39 DIFFERENCE AND FUNCTIONAL EQUATIONS

39A Difference equations

MR2864818 39A05 37B99
Sacker, Robert J. (1-SCA; Los Angeles, CA)
An invariance theorem for mappings. (English summary)
J. Difference Equ. Appl. **18** (2012), *no.* 1, 163–166.

The following theorem is proved.

Theorem 2.1. Let $D\subset\mathbb{R}^n$ be a bounded subset and $f\colon\overline{D}\to\mathbb{R}^n$ be continuous. Suppose $f\colon\mathring{D}\to\mathbb{R}^n$ is injective (one-to-one) and $f(\partial D)\subset\overline{D}$.

If $\overline{D}^{\mathsf{c}}:=\mathbb{R}^n\smallsetminus\overline{D}$ has no bounded components, then $f(\overline{D})\subset\overline{D}$.

Here $\overline{D}$ is the closure of set D, $\mathring{D}$ is the interior of D, $\partial D=\overline{D}\smallsetminus\mathring{D}$. A simplified application is given to a biological migration-selection model. *Vladimir Sh. Burd*

MR2846903 39A06 39A45 68W30
Abramov, S. A. (RS-AOS-C; Moscow)**; Barkatou, M. A.** (F-LIMO-IXL; Limoges)**;**
van Hoeij, M. (1-FLS; Tallahassee, FL)**; Petkovšek, M.** (SV-LJUBMP-NDM; Ljubljana)
Subanalytic solutions of linear difference equations and multidimensional hypergeometric sequences. (English summary)
J. Symbolic Comput. **46** (2011), *no.* 11, 1205–1228.

Summary: "We consider linear difference equations with polynomial coefficients over $\mathbb{C}$ and their solutions in the form of sequences indexed by the integers (sequential solutions). We investigate the $\mathbb{C}$-linear space of subanalytic solutions, i.e., those sequential solutions that are the restrictions to $\mathbb{Z}$ of some analytic solutions of the original equation. It is shown that this space coincides with the space of the restrictions to $\mathbb{Z}$ of entire solutions and that the dimension of this space is equal to the order of the original equation.

"We also consider d-dimensional ($d\ge 1$) hypergeometric sequences, i.e., sequential and subanalytic solutions of consistent systems of first-order difference equations for a single unknown function. We show that the dimension of the space of subanalytic solutions is always at most 1, and that this dimension may be equal to 0 for some systems (although the dimension of the space of all sequential solutions is always positive).

"Subanalytic solutions have applications in computer algebra. We show that some implementations of certain well-known summation algorithms in existing computer algebra systems work correctly when the input sequence is a subanalytic solution of an equation or a system, but can give incorrect results for some sequential solutions."

MR2838429 39A10 39A12
Lin, Xiaojie [Lin, Xiao Jie] (PRC-XUNU-SM; Xuzhou)**;**
Liu, Wenbin [Liu, Wen Bin^4] (PRC-CUMT; Xuzhou)
Positive solutions to a second-order discrete boundary value problem. (English summary)
Discrete Dyn. Nat. Soc. **2011,** *Art. ID* 596437, 8 *pp.*

The paper considers the following 3-point boundary value problem (BVP) for a second-order difference equation:

$$\Delta^2 y(k-1)+h(k)f(y(k))=0,\quad k\in\{1,\dots,T\},$$
$$y(0)-\alpha\Delta y(0)=0,\quad y(T+1)=\beta y(n).$$

Using the Krasnosel′skiĭ fixed point theorem in a cone, the existence of at least one positive solution for the BVP is established. *Patricia J. Y. Wong*

MR2825243 39A10 39A12

Ma, Ruyun [**Ma, Ru Yun**] (PRC-NWNU; Lanzhou);
Ma, Huili [**Ma, Hui Li**] (PRC-NWNU-EMG; Lanzhou)

Global structure of positive solutions for superlinear discrete boundary value problems. (English summary)

J. Difference Equ. Appl. **17** (2011), *no.* 9, 1219–1228.

This paper is devoted to investigating the global structure of positive solutions of superlinear discrete boundary value problems of the form

$$\Delta^2 u(t-1)+\lambda a(t)f(u(t))=0,\quad t\in\mathbf{T},$$
$$u(0)=u(T+1)=0.$$

