing calculators on graded papers, you may want to indicate the level of work that must be provided, since these calculators perform many of the operations (such as differentiation and graphing) on which you may be testing the students. Again, you may want to discuss calculator policy with your colleagues. If they do not allow graphing calculators, make sure to test that your students can work independently of these instruments as well. Otherwise, you may be sending your class off unprepared if their future instructors do not allow graphing calculators.

Participation & Attendance

In most cases, a student's participation and attendance affect their final course grade whether or not it is actually a formal part of the grade. You should decide whether you want either of these to formally factor into your grade calculation and, if so, how you will assess each. This should be clearly indicated in the syllabus to avoid any misunderstandings.

In the case of participation, you may want to respect the shyness a student may feel towards speaking out in class. One way to account for this is to allow attendance in office hours (when they can ask questions in private) to count toward participating in class. Also a student who does not know the answer to a question on the spot, nor is able to immediately formulate a question, is not necessarily removed from the course. This can be especially true for students with learning disabilities, who may require taping a lecture or having another student take notes for review at a later time.

With regard to attendance, students who frequently skip classes usually do not do particularly well, especially if the lectures and homework assignments relate strongly to the test material and format. In addition, students in lower-level courses often have difficulty learning from the text or classmates' notes. Students in higher-level courses generally have an interest in the subject and are far less likely to miss class regularly. In a rare instance, you may have a student that is gifted enough, or simply already knows the material well enough, to earn the desired grade without good attendance.

It should be said that some first-year students (freshmen) lack the discipline to attend class if there is no immediate consequence for an absence. These students often realize too late that they will not succeed in class without proper attendance. If attendance does not formally enter your course grade, you may want to keep a watchful eye on first-year students, especially in the fall semester. Giving frequent quizzes, announced or unannounced, is one way to curb excessive absences without formally incorporating attendance in your course grade.

Taking attendance every day, even if you do not plan to use it in any formal way in your grades, sends the message that you value attendance. It can also be valuable to know who has missed a quiz, handout, or an important announcement – or who did *not* miss one of these. If you are going to make attendance a formal part of your grade, you should be specific about the number of absences that will affect the grade and how.

You may also want to note students who are tardy and factor this into participation and attendance. The distraction of a student entering class late can be a detriment to your concentration as well as that of the students. You may want to encourage students to alert you to problems that may cause frequent tardiness, such as a prior class reasonably far away from yours or one that frequently dismisses late.

Extra Credit

You may find that extra credit in a college math class is difficult to make meaningful. Anything beyond the scope of the course will distract the student from the work that needs to be done to master the skills in your course. This is not only a waste of time, but actually detrimental to future performance. In all likelihood, the students who will be most successful in such endeavors are not in need of extra credit.

You could offer extra credit for completing extra math problems as is sometimes done in high school, but this essentially amounts to awarding extra credit for studying and practicing problems. This could be considered as part of an effort or participation grade.

Another option is to allow students to earn points back on exams by doing corrections. You may want to reserve this practice for the rare occasion when the bulk of a class does poorly on an exam. In this instance, the students need to spend time redoing the material so that they can proceed with the course successfully. In addition, this allowance helps build morale back up and lets the students see that your goal is their learning.

If you do not plan to offer any extra credit in your course, alert your students (especially first-year students) to this policy, since many of them are used to having extra credit as an option in high school. If you will offer extra credit, be sure to clearly outline how students may earn this extra credit and its maximum effect it could have on a student's final grade.

Syllabi

Your institution likely has a set of guidelines and requirements for your syllabi. If so, this is a great place to get started. You can start writing up your syllabi by simply handling all of the mandated material first. After that, go through each portion and decide if there is anything that you want to add or elaborate on. If you have not been given any guidelines, provided below are some basics you may want to cover.

At the top of the page:

- Course & meeting time (and place, if desired)
- Contact information (name, office location, office phone number, e-mail address)
- Office hours

Then the general course information:

- Course topics (general list of the material to be covered)
- Textbook
- Required materials (such as a calculator)

Finally, the specifics of the course:

- Grades: How grades will be broken down and how letter grades are determined from the course percentage (such as A = 90-100, B = 80-89…)
- Policies on missed quizzes/tests or late homework assignments.

- Calculator policy and specific description of the type of calculator allowed and functions required.
- Procedure for cancelled classes – Indicate if you will have a web page or voice mail that you want students to check for instructions should class be cancelled due to illness or weather.
- Accommodations for students with documented disabilities. Your institution likely provides accommodations such as providing class notes to such students or allowing extra time on tests. You may want to indicate how students should pursue these accommodations and how you will handle such requests. You may want to require that the student must personally discuss arrangements for accommodations in advance to avoid any misunderstandings of how needs will be addressed. Students will sometimes assume incorrectly that you have already been notified of their needs or do not understand that they have to submit official paperwork.

In general, as you write your syllabus, give yourself flexibility where possible, but also be very clear and direct, so that there is little room for misunderstandings and arguments.

Course Grades & "Kindness"

While you may want to reward a diligent student in your course with a little latitude in his or her grade, you may also be doing your hard-working student a disservice if he or she is taking another math course beyond yours. This is especially true if the student has a borderline failing grade. If a student really needs the material in your course to succeed in the next course, you may be setting him or her up for failure in the next course as well by "kindly" awarding a passing grade.

It can be difficult to be objective when you have worked all semester helping a diligent, but mathematically-weak student. You have hopefully built a rapport and camaraderie by the end of the semester and you truly want this student to succeed. But, be honest if the student falls too short of the cut-off point.

Along this line, if you have worked extensively with such a student, be sure to be clear what your standard will be for a passing grade. The student could misinterpret your encouragement and acknowledgement of hard work as an assurance that you will not award an F.

Chapter 3: Courses

Discussion Sections

If you are a graduate teaching assistant, you may have been assigned to teach a discussion section (or "fourth hour") of a course in which you answer homework questions. This is a nice way to ease into teaching if you have not taught before, since you will practice presenting math problems, but may not be responsible for presenting any lectures. These sections present their own set of challenges.

You may have been asked to teach the discussion section for a course you have never taken yourself. It is certainly most likely the case that you are a competent enough mathematician to handle the task of running the discussion section, but it can be a little daunting. One way to prepare and to alleviate any nervousness you may feel is to complete all of the assigned homework. Before your class meeting, follow up with the professor teaching the course with any questions you may have on the assignment or material in general. The more prepared you can be for the class, the more relaxed and competent your presentation will be. If you are truly intimidated by the course assigned to you, then attend the lecture if possible.

Another challenge you may encounter when teaching a discussion section occurs when the students are unhappy with the instruction from the professor for the course. If they feel exceptionally lost in the class, they may look to you for all the answers. On one hand, this is essentially your assignment. On the other hand, if you find as you attempt to go over the homework that the class as a whole is truly lost, then you may find yourself in the middle of an impromptu lecture. Again, having a high degree of comfort with the material covered in the assignment is extremely important. You may want to have a list of relevant formulas and definitions on hand for such an occasion. If you find the class often requires a significant amount of help on the assignments, you may want to be prepared to address common areas of confusion in the material.