Here $T>1$ is a fixed integer, $\mathbf{T}:=\{1,\dots,T\}$, $\widehat{\mathbf{T}}:=\{0,1,\dots,T+1\}$, $a\colon\widehat{\mathbf{T}}\to\mathbf{R}^+$ and $f\in C([0,\infty),[0,\infty))$ satisfies $f(s)>0$ for $s>0$ and $\lim_{s\to 0^+} f(s)/s=\infty$. The main results are obtained by using Rabinowitz's global bifurcation theorem. *Michail G. Blizorukov*

MR2870025 39A10

Stević, Stevo (SE-SAOS; Belgrade)

On some solvable systems of difference equations. (English summary)

Appl. Math. Comput. **218** (2012), *no.* 9, 5010–5018.

Summary: "We show that the following systems of difference equations

$$x_{n+1}=\frac{u_n}{1+v_n},\ y_{n+1}=\frac{w_n}{1+s_n},\ n\in\mathbb{N}_0,$$

where u_n, v_n, w_n, s_n are some of the sequences x_n or y_n, with real initial values x_0 and y_0, are solvable in fourteen out of sixteen possible cases. Two cases are left unsolved. Probably the most interesting is the result in the case $u_n=x_n$, $v_n=x_n$, $w_n=x_n$, $s_n=y_n$, where a fascinating formula is obtained in an elegant way by using some ad hoc ideas."

MR2775993 39A13 39A06 39B52 68W30

Barkatou, Moulay A. (F-LIMO-IXL; Limoges);
Broughton, Gary (4-KNSTCM; Kingston upon Thames);
Pflügel, Eckhard (4-KNSTCM; Kingston upon Thames)

A monomial-by-monomial method for computing regular solutions of systems of pseudo-linear equations. (English summary)

Math. Comput. Sci. **4** (2010), *no.* 2-3, 267–288.

The paper presents a local analysis of systems of pseudo-linear equations of the form $\delta Y=AY$, where δ is a pseudo-derivation and A is a given matrix. The notion of regular solution is defined and the structure and existence of regular solutions of various systems are discussed. Moreover, an algorithm for the computation of a basis of regular solutions is presented, together with its implementation by using the computer algebra system Maple. *Gian-Luigi Forti*

MR2838504 39A21

Thandapani, Ethiraju (6-MADR-R; Chennai);
Vijaya, Murugesan (6-MADR-R; Chennai);
Li, Tongxing [Li, Tong Xing²] (PRC-SHAN-CNE; Jinan)

On the oscillation of third order half-linear neutral type difference equations. (English summary)

Electron. J. Qual. Theory Differ. Equ. **2011,** *No.* 76, 13 *pp.*

In this paper, oscillations of solutions of the third-order neutral half-linear difference equation

$$\Delta(a_n(\Delta^2(x_n+p_nx_{n-\delta}))^\alpha)+q_nx^\alpha_{n-\tau}=0$$

are considered. By investigating the properties of positive solutions of this equation, taking the neutral term as a whole part and using the Riccati transformation, oscillation criteria similar to those of Kamenev type are obtained. *Zhaowen Zheng*

MR2884267 39A21

Wang, Dong Mei (PRC-HAINU-CIT; Haikou);
Xu, Zhi Ting (PRC-SCN-SM; Guangzhou)

Oscillation criteria for second-order quasilinear neutral delay difference equations. (Chinese. English and Chinese summaries)

Acta Math. Appl. Sin. **34** (2011), *no.* 3, 537–553.

Summary: "By using Riccati transformation, averaging technique and a lot of inequality techniques, some sufficient conditions are obtained for oscillation of the second-order quasi-linear neutral delay difference equations

$$\Delta[r_n|\Delta z_n|^{\alpha-1}\Delta z_n]+q_nf(x_{n-\sigma})=0,$$

where $z_n=x_n+p_nx_{n-\tau}$, under the conditions $\alpha\geq\beta\geq 1$ or $\alpha\geq 1$, $0<\beta<1$, where β is a constant in condition $A(4)$ of this paper, and give some examples to explain."

Qi Ru Wang

MR2884758 39A22 34L05 49M29

Galewski, Marek (PL-PLDZ-IM; Łódź)

On the dual variational method for a system of nonlinear equations with a parameter. (English summary)

Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **18** (2011), *no.* 5, 699–704.