Even if the class adores the professor for the course and has learned the material well, you may still face some difficulties in reviewing the assignment. Any variation on your part from the notation or method used by the professor may be met with dismay. Students in the first year or two of undergraduate study often fail to recognize when methods are equivalent and minor differences may confuse them. Check the text to see if the students' book uses the notation and method you would naturally use. If it differs, you may want to check with the professor teaching the course to see what was presented in class. If students question your notation or method, attempt to switch to the method they have seen in class, if it can be done smoothly and quickly. Your goal is to be cohesive with the professor and to help the students learn the material as originally presented, if possible.

You can learn a lot from teaching discussion sections. Perhaps, the most important lesson is to be prepared. Regardless of the environment in the class, the students' skill level, or your comfort with the material, do not underestimate the value of your preparedness to handle the homework as well as the material in general.

Algebra – Is it too late?!

By “algebra,” I mean the skills *hopefully* learned in high school algebra courses. Unfortunately, the skills possessed in this area by some high school graduates can be appalling to those teaching mathematics at the college level. The question, which begs to be asked, is *“Is it too late to learn algebra skills in college?”*

It is the rare college student who is actually *incapable* of learning this material with the right amount of dedication, time, and materials, paired with the right teacher for that student’s learning style. However, students possessing *extremely* sub-standard algebra skills upon reaching college may feel a future in math is unlikely. Therefore, a responsible prioritizing of their workloads may require that they limit the amount of time spent attempting to master skills they struggled with at length in high school. This is especially true when the material in question is not a section of the current course, but presumed knowledge.

The reality is that it takes much more work to deprogram incorrect notions about algebra than to learn it correctly in the first place. It can require extensive *drill work* to accomplish this deprogramming – something you generally do not have time for in the college semester. So, finally we reach the question. *“What do I do as an instructor faced with teaching students who lack basic algebra skills?”*

It depends on the focus of the course at hand. If you are actually teaching a remedial college math course, which is *intended* to patch up these holes in the students’ mathematical backgrounds, then you must go back to basics and go slowly. Remember, it takes time to uproot all of the weeds of misconception and replant the knowledge correctly. You may need to take what you might feel is a high school approach –

worksheets, group work, and plenty of repetition of the processes at hand. This type of course can be just as rewarding as those you might find more mathematically interesting; the challenge is pedagogical, not mathematical. The teacher in you will be working overtime, using a few notes the mathematician left behind. While this may not be your dream assignment, you can experience something powerful that you may not get with your more advanced students. If you can take a concept that has haunted a student and turn the light on for them, you will experience an addictive exhilaration. When your student gets a piece of self-esteem back, it is priceless for both of you.

The more common experience you will likely have is teaching a course like pre-calculus or calculus, in which the algebra background is supposed to already be in place. While you should be able to assume that most students in your class can do basic processes, like FOIL'ing or factoring, you should also assume that some cannot do this well or with ease. In pre-calculus, you may need to go slowly through the first couple of examples that involve prerequisite algebra skills of this nature. Before you begin a problem like this, you might say that it has been awhile since anyone has done this (such as over the summer), so you will just do a couple to get everyone back in the groove and from there on out, you will do those prerequisite skills more quickly. You will need to balance between appeasing students who need a lot of background detail and those who have rock solid prerequisite skills and understandably want you to focus on new material. The first group may be lost and frustrated if you presume too much and the latter group may get bored and less attentive if you presume nothing.

Always encourage students to ask questions if they do not understand something. Confusion is not usually isolated - if one student is confused, often so is another. Not only will you be clarifying your lesson, but also you will be keeping the door open for communication. Handling questions graciously will allow your students to see that you really want to help them learn and you are not “out to get them.” If you are faced with the rare student who lacks the appropriate background for your course, is truly hindering the progress of the course, and you feel you *must* request that student cut back on interruptions, do so outside of class. Do not take the chance that another student will misinterpret your desire to handle *extra* questions after class or in office hours as a general statement that you do not want any interruptions.

In a calculus course, you can take the same approach outlined above for pre-calculus or cut back on your explanations. The former will allow for a wider audience and the latter may weed out students who belong in a lower-level course. Regardless of your approach, *always remember to respond kindly to questions* – it takes courage to ask questions, especially on material that is presumed knowledge, and your class may shut down on you if they think you will constantly lecture them on what they should already know how to do. To avoid having to make a comment of this nature, try to make a statement before you begin the algebra work such as “We need to factor this. You may be a bit rusty, but this should be a familiar skill with which you were once very comfortable. If that is not the case, you might want to talk to me about your background and make sure you are in the right course.” Preparing the class ahead of time for a “quick” move on your part may make it less stressful for them. After working through the algebra step briskly, reiterate to the class that it is okay to be a bit rusty, but encourage students to check in with you during office hours if they find the algebra to be a struggle during the course of completing the homework. Office hours is the place to advise students who have revealed a low ability level on a course that would be better suited for them. If you are encouraging a student to drop to a lower course, it may help to emphasize that you simply want him or her to have the tools that will make your course much easier. Point out how much more enjoyable it will be to *succeed* in two math classes as opposed to *failing and repeating* one course because they lack the necessary background.

Depending on the level of the institution where you are working, you may even encounter math majors with frightening algebra notions, but generally they should only have a few isolated misconceptions that they have been unable to correct. The upside is that these students are usually aware that they are expected to know these concepts and generally do not fault you for omitting some steps. These students are generally confident enough to ask questions, though, and they should be handled courteously.

Pre-calculus

Pre-calculus not only introduces students to functions and limits, but may also be a catch-all course for whatever algebra and trigonometry we think our students need in a calculus course. You should assume that some students placed in this course are still struggling with algebra skills. You should also assume that a few calculus-competent students are hiding out in your class for an easy ‘A’. For this reason, this course can be one of the hardest to teach and to pace appropriately.

Many students will have seen a majority of the topics in this class previously and not fared well with at least some of them. I recommend you draw parallels whenever possible back to material that they have learned in the past. For example:

- Simplifying rational expressions: Start with an example of reducing a fraction by factoring the numerator and denominator and canceling common factors. Then when you do this with polynomials instead of numbers, it is not *quite* as foreign. Make sure to leave your example with numbers on the board, pointing out parallels, while you simplify the quotient of polynomials.

- Polynomial division: First, do one example of long division with numbers. Your students have not done this in years, so remind them how the process flowed. (To appease your more advanced students, you may want to alert the class that they do not have to write this down; we are just reminding ourselves how this worked.) Then move on to polynomial long division, showing the students how it bears resemblance to something familiar. If you teach synthetic division after students have mastered long division, you can do the same problem with long division first and leave it on the board. After performing the synthetic division, show them where the same numbers appeared in the long division. I would not *detail* how the synthetic is working, unless you have a very strong group of students, but revealing the parallel takes away some of the mystery and demonstrates that this method is a shorthand, not a trick.

This brings us to the issue of how to handle multiple approaches to material. When two or more methods exist, decide whether it is beneficial to teach multiple methods. On one hand, a student may be completely confused by one approach and find another easy. On the other hand, seeing more than one solution may be confusing and your students may have trouble separating the two methods. For example:

- Simplifying compound rational expressions: This is a situation when it may be beneficial to teach more than one method. Method 1: Multiply the top and bottom of the fraction by the least common denominator of all the sub-denominators. This clears all sub-fractions and leaves one clean fraction to simplify. For example:

$$\frac{\frac{1}{x^2}+\frac{x}{2}}{\frac{5}{(x-2)}-\frac{3}{x}} = \frac{\frac{1}{x^2}+\frac{x}{2}}{\frac{5}{(x-2)}-\frac{3}{x}}\cdot\frac{2x^2(x-2)}{2x^2(x-2)} = \frac{2(x-2)+x^3(x-2)}{10x^2-6x(x-2)} = \frac{(2+x^3)(x-2)}{4x^2+12x} = \frac{(2+x^3)(x-2)}{4x(x+3)}$$

$$for\;\; x \neq 2$$

Method 2: Achieve one fraction in the numerator and one fraction in the denominator by combining sub-fractions. Then multiply by the reciprocal of the simplified denominator fraction.

$$\frac{\frac{1}{x^2}+\frac{x}{2}}{\frac{5}{(x-2)}-\frac{3}{x}}=\frac{\frac{2}{2x^2}+\frac{x^3}{2x^2}}{\frac{5x}{x(x-2)}-\frac{3(x-2)}{x(x-2)}}=\frac{\frac{2+x^3}{2x^2}}{\frac{2x+6}{x(x-2)}}=\frac{(2+x^3)}{2x^2}\cdot\frac{x(x-2)}{2x+6}=\frac{(2+x^3)(x-2)}{4x(x+3)}$$

$$for\ \ x \neq 2$$

You could teach both methods, showing how to do the same problem both ways and reveal the parallels when you reach similar places in each problem. When explaining a homework problem, you might ask the student which method he or she wants to see rather than continuing to show both approaches.

- Completing the square to solve an equation: In this case, one method has an advantage over another. You have the option of moving everything not involving a variable to the right side of the equation, completing the square on the left side and solving for the variable.

$$x^2+2x-4=0$$
$$x^2+2x=4$$
$$x^2+2x+1=4+1$$
$$(x+1)^2=5$$
$$x=-1\pm\sqrt{5}$$

Using this approach can be problematic if you will be using the method of completing the squares for applications beyond solving equations. Instead, you might prefer to pull everything to the left (if not already there) and complete the square. Then to solve the equation, begin to pull terms to the right.

$$x^2 + 2x - 4 = 0$$
$$x^2 + 2x + 1 - 1 - 4 = 0$$
$$(x+1)^2 - 5 = 0$$
$$(x+1)^2 = 5$$
$$x = -1 \pm \sqrt{5}$$

Using the first method often causes problems that arise when students begin to work with functions instead of equations. For example, the students will need to complete the square to put functions representing parabolas and circles into forms revealing vertices and centers. Students who have been performing the first option when completing the square in an equation quickly encounter a problem - the lack of an equals sign! When asked to put the function into standard form and find, for instance, the vertex of a parabola, these students invariably try to *solve* the function for *x* by setting the function equal to zero. You can see from the work below that following the second method results in a seamless transition between functions and equations.

$$f(x) = x^2 + 2x - 4$$
$$f(x) = x^2 + 2x + 1 - 1 - 4$$
$$f(x) = (x+1)^2 - 5$$

> This is not to say that some students will not continue their work too far and solve this correct function set equal to zero, but at least they do cross the correct answer along the way. When presenting the original material on completing the square, you may want to only *verbally* acknowledge the first method. There is little harm in allowing students who are already comfortable with that method to continue performing it when dealing with equations, but you may want to explain that they will have to make adjustments when you use completing the square in other applications later in the course.

Predicting problems such as this is key to a course like pre-calculus, because the students in such a course often have a low threshold for even a slight change in methodology. It is impossible the first time teaching a course for you to foresee all possible areas of confusion. There are seemingly small choices you will make along the way that will have unforeseen consequences later, hopefully all minor. Pay attention to the nuances of these effects for the next time you teach the course. It is often helpful to jot a quick note to yourself onto the first page of your lecture to alert yourself to changes that you may want to make when prepping that topic again.

If possible, give students leeway to perform the methods that they are most comfortable with, as long as they are demonstrating skills on level with the course. But, when it really matters and may ultimately otherwise hinder their learning, strongly encourage (or require) the method you feel is most appropriate. If you do require one method over another equally valid mathematical operation, explain your reasoning to the class. Also, make sure that your expectations of the method you will be accepting for credit is clear to students prior to any graded paper.

One final note on a course of this level is to quiz the students early and often. These students need immediate and frequent feedback throughout the course to keep them on track. Graded homework provides one level of feedback, but it is not a substitute for a testing environment. A student who has little ability on his or her own could receive good homework scores if working with a tutor or classmate or simply through many trials to achieve answers at which other students arrived.   Such students may feel a false sense of security about their comprehension levels. Quizzes will keep them honest with themselves and give you a chance to assess areas where the class is struggling.

Calculus I: Differential Calculus

Class composition is one challenge when teaching calculus courses. In Calculus I (differential calculus), you will likely encounter a mix of students who have had a year of calculus in high school (possibly Advanced Placement) and those who have only taken pre-calculus previously. It can be difficult when you have overly prepared students in the class, but teach the course as though it is all new material. Of course, when you graze or review pre-calculus topics, such as equations of lines, you may want to assume some knowledge, but make sure to cover the basics. Always remember that math is easy to forget when left unpracticed and few students are thinking about calculus over the summer.

It may be helpful to poll your students (on paper, not by a hand-raise) as to the last math class they have taken and how they did in the course. Instead of asking students to give the grades they received in the courses they took, you may find it more useful to ask them to grade the level of comprehension they feel they took away from the course. This eliminates the need a student may feel to defend a poor performance or explain that a high letter grade was not representative of his or her understanding. (Students can earn an A in A.P. Calculus, but feel they learned nothing at all.) Unfortunately, when fielding questions in class, you may find that your most vocal students do not always represent the class as a whole and this type of poll can reveal where the bulk of your class stands as you start the semester.

As with pre-calculus, it is important to provide constant and immediate feedback, so quiz early and often with calculus students. As with pre-calculus, relying solely on collected homework may not provide sufficient feedback since students should be *encouraged* to work together and seek help from friends, a tutor, or you, as they work through assignments. At this level, the purpose of homework is to help the students learn and practice the material, not to test them.

A point you will want to emphasize in Calculus I is that students need to understand the basic differentiation rules well and relatively quickly. Make it clear that what you cover in one or two days lecture will be used *every day* for the rest of the semester. Any confusion should be addressed immediately and thoroughly. Suggest any student who experiences any difficulty with the assignment see you prior to the next lecture. If a student's mastery of the basic skills is insufficient, his or her flawed techniques will greatly hinder attempts to master the product, quotient, and chain rules.