The author discusses the existence of solutions of the nonlinear system

$$(1)\qquad Ax=\lambda f(x),\quad x\in\mathbb{R}^N,$$

where λ is a positive parameter, A is a real positive definite $N\times N$ matrix, and $f(x)=(f_1(x_1),\dots,f_N(x_N))^T$ is a continuous function. Let S be the square root of A: $S^2=A$. Define the objective function

$$J_\lambda(z)=\frac{1}{2}\langle Sz,Sz\rangle-\lambda\sum_{i=1}^N\int_0^{z_i}f_i(t)\,dt.$$

By using duality, the author proves that the nonlinear system (1) has at least two nontrivial bounded solutions which are a minimizer and a maximizer of the objective function. *Lianwen Wang*

MR2872215 39A22 39A21
Li, Xianyi [Li, Xian Yi[2]] (PRC-SZU-MCP; Shenzhen)**;**
Zhou, Li [Zhou, Li[5]] (PRC-SZU-MCP; Shenzhen)
A note for "On the rational recursive sequence $x_{n+1}=\frac{A+\sum_{i=0}^k\alpha_ix_{n-i}}{\sum_{i=0}^k\beta_ix_{n-i}}$" [MR2494777]. (English summary)
Arab J. Math. Sci. **18** (2012), *no.* 1, 15–24.

This note is essentially a critique of the paper [E. M. E. Zayed and M. A. El-Moneam, Math. Bohem. **133** (2008), no. 3, 225–239; MR2494777 (2009m:39026)]. In Theorem 5, Zayed and El-Moneam claimed that if all roots of the polynomial $\lambda^{k+1}-\sum_{j=0}^k b_j\lambda^{k-j}$ lie in the open unit disk, then $\sum_{j=0}^k|b_j|<1$. This note gives the well-known example, for the case $k=1$, namely $\lambda^2+\lambda+a$ for $a\in(0,1/4]$, where both roots of a quadratic polynomial belong to the unit disk, but $|b_0|+|b_j|>1$, which refutes the above claim.

In Theorem 7, Zayed and El-Moneam claimed that positive increasing solutions of the difference equation

$$x_{n+1}=\frac{A+\sum_{i=0}^k\alpha_ix_{n-i}}{\sum_{i=0}^k\beta_ix_{n-i}} \tag{1}$$

are bounded and persist. In this note X. Li and L. Zhou prove that positive solutions of equation (1) are bounded and persist. However, this result is also well known and can be found, for example, in the reviewer's paper [J. Appl. Math. Comput. **24** (2007), no. 1-2, 295–303; MR2311966 (2008f:39018)].

In Theorem 2.1, Li and Zhou also prove the existence of non-oscillatory solutions of equation (1) by using an inclusion theorem by L. Berg. They modify the proofs of similar results presented in some papers by this reviewer, for example, in [J. Math. Anal. Appl. **316** (2006), no. 1, 60–68; MR2201749 (2006i:39026); Appl. Math. Lett. **20** (2007), no. 1, 28–31; MR2273123 (2007f:39031); C. Çinar, S. Stević and İ. Yalçınkaya, Rostock. Math. Kolloq. No. 59 (2005), 41–49; MR2169497 (2006d:39011)], but do not cite any of these papers. *Stevo Stević*

MR2846501 39A23
Liu, Shibo [Liu, Shi Bo] (PRC-SHTO; Shantou)
Multiple periodic solutions for non-linear difference systems involving the p-Laplacian. (English summary)
J. Difference Equ. Appl. **17** (2011), *no.* 11, 1591–1598.

The author considers nonlinear difference systems, involving the p-Laplacian, of the form

$$\Delta(|\Delta x_{n-1}|^{p-2})\Delta x_{n-1}+f(n,x_{n+1},x_n,x_{n-1})=0,\quad x_{n+m}=x_n.$$

Using the three critical points theorem, Clark's theorem and Morse theory the author presents conditions under which there exist multiple periodic solutions of the above difference systems. *Ewa L. Schmeidel*

MR2855255 39A23
Tang, Mei-Lan [Tang, Mei Lan] (PRC-CSU-SMC; Changsha)**;**
Liu, Xin-Ge (PRC-CSU-SMC; Changsha)
Positive periodic solution of higher-order functional difference equation. (English summary)
Adv. Difference Equ. **2011,** 2011:56, 8 *pp.*