One difficulty of teaching calculus is that some topics are very quick to put on the board (such as product or quotient rule), but most students will require quite a bit of practice before these skills are mastered. Do as many examples as you feel your class can tolerate, starting with the most basic and <u>gradually</u> building up to more involved examples. Recognize the need for practice before moving on to another rule or examples involving multiple rules at once. Stress to your students that these problems will become more complicated soon and they should address any confusion immediately.

If your Calculus I course includes ε - δ proofs for the formal definition of a limit, expect confusion. Students will generally get the hang of the proof process after awhile, but may still struggle with what is really happening. Do not underestimate the need to thoroughly (and repeatedly) explain what ε and δ represent and why the ability to achieve an ε–distance by controlling a δ–distance tells you anything about a limit!

Curve-sketching provides a wonderful summary of derivative applications in this course. Students generally enjoy how they are able to use everything they have learned over the past couple of months to create a graph. Testing can present problems if a student makes an error early. Naturally, the graphs are more interesting for functions that require more complicated differentiation, but your students may get snarled in the derivative and either never complete the application or overly simplify/complicate the problem you have given. You may prefer to test differentiation skills separately from graphing applications and either use functions with relatively easy derivatives or provide the necessary derivatives for a more complicated function. For example, you can ask students to graph a polynomial such as $g(x) = 4x^3 - 6x^2 + 5$ first and then give another graphing problem for a rational function such as $f(x) = \frac{3x}{x^2 - 9}$. In the first problem, the coefficients have been chosen so that the critical and inflection points do not contain overly messy values. In the second problem, you may want to provide the students with the first and second derivatives of $f(x)$. This approach allows the students to demonstrate the methods of graphing even if they are not as masterful with differentiation rules. Another helpful tool in testing graphing abilities is to break the problem down into a number of sub-problems: intercepts, domain, asymptotes, extrema, intervals of monotonicity, inflection points, concavity, and finally the graph. This will be a great aid to you in grading, since the work is separated and organized into distinct parts. This organization helps your students as well as they attempt to construct their graphs.

Calculus II: Integral Calculus

In Calculus II (integral calculus), students are faced with a wide variety of integration techniques. For this reason, many students who have had high school calculus still see a decent amount of new material in this course. Most students are a little overwhelmed at the point in the semester when all the techniques are merged and they are faced with a list of integrals and no hints as to which method to apply. As you proceed through the techniques, it may be helpful to keep a running list of the tools at hand and how to identify when you might use each. Depending on the scope of your course, you may have a list like the following.

- Expand simple products and simplify fractions when possible.
- Power Rule
- *u*-substitution
- Partial Fractions
- Trigonometric Integrals
- Trigonometric Substitutions
- Integration by Parts

For a detailed handout on these methods, see the appendix.

Show your students how certain integrands should cry out for a method (such as xe^x), so that they look for this before blindly going through the list trying each method available. Encourage students to scramble problems and try to solve them when they do not know the category of the integral.

When you cover volumes of solids of revolution, make sure to carefully address and diagram how the solid is sliced into disks/washers or built with shells and how this relates to the differential. If you are artistically challenged, you may want to prepare overheads or use physical models in place of drawing on the board. Encourage your students to avoid blind memorization as to when to integrate with respect to *x* or *y*. Your goal is to foster an understanding of what is happening and what is being summed, not to just get them through the computation.

When addressing the method of cylindrical shells, make sure to demonstrate why this method is beneficial and when preferable over the disk/washer approach. For example, do a problem that requires two integrals when done by the washer method but only one with cylindrical shells, such as the volume of the solid generated by revolving the region bounded by $y = x^2$, $y = 0$, $x = 1$, and $x = 3$ around the *y*-axis.

Sophomore Calculus

The contents and order of material taught in sophomore calculus will vary based on the institution and text used for the course. This section will discuss the variety of topics that may occur in such courses. At this point, your students should be either minoring or majoring in math or a math-related discipline. It becomes even more essential at this point that students are responsible for understanding how techniques work and *why*. While you may not have time to prove every theorem and formula you come across, you should definitely at least discuss the crux of most. It does little to further the students' mathematical education, if time only allows you to jot a formula on the board.

Topics such as ***arc length*** and ***surface area for a solid of revolution*** lend themselves to direct derivation of the formula. Deriving the formula for arc length utilizes the Pythagorean Theorem, which the students should be comfortable with, and also allows you to review and apply the mean value theorem. When initially approaching the formula for the surface area for a solid of revolution, start with an example of a right circular cylinder. Show how this cylinder can be cut up into the rectangle of height h and length $2\pi r$, to ensure that students comprehend the origin of the surface area formula. Then move on to a similar object, like a parabola revolved around its axis of symmetry. Slicing this object into bands and cutting a sample band just as you did the cylinder, you arrive at the same rectangle as before of height h (the arc length of your curve) and length $2\pi r$. Completing this derivation by finding the formulas for h and r demystifies the formula and encourages your students to avoid blind memorization. They will hopefully begin to realize that comprehending the origin of formulas makes their recollection much easier.

Parametric equations may be your students' first exposure to any graphing that is outside the Cartesian box. They may find this tedious since they already know a method of graphing, so begin with examples that exhibit the ability to graph rambling, non-functions, such as those representing a flight-path of a bug. Also, indicate the ability to travel a path repeatedly (such as around a circle) to illustrate that we are increasing our capabilities with this new method, not just thinking of the same old graphs with different formulas.

Polar coordinates provide another opportunity to illustrate how restrictive our previous graphing abilities were. Motivate your discussion by drawing a cardioid, 3-leaved rose, or open washer and asking your students to think about how they would write an equation to represent the graph. Illustrating how polar coordinates can simplify expressions, even for just a circle, may help you draw in your students. They can often be so overwhelmed with the new material that they miss the beauty of it.

When teaching ***sequences and series***, one of the obstacles is simply getting your students to understand the difference between the two, oddly enough. Test your students not only on whether they know the distinction between sequences and series, but also if they know the relationship between them. Push your students to avoid memorization. Do they understand what being monotonic and bounded really mean about a sequence and why we can say a sequence converges or diverges? Sometimes students will just blindly try to pigeonhole a sequence instead of getting an overview of how it behaves. The same applies to the tests for series convergence. Make sure to justify why these tests work versus simply stating the tests and using them. Just as you did with integration techniques, make sure to guide your students on the intuition as to when to use which test and why. Do some clean applications, like writing a repeating decimal as a fraction, to make this process less abstract.

Vectors and all that come with them may appear easier to your students than some of the previously discussed topics. While you may find students thrilled to be able to revert back to memorization of formulas and procedures, again do your best to focus in on the concepts behind the various computations you may perform. When possible, utilize applications, such as that involving the volume of a parallepiped, to relate these calculations to something tangible.

When you present ***partial derivatives***, you have the earlier derivative background to build upon. Draw the parallels between the tangent lines found in Calculus I and the tangent planes you wish to produce now. Remind students of the chain rule for one variable before you produce the rule for a function of two variables. Some students may not have learned the second derivative test in Calculus I, but it is easy enough for students at this level to grasp. So, "remind" them of that procedure before teaching the second partials test.

These courses are enjoyable both for the material and caliber of student they bring. Do them both justice by giving the material as much depth as time allows and encouraging the students to really think about the math they perform. Show your enthusiasm for the subject you have chosen to pursue.

Linear Algebra

Expect to largely be dealing with math-related majors and minors in this course. This is usually enjoyable, since potentially your whole class enjoys the subject of mathematics, instead of a few select members. This course is challenging from a presentation perspective, but also rewarding as the world of mathematics begins to unfold before your students.

One of the challenges you will face in this course is the excessive writing it seems to take to get the material on the board. You have the option of referring your students to their text when referencing a list of axioms. If you plan to use these axioms to any degree, they should be visually presented on the board or on an overhead. To help ease any stress your students may feel in trying to write everything down, you can tell them where this list can be found in their texts. Many students will still write it along with you, but you have relieved them from the need to try to keep up. Students who fall behind or tire of writing out the long list do not feel as flustered because they can look it up later. There is value in actually writing out a list of axioms instead of simply reading it, so you might encourage students to write out important axioms in a special section of their notebooks or to make up their own study guides for the axioms. As this course is heavy in axiom lists and formulas, you should be clear as to what you expect the students to know for exams.

Another challenge is dealing with lengthy matrix problems, such as finding an inverse. As tedious as it may be, I strongly encourage you to put all details of row-reduction on the board for a while. It is not that the concept will be difficult for your students to follow, but if you do everything in your head and they miss a moment of lecture they may quickly become lost and panicked. In addition, when they look back at the example later, they may not remember what you did in class. An example demonstrating full details follows. You should be able to wean the details written in curly brackets as the students practice the material and become comfortable performing these operations in their heads, but the directions written above the arrows are important to maintain.

Find the inverse of $A = \begin{bmatrix} 1 & 8 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 8 & 0 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & -2 & 0 \\ 0 & 4 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left\{\begin{array}{r|rrrrrr} -2R_2: & 0 & -8 & -4 & 0 & -2 & 0 \\ +R_1: & 1 & 8 & 0 & 1 & 0 & 0 \\ \hline new\,R_1: & 1 & 0 & -4 & 1 & -2 & 0 \end{array}\right\}$$

$$\xrightarrow[-R_3+R_2\to R_2]{2R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 4 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left\{\begin{array}{r|rrrrrr} 2R_3: & 0 & 0 & 4 & 0 & 0 & 2 \\ +R_1: & 1 & 0 & -4 & 1 & -2 & 0 \\ \hline new\,R_1: & 1 & 0 & 0 & 1 & -2 & 2 \\ -R_3: & 0 & 0 & -2 & 0 & 0 & -1 \\ +R_2: & 0 & 4 & 2 & 0 & 1 & 0 \\ \hline new\,R_2: & 0 & 4 & 0 & 0 & 1 & -1 \end{array}\right\}$$

$$\xrightarrow[\frac{1}{2}R_3 \to R_3]{\frac{1}{4}R_2 \to R_2} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$$\left\{\begin{array}{l} \frac{1}{4}R_2 : 0 \quad 1 \quad 0 \quad 0 \quad \frac{1}{4} \quad -\frac{1}{4} = new\, R_2 \\ \frac{1}{2}R_3 : 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad \frac{1}{2} = new\, R_3 \end{array}\right\}$$

$$\longrightarrow \quad A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Requiring that students use similar notes as those written above the arrows when performing their own work will greatly aid your ability to follow their work and award partial credit. Combining steps in the first lecture may be confusing, but after that ask your class if a particular combination is okay. This will help you feel out where they stand. You want to eliminate as much tedious note taking as possible, but do not sacrifice clarity for speed.

Linear algebra is another subject that greatly lends itself to the practice of drawing parallels. When computing matrix inverses, you can liken the inverse of a matrix to the inverse of a real number. Additionally, we can understand that the inverse of the real number 0 does not exist in the same way that the inverse of a matrix with determinant zero will not exist, since both problems involve a problem of dividing by zero. When discussing vector spaces, analyzing the typical spaces we work with, like $\mathbf{R}^n$ with the usual addition and scalar multiplication, and comparing and contrasting them with spaces previously foreign, such as the set of *n* x *n* matrices under the natural matrix addition and scalar multiplication, helps to deepen the understanding of the new spaces. The parallels allow students to see the similar spaces as less intimidating, and the contrasts help to further understand the complexities of these spaces and the axioms we are attempting to satisfy. This process of drawing parallels not only helps your students learn and retain the material more quickly, but it shows them how they are broadening their previous understanding of mathematics. The math that they have learned so far is largely one dimensional and based on standard arithmetic. If this class is the students' first exposure to vector spaces, you will be revealing a much larger mathematical world than they may have previously considered.

Depending on the scope of the course you are teaching, you may be asking students to construct proofs beyond the basic. Know your audience. Many students in a linear algebra course may not have had extensive courses utilizing proofs. If this is the case, they may be talented mathematicians and still have yet to develop good proof-sense. Many students need a bit of practice before they master the correct flow, detail, and precision of a good proof.

Proofs

A discrete mathematics course may be the first exposure your students have to a non-computational view of mathematics. It may also be the first time some students have struggled even slightly in a math course. This course helps to begin the process of sorting out the math majors from the minors and may send some students to pursue another subject altogether. Just as you can have a hierarchy in approaching integration techniques, you can demystify approaching a proof by organizing the process. Of course, you should also acknowledge that some problems call out for a particular method, such as induction, and that should be considered before proceeding through the list. A sample list follows:

- Direct Proof
- Indirect Proof
    - Contradiction
    - Contrapositive
- Counterexample (for disproof)
- Induction

You may want to discourage the over-use of proof by contradiction. If a direct proof can be accomplished, it generally is more illustrative of the concepts at hand than a proof by contradiction. Proof by contradiction can be a helpful tool in identifying the crux of a proof and may provide the starting point for a stumped student, but students need to develop the ability to think about the issues at hand without always relying on this approach. When appropriate, encourage students stuck in the contradiction rut to take their contradiction proofs and re-write them as direct proofs. The following illustrates how an unnecessary proof by contradiction can be reworked into a more succinct, direct proof.

Prove the product of rational numbers is rational.

By contradiction: Let x and y be rational numbers. Then there exists integers m, n, p, and q, with n and q nonzero, such that $x = \frac{m}{n}$ and $y = \frac{p}{q}$. Suppose that xy is irrational. Then xy must not be a fraction of integers of nonzero denominator. But $xy = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$ and since m, n, p, q are integers, mp and nq are integers. In addition, since neither n nor q is zero, nq must be nonzero. Thus xy is a fraction of integers of nonzero denominator and we reach a contradiction.