The paper concerns the higher-order functional difference equation

$$x_{n+m+k}-a_{n+m}x_{n+m}-bx_{n+k}+a_nbx_n=f_n(x_{n-\tau_n}),\quad n\in\mathbb{Z}, \tag{E}$$

where $k, m \in \mathbb{N}$, $b \neq 1$ is a positive constant and, for all $n \in \mathbb{Z}$, the sequence $\{a_n\}_{n\in\mathbb{Z}}$ satisfies $a_n \geq 0$ with $a_n \neq 1$, the sequences $\{\tau_n\}_{n\in\mathbb{Z}} \subset \mathbb{N}$ and $\{a_n\}_{n\in\mathbb{Z}}$ are ω-periodic sequences for a fixed $\omega \in \mathbb{N}$, and $f\colon \mathbb{R}\times(0,\infty)\to[0,\infty)$ is continuous with $f_{n+\omega}(u) = f_n(u)$ for $n \in \mathbb{N}$, $u \in \mathbb{R}_+$. By using a fixed point theorem in cones, the authors establish sufficient conditions for the existence of a positive periodic solution of equation (E). The main result of the paper removes constraints appearing in a result due to W. B. Wang and X. Chen [Appl. Math. Lett. **23** (2010), no. 12, 1468–1472; MR2718532 (2011h:39016)]. An example illustrating the main result of the paper is provided. *Ioannis K. Purnaras*

MR2854828 39A28 39A30 92D25

Guzowska, Malgorzata [Guzowska, Małgorzata] (PL-SZCZ-EMS; Szczecin);
Luís, Rafael (P-TULT-CAN; Lisbon);
Elaydi, Saber [Elaydi, Saber N.] (1-TRI; San Antonio, TX)

Bifurcation and invariant manifolds of the logistic competition model. (English summary)

J. Difference Equ. Appl. **17** (2011), *no.* 12, 1851–1872.

In this paper, a new logistic competition model is proposed. Stability and oscillation patterns are discussed. The invariant manifolds, including the important center manifolds, are calculated. Saddle-node and period-doubling bifurcations are analyzed and the route to chaos is exhibited via numerical simulations. *Ming Shu Peng*

MR2874696 39A28 37C29 39A12

He, Xiaofei [He, Xiaofei[1]] (PRC-JISH-CS; Jishou)

Infinitely many homoclinic orbits for $2n$th-order nonlinear functional difference equations involving the p-Laplacian. (English summary)

Abstr. Appl. Anal. **2012,** *Art. ID* 297618, 20 *pp.*

Conditions which guarantee the existence of infinitely many homoclinic orbits of the linear difference equation

$$\Delta^n(r(t-n)\varphi_p(\Delta^n u(t-1))) + q(t)\varphi_p(u(t)) = f(t, u(t+n), \dots, u(t), \dots, u(t-n)), \quad n \in \mathbb{Z}(3),\ t \in \mathbb{Z},$$

are established. Three examples illustrating the results are given. *I. P. Stavroulakis*

MR2846500 39A30

Berenhaut, Kenneth S. (1-WKFR; Winston-Salem, NC);
Guy, Richard T. (1-WKFR; Winston-Salem, NC);
Barrett, Christa L. (1-WKFR; Winston-Salem, NC)

Global asymptotic stability for minimum-delay difference equations. (English summary)

J. Difference Equ. Appl. **17** (2011), *no.* 11, 1581–1590.

The authors obtain global asymptotic stability results for the positive solutions of the minimum-delay difference equation

$$y_n = \min\{f(y_{n-k_1}, y_{n-m_1}), \dots, f(y_{n-k_L}, y_{n-m_L})\}, \quad n \geq 0,$$

where $f\colon \mathbb{R}_+^2 \to (a,\infty)$ with $a := \inf_{x,y\in\mathbb{R}_+} f(x,y) \geq 0$ is continuous, for $i = 1, 2, \dots, L$ the integers $k_i, m_i \geq 1$ are such that $\gcd(k_1, m_1, k_2, m_2, \dots, k_L, m_L) = 1$ and $y_{-s}, y_{-s+1}, \dots, y_{-1} \in (0,\infty)$ in which $s = \max\{k_1, m_1, k_2, m_2, \dots, k_L, m_L\}$.

The results generalize recent work in the literature for equations of the form

$$y_n = f(y_{n-k_1}, y_{n-m_1}).$$

Sung Kyu Choi

MR2884324 39A30
Hamza, Alaa E. (ET-CAIRS; Cairo);
Ahmed, A. M. [**Ahmed El-Kennib, Ahmed Mohamed**[2]]
(SAR-RYAD-NCC; Riyadh);
Youssef, A. M. [**Youssef, Amr Mohamed**] (SAR-RYAD-NCC; Riyadh)
On the recursive sequence $x_{n+1} = \frac{a+bx_n}{A+Bx_{n-1}^k}$. (English summary)
Arab J. Math. Sci. **17** (2011), *no.* 1, 31–44.
The authors study the rational difference equation

$$x_{n+1} = \frac{a+bx_n}{A+Bx_{n-1}^k}, \quad n \geq 0, \tag{1}$$

with positive parameters $a, b, A, B \in (0, \infty)$ and nonnegative initial conditions $(x_{-1}, x_0) \in [0, \infty)^2$. They determine the boundedness of solutions of the above equation. Then they use their results to demonstrate the global stability for a range of the parameters. The convergence arguments rely heavily on the work in [K. A. Cunningham et al., Nonlinear Anal. **47** (2001), no. 7, 4603–4614; MR1975854 (2004d:39009); R. DeVault et al., Nonlinear Anal. **47** (2001), no. 7, 4743–4751; MR1975867 (2004d:39010)]. One should note that many of the convergence results uses hypotheses similar to the hypotheses of the well-known m-M theorem (see [M. R. S. Kulenović, G. Ladas and W. S. Sizer, Math. Sci. Res. Hot-Line **2** (1998), no. 5, 1–16; MR1623643 (99g:39008); M. R. S. Kulenović and O. Merino, Discrete Contin. Dyn. Syst. Ser. B **6** (2006), no. 1, 97–110; MR2172197 (2006g:39029); G. Lugo and F. J. Palladino, J. Difference Equ. Appl. **18** (2012), no. 5, 941–945, doi:10.1080/10236198.2011.555764] for more information regarding the m-M theorem).

The authors begin the article by discussing the special case of equation (1) for $k = 1$. This special case is known as "the y2k equation" and has been studied by numerous authors. The latest reference to this topic provided in the paper is [J. H. Jaroma, in *Proceedings of the First International Conference on Difference Equations (San Antonio, TX, 1994)*, 283–295, Gordon and Breach, Luxembourg, 1995; MR1678718]; however, there has been subsequent work on the problem culminating in the complete description of the qualitative behavior of "the y2k equation" in [O. Merino, J. Difference Equ. Appl. **17** (2011), no. 1, 33–41; MR2753043 (2011k:39031)]. Notably, [O. Merino, op. cit.] was published nearly simultaneously with the publication of this article so it is understandable that the authors were not aware of this recent work.

Frank J. Palladino

MR2860634 39A30 45P05 92D25
Zhang, Yuxiang (3-NF; St. John's, NL); **Zhao, Xiao-Qiang** (3-NF; St. John's, NL)
Bistable travelling waves in competitive recursion systems. (English summary)
J. Differential Equations **252** (2012), *no.* 3, 2630–2647.
The authors study the discrete-time two-species competition system

$$\begin{aligned} p_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_1)p_n(x-y)}{1+r_1(p_n(x-y)+a_1q_n(x-y))} k_1(y)dy, \\ q_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_2)q_n(x-y)}{1+r_2(q_n(x-y)+a_2p_n(x-y))} k_2(y)dy, \end{aligned} \tag{1}$$

where $p_n(x)$ and $q_n(x)$ are the population densities of two species at time n and position x; $k_1(y)$ and $k_2(y)$ are the dispersal kernels of the two species which satisfy

$$\int_{\mathbb{R}} k_i(y)dy = 1 \quad \text{and} \quad \int_{\mathbb{R}} e^{\alpha y} k_i(y)dy < \infty,$$

for all $\alpha \in \mathbb{R}$, $i = 1, 2$. Introducing the change of variables, $u_n = p_n$ and $v_n = 1 - q_n$, the equations in (1) are changed to

$$(2) \qquad \begin{aligned} u_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_1)u_n(x-y)}{1+r_1(u_n(x-y)+a_1(1-v_n(x-y)))} k_1(y)dy, \\ v_{n+1}(x) &= \int_{\mathbb{R}} \frac{a_2 r_2 u_n(x-y)+v_n(x-y)}{1+r_2((1-v_n(x-y))+a_2 u_n(x-y))} k_2(y)dy. \end{aligned}$$