Direct: Let x and y be rational numbers. Then there exists integers m, n, p, and q, with n and q nonzero, such that $x = \frac{m}{n}$ and $y = \frac{p}{q}$. Thus, $xy = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$ and since m, n, p, q are integers, mp and nq are integers. In addition, since neither n nor q is zero, nq must be nonzero. Hence xy is rational.

Often the direct proof is much cleaner, shorter, and overall more pleasant to read, so the students may begin to see the value of this revision. Of course, you should give the disclaimer that some problems must be approached (and completed) with the contradiction method. Examples include the proofs that there is no largest integer and that some given entity is irrational.

While the mathematical shorthand symbols ($\exists$, $\forall$, $\ni$, $\therefore$) are excellent tools for saving time in both lecture and proof attempts, you may choose to disallow these symbols on final graded responses. One option is to tell students that a fellow college student, who is not studying math, should be able to read and reasonably understand what they have written, given necessary definitions of terms. This rule is especially helpful when students first begin working with these notions of "for all" and "there exists" and their shorthand notations. They often confuse when to use each quantifier and also confuse the symbols. Putting everything into words helps clarify precisely what the student means. Keep this in mind as you lecture. You will want to use the shorthand notations to cut down on the writing for you and your students, but make sure that everyone knows what the symbols mean, especially in the first couple of weeks that you use them. It may help to leave their definitions on the board somewhere. You may also remind students when you finish a problem what symbols would need to be replaced for a graded response.

The notion that a non-mathematician should be able to read and comprehend what your student has written also addresses another frustration that your students may face. Often students expect that you will know what they "meant" by a proof. Remind them that, most frequently, the answer is already provided and they have to prove to you that they know the path to conclude that answer. Simply writing a few steps and concluding the desired result does not evidence knowledge.

Senior-level Math

Topology, Real Analysis, Algebra, & Complex Variables

Generally, these courses are unique and refreshing to teach, in part due to the fact that the students' interests in math are sufficient to push them into pursuing topics they do not understand. Of course, the level of the material makes the lessons more interesting as well.

One of the especially enjoyable aspects of a course of this nature is that you are showing the students material that they have likely seen virtually none of before. That freshness is exciting after so many pre-calculus classes! Not only are the students listening with a different level of care, but you most likely have a blank slate to work with in regard to notation and terminology.

As the topic of complex variables is a bit unique, it deserves a few specific words. Again, the practice of providing parallels will serve you well in this course. In complex variables, freshman-level calculus can be directly drawn upon when you approach limits, the limit definition of the derivative, and differentiation formulas. Sophomore-level calculus courses provide the background in sequences, series, convergence/divergence, and polar coordinates. Of course, vector calculus or linear algebra introduces the students to vectors, which is helpful as you begin graphing. The only snag in drawing on the earlier courses is that some students may have largely forgotten the material learned in previous semesters!

A nice feature of a complex variables course is the ability to do something different and yet not too advanced or difficult. For example, graphing with exponential form, integration along contours, and roots of complex numbers are all concepts that can be challenging at first, but appealing in their uniqueness. Revealing simple approaches to problems that were quite challenging in calculus also reveals yet another beauty of mathematics. For example, applying the residue theorem to an integral for which we cannot find the antiderivative helps to expose students to the value of this subject.

In these senior-level courses, still remember your audience. Do not take this opportunity to demonstrate some of your research material, unless you truly have an audience of that level. Quiz your students frequently and interact with them during class to get a feel of where the class stands conceptually.

It is often best to start slow in these more advanced classes. If you start off a little easier than you would like, you have room to increase the difficulty if the class is responding well to the material. If you begin at a level that is too sophisticated, the students may comprehend little of the foundations of the course and quickly become frustrated. Because of the nature of the material, students may find it overwhelming. Try to give them a few easy successes early on and gradually increase the challenges as their confidence levels build. This practice may also help you determine the caliber of students you have in your course, if the “gimmes” do not go well.

Chapter 4: Documenting Your Teaching

Creating the practice of documenting your teaching will not only help you become a better, more focused and efficient professor, but will also aid you in future job searches, reviews, and tenure and promotion applications. This process can be initiated quite simply in your day-to-day experiences of teaching and will naturally evolve into a precise refinement of your craft.

Begin by taking the time as you complete a lecture to note what you feel worked and what did not. This will require that you watch your students as you lecture and tune into what material they grasp. Consider how questions arose in lecture and where the students had difficulty comprehending your points. Also, make notes after you have taken homework questions on that material as to what may need further attention the next time you present this lecture. You may also want to indicate if you need more or varied examples for a topic. This process of noting problem areas need not take an extensive amount of time. Even just a minute after class to reflect and jot down any thoughts you have may be useful. You also do not need to have all the answers to the problems you identify in the lecture immediately. You may find some fresh ideas occur to you when you approach the material the next time and read over the notes you made for yourself.

At the end of the semester, take an overall review of the course. Consider which topics went well and which need revision. Were there unforeseen consequences of choices made early in the semester that you want to avoid next time? Consider the pacing of the course. Are there any topics you would like to add or delete? Are there topics you would like to explore in more depth or dwell on less in the future? Make clear notes and state your reasoning for these changes. Make a note of the general skill-level of the students enrolled in the particular section you just taught. You may want to make different choices if the next class is more or less advanced. Keeping these reviews of your teaching will help you later document your progression as a teacher when you apply for jobs or tenure.

Keep a sample of all handouts, overheads, quizzes, and exams for each course you teach. You may reuse some and use others as a basis for an improved version. These items are also helpful as you prepare a teaching portfolio or apply for tenure. Reviewing these papers at the end of the semester may also reveal a gap you would like to fill the next time you teach the course.

The day-to-day and semester-end review of your work will not only hone your presentations and course content, but will develop your senses as an instructor. You will begin to gain a sense of what you want to achieve in the classroom and how you best accomplish those goals. This will be helpful as you approach each new teaching endeavor and as you attempt to explain your philosophies to others.

As you prepare to apply for jobs, encounter reviews and apply for tenure or promotion, it will be increasingly important to have a definite sense of your distinct approach to the classroom. The process of constantly analyzing your teaching and striving to improve your techniques will not only serve to enhance your abilities in the classroom, but also to develop your capacity to communicate your philosophies to others.

Appendix

Continuity Handout

Use the formal definition of continuity to prove the following functions are either continuous or discontinuous at a particular point.

1. $f(x) = \dfrac{x^2 - 16}{x + 4}$. This is discontinuous – where? Now prove you are right.

2. $f(x) = x^4 + 5x - 3$ at the point $x = 1$.

3. $f(x) = \begin{cases} x + 4 & if \quad x \le 1 \\ 2x - 3 & if \quad x > 1 \end{cases}$ at the point $x = 1$.

4. $f(x) = \begin{cases} 3x & if \quad x \ne 2 \\ 7 & if \quad x = 2 \end{cases}$ at the point $x = 2$.

5. $f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 1} & if \quad x \ne 1 \\ -1 & if \quad x = 1 \end{cases}$ at the point $x = 1$.

<u>Solutions</u>

1. $f(x) = \dfrac{x^2 - 16}{x + 4}$. This is discontinuous – where? Now prove you are right.

$$f(x) = \frac{x^2 - 16}{x + 4} = \frac{(x+4)(x-4)}{x + 4} = x - 4 \text{ for } x \ne -4.$$

Since $f(-4)$ does not exist, $f(x)$ is discontinuous at x = -4.

2. $f(x) = x^4 + 5x - 3$ at the point $x = 1$.

a) $f(1) = 1^4 + 5(1) - 3 = 3$, so $f(1)$ exists.

b) $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}(x^4 + 5x - 3) = 1^4 + 5(1) - 3 = 3$, so $\lim\limits_{x\to 1} f(x)$ exists.

c) $\lim\limits_{x\to 1} f(x) = 3 = f(1)$

So, $f(x)$ is continuous at x = 1.

3. $f(x)=\begin{cases} x+4 & if \quad x\le 1 \\ 2x-3 & if \quad x>1 \end{cases}$ at the point $x = 1$.

a) $f(1)=1+4=5$, so $f(1)$ exists.
b) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+4) = 1+4=5$
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x-3) = 2(1)-3=-1$
$\lim_{x\to 1^-} f(x) = 5 \ne -1 = \lim_{x\to 1^+} f(x)$, so $\lim_{x\to 1} f(x)$ does not exist.

Since $\lim_{x\to 1} f(x)$ does not exist, $f(x)$ is discontinuous at $x = 1$.

4. $f(x)=\begin{cases} 3x & if \quad x\ne 2 \\ 7 & if \quad x=2 \end{cases}$ at the point $x = 2$.

a) $f(2)=7$, so $f(2)$ exists.
b) $\lim_{x\to 2} f(x) = \lim_{x\to 2} 3x = 3(2) = 6$, so $\lim_{x\to 2} f(x)$ exists.
c) $\lim_{x\to 2} f(x) = 6 \ne 7 = f(2)$

Since $\lim_{x\to 2} f(x) \ne f(2)$, $f(x)$ is discontinuous at $x = 2$.

5. $f(x)=\begin{cases} \dfrac{x^2-3x+2}{x-1} & if \quad x\ne 1 \\ -1 & if \quad x=1 \end{cases} = \begin{cases} \dfrac{(x-1)(x-2)}{x-1} & if \quad x\ne 1 \\ -1 & if \quad x=1 \end{cases}$
$=\begin{cases} x-2 & if \quad x\ne 1 \\ -1 & if \quad x=1 \end{cases}$

a) $f(1)=-1$, so $f(1)$ exists.
b) $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x-2) = 1-2=-1$, so $\lim_{x\to 1} f(x)$ exists.
c) $\lim_{x\to 1} f(x) = -1 = f(1)$

So, $f(x)$ is continuous at $x = 1$.

Graphing Techniques Handout: Polynomials and Rational Functions

Extrema, Increasing/Decreasing, and Concavity

Relative Extrema and Critical Numbers

- First note any discontinuities of your function. These will not be extrema!*
- Solve $f'(x)=0$ for x and find any x-values that make $f'(x)$ undefined. The solutions which are **in the domain** of $f(x)$ are your critical numbers.
- Perform either the first or second derivative test:

  First derivative test:

  Make a sign chart for $f'(x)$, using the critical numbers and discontinuities.

  If the sign of $f'(x)$ changes from + to –, you have found a relative max (unless the number was a discontinuity of $f(x)$).

  If the sign of $f'(x)$ changes – to +, you have found a relative min (unless the number was a discontinuity of $f(x)$).

  Second derivative test:

  Plug each critical number into $f''(x)$. If $f''(cn)>0$, then you have a relative min at the critical number. If $f''(cn)<0$, then you have a relative max at the critical number. If $f''(cn)=0$ or does not exist, use the first derivative test.
- Find the y-value of the extrema by plugging the x into the original function, $f(x)$. State your final answer as an (x, y) point.

Absolute Extrema on a Closed Interval $[a,b]$ ($f(x)$ must be continuous on $[a,b]$.)

- Solve $f'(x)=0$ for x and find any x-values that make $f'(x)$ undefined. The solutions which are **in the domain** of $f(x)$ are your critical numbers. (You only need to consider critical numbers in the interval.)
- Plug each critical number into the original function $f(x)$. Plug the interval endpoints into the original function $f(x)$ (i.e. find $f(a)$ and $f(b)$.) The largest answer tells you the absolute max and the smallest answer tells you the absolute min. Make sure to state your final answer as an (x, y) point.

Determining Intervals Where a Function Increases/Decreases

- First note any discontinuities of your function.
- Solve $f'(x)=0$ for x and find any x-values that make $f'(x)$ undefined. The solutions which are **in the domain** of $f(x)$ are your critical numbers.
- Make a sign chart for $f'(x)$, using the critical numbers and discontinuities found above.

  $f(x)$ increases on the intervals where $f'(x)>0$.

  $f(x)$ decreases on the intervals where $f'(x)<0$.
- Make sure to give one answer for increasing (connecting intervals together with union symbols) and one answer for decreasing (connecting intervals together with union symbols).

Determining Inflection Points and Intervals Where a Function is CCU/CCD

- First note any discontinuities of your function. These will not be inflection points!
- Solve $f''(x) = 0$ for x and find any x-values that make $f''(x)$ undefined. The solutions which are **in the domain** of $f(x)$ are your possible inflection points.
- Make a sign chart for $f''(x)$, using the possible inflection points and discontinuities found above.

  $f(x)$ is concave up (CCU) on the intervals where $f''(x) > 0$.

  $f(x)$ is concave down (CCD) on the intervals where $f''(x) < 0$.
- Make sure to give one answer for CCU (connecting intervals together with union symbols) and one answer for CCD (connecting intervals together with union symbols).
- Inflection points occur only for x-values in the domain of $f(x)$ where concavity changes (i.e where the sign changes on your sign chart for $f''(x)$.) Find the y-value of the inflection points by plugging the x-value into the original function, $f(x)$. State your final answer as an (x, y) point.

*Note: This handout is written for polynomials and rational functions. Other functions, such as those with jump discontinuities, may have extrema at discontinuities.

Integration Strategy Handout

1. **Simplify** the integrand when possible.
   - Products: If the product is one that is easily multiplied out, this may be the simplest (or only) way to solve the integral. For example:
     $\int x^2(2-x)dx = \int (2x^2 - x^3)dx = \frac{2}{3}x^3 - \frac{1}{4}x^4 + C$.
   - Fractions: If the denominator is one term, you can break the fraction apart, writing each piece of the numerator over the denominator. For example:
     $\int \frac{1-x^5}{6x^2}dx = \int \left(\frac{1}{6x^2} - \frac{x^5}{6x^2}\right)dx = \int \left(\frac{1}{6}x^{-2} - \frac{1}{6}x^3\right)dx = -\frac{1}{6}x^{-1} - \frac{1}{24}x^4 + C = -\frac{1}{6x} - \frac{1}{24}x^4 + C$.

2. Look for a basic ***u*-substitution**.
   - Powers: If you have an expression raised to a power, try a *u*-substitution for the expression. For example, letting $u = 6-x$ in the following:
     $\int (6-x)^{10}dx = \int -u^{10}du = -\frac{1}{11}u^{11} + C = -\frac{1}{11}(6-x)^{11} + C$.
   - Trigonometric Functions:
     - Try a *u*-substitution for the angle. For example, letting $u = 6x$ in the following:
       $\int \sin(6x)dx = \int (\frac{1}{6}\sin u)du = -\frac{1}{6}\cos u + C = -\frac{1}{6}\cos(6x) + C$.
     - Try a *u*-substitution for an entire trig function. For example, letting $u = \sin(3x)$ in the following:
       $\int \sin(3x)\cos(3x)dx = \int \frac{1}{3}u\,du = \frac{1}{6}u^2 + C = \frac{1}{6}\sin^2(3x) + C$.
       Note: For the above example, you could choose $u = \sin(3x)$ or $u = \cos(3x)$. If one of the functions in the example above is raised to a power, then *u* will have to be that function. So, in the following example, we must choose $u = \sin(3x)$. For more complicated problems, see #4.
       $\int \sin^2(3x)\cos(3x)dx = \int \frac{1}{3}u^2du = \frac{1}{9}u^3 + C = \frac{1}{9}\sin^3(3x) + C$.
     - Try rewriting functions in terms of sine and/or cosine. (The *u*-substitution $u = \cos x$ is also used below.)
       $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \int \frac{-1}{u}du = -\ln|u| + C = -\ln|\cos x| + C$.
       (Equivalently, this answer can be written as: $\ln|\sec x| + C$.)
   - Exponentials:
     - Try a *u*-substitution for the exponent. For example, letting $u = 3x+1$ in the following:
       $\int e^{3x+1}dx = \int \frac{1}{3}e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x+1} + C$.
       Or letting $u = 2x^2 - 4$ in the following:
       $\int x7^{2x^2-4}dx = \int \frac{1}{4}7^u du = \frac{1}{4\ln 7}7^u + C = \frac{1}{4\ln 7}7^{2x^2-4} + C$
     - Try a *u*-substitution that includes the entire exponential. For example, letting $u = e^x + 1$ in the following:

$$\int \frac{e^x}{e^x+1}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|e^x+1| + C = \ln(e^x+1) + C\,.$$

- Logarithms: Try a *u*-substitution for the entire log term. For example, letting $u = \ln x$ in the following:
  $$\int \frac{\ln x}{x}\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}(\ln x)^2 + C\,.$$
- Products: If the product cannot be multiplied out, or that would require much work, then try a substitution. For example, letting $u = 2 - x^2$ in the following:
  $$\int x\sqrt{2-x^2}\,dx = \int -\tfrac{1}{2}u^{1/2}\,du = -\tfrac{1}{3}u^{3/2} + C = -\tfrac{1}{3}(2-x^2)^{3/2} + C\,.$$
- Fractions: If the fraction cannot be simplified as discussed in #1, look for the following substitutions as a possible way to solve the integral.
  - If there is an expression raised to a power, try a *u*-substitution for the expression. For example, letting $u = 4 - x$ in the following:
    $$\int \frac{2}{(4-x)^{10}}\,dx = \int -2u^{-10}\,du = \tfrac{2}{9}u^{-9} + C = \tfrac{2}{9}(4-x)^{-9} + C = \frac{2}{9(4-x)^9} + C$$
  - If it appears that the numerator may be the derivative of the denominator, try letting *u* = denominator. For example, letting $u = x^5 - 1$ in the following:
    $$\int \frac{3x^4}{x^5-1}\,dx = \int \frac{3}{5}\frac{1}{u}\,du = \tfrac{3}{5}\ln|u| + C = \tfrac{3}{5}\ln|x^5-1| + C\,.$$
  - If the denominator factors, consider partial fractions (see #3).

3. If the integrand is a proper rational function (i.e. $\frac{polynomial}{polynomial}$ with the degree of the numerator less than the degree of the denominator), try simplifying the integrand using **partial fractions**. Factor the denominator and follow the guidelines below.
   - Linear factors which are not repeated will have a single constant in the numerator.
     $$\frac{x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
   - Linear repeated factors will have a single constant in the numerator and each power of the factor must be represented in the denominators.
     $$\frac{x}{(x-1)(x+2)^3} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$
   - Irreducible quadratic factors which are not repeated will have linear numerators.
     $$\frac{x}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$$
   - Irreducible quadratic repeated factors will have linear numerators and each power of the factor must be represented in the denominators.
     $$\frac{x}{(x^2+1)^2(x+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x+2}$$
   - You will use $\int \frac{1}{u}\,du = \ln|u| + C$ and $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ for many of these problems.
   - These integrals will often have answers involving the natural log – clean up using log rules.

4. If the integrand consists of **powers of trig functions**, try the following substitutions.
   - Integrals with sine and cosine:

- ♦ If the power on either sine or cosine is odd, pull off one copy of that trig function and convert the rest to the other trig function, using $\sin^2 x + \cos^2 x = 1$. If both powers are odd, just pick one to convert.
  - ♦ If both are raised to even powers, convert everything to cosines using $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ and $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$
- Integrals with tangent and secant:
  - ♦ If the power on secant is even, pull off one copy of $\sec^2 x$, and convert the rest of the secants to tangents, using $\sec^2 x = 1 + \tan^2 x$.
  - ♦ If the power on tangent is odd, pull off one copy of secant and one copy of tangent, and convert the rest of the tangents to secants, using $\tan^2 x = \sec^2 x - 1$.
  - ♦ If neither is true, just experiment.
- Integrals with cotangent and cosecant:
  - ♦ If the power on cosecant is even, pull off one copy of $\csc^2 x$, and convert the rest of the cosecants to cotangents, using $\csc^2 x = 1 + \cot^2 x$.
  - ♦ If the power on cotangent is odd, pull off one copy of cosecant and one copy of cotangent, and convert the rest of the cotangents to cosecants, using $\cot^2 x = \csc^2 x - 1$.
  - ♦ If neither is true, just experiment.

5. If the integrand involves **roots** and the standard *u*-substitution fails, consider the following trigonometric substitutions.
   - $\sqrt{a^2 - x^2}$: $x = a\sin\theta$. Where is θ?
   - $\sqrt{a^2 + x^2}$: $x = a\tan\theta$. Where is θ?
   - $\sqrt{x^2 - a^2}$: $x = a\sec\theta$. Where is θ?

6. Try **integration by parts**. $\int u\,dv = uv - \int v\,du$ or $\int_a^b u\,dv = uv\,|_a^b - \int_a^b v\,du$
   - Recognize integrals of the form $\int x^n e^x dx$, $\int x^n \sin(x)dx$, and $\int x^n \ln(x)dx$ are solved using integration by parts.
   - Remember the special approach required for an integral such as $\int e^x \sin(x)dx$.

7. Try, try again!