Using the upper and lower solutions method, the spreading speeds of monostable systems and the monotone semiflow approach, the authors obtain the existence and global stability of bistable travelling waves of (2). As an example, they consider (2) with $a_1 = 6/5$, $a_2 = 10$, $r_1 = 1/9$, $r_2 = 1/10$, $k_1 = (1/\sqrt{2\pi})\exp(-y^2/2)$, and $k_2 = (1/\sqrt{4\pi})\exp(-y^2/4)$.
Tingxiu Wang

MR2816829 39A70 39A12 41A55

Azamov, S. S. (UZ-AOS-IFT; Tashkent)

The discrete analogue of the differential operator $\frac{d^{2m}}{dx^{2m}} + \frac{d^{2m-2}}{dx^{2m-2}} + \frac{d^{2m-4}}{dx^{2m-4}}$. (Russian. English and Uzbek summaries)

Uzbek. Mat. Zh. **2011,** *no.* 2, 15–28.

Summary (reviewer's translation): "In this paper, the discrete analogue of the operator

$$\frac{d^{2m}}{dx^{2m}} + \frac{d^{2m-2}}{dx^{2m-2}} + \frac{d^{2m-4}}{dx^{2m-4}}$$

is constructed. It is used to derive optimal quadrature formulas." *Fozi M. Dannan*

39B Functional equations and inequalities

MR2867208 39B12 26A18 37E05 39B22

Reich, Ludwig (A-GRAZ; Graz); **Smítal, Jaroslav** (CZ-SIL-IM; Opava);
Štefánková, Marta (CZ-SIL-IM; Opava)

Functional equation of Dhombres type in the real case. (English summary)

Publ. Math. Debrecen **78** (2011), *no.* 3-4, 659–673.

In this paper the authors, using some notions and results from dynamical systems, continue [see, for instance, L. Reich and J. Smítal, J. Difference Equ. Appl. **15** (2009), no. 11-12, 1179–1191; MR2569140 (2011c:39026)] the interesting investigations of solutions f of the generalized Dhombres functional equation

$$(1) \qquad f(xf(x)) = \varphi(f(x)), \quad x \in (0, \infty),$$

where φ is a given continuous map.

Under the assumption that the set of periodic points $\mathrm{Per}(\varphi)$ equals the set of fixed points $\mathrm{Fix}(\varphi)$ and $f\colon (0,\infty) \to (0,1]$ is a non-constant continuous solution of equation (1), it is shown that $\mathrm{Fix}(\varphi) \cap R_f \in \{\varnothing, \{1\}, \{p,1\}, \{p\}\}$, where R_f denotes the range of f and $p \in (0,1)$. If $\mathrm{Fix}(\varphi) \cap R_f = \varnothing$, then R_f is an open interval with endpoints in $\mathrm{Fix}(\varphi)$. If $\mathrm{Fix}(\varphi) \cap R_f = \{1\}$, then $R_f = (u,1]$ for some $u \in (0,1)$. If $\mathrm{Fix}(\varphi) \cap R_f = \{p,1\}$, then there is a $u \in (0,1)$ such that $R_f = (u,1]$ or $R_f = [u,1]$, and p is a boundary (i.e., $p = u$) or interior point of R_f. If $\mathrm{Fix}(\varphi) \cap R_f = \{p\}$, then R_f is a non-closed interval, p is a boundary or interior point of R_f, and $\overline{R_f} \setminus R_f \subseteq \mathrm{Fix}(\varphi)$. Moreover, every type of the behavior described above can be realized by suitable choice for φ and a corresponding continuous solution f of equation (1).

The results obtained solve one of the problems from the mentioned paper of Reich and Smítal. They also give credibility to the conjecture that the answers to some other problems posed there are negative. *Krzysztof Ciepliński*

MR2866781 39B12

Zhong, Ji Yu (PRC-ZJN; Zhanjiang);
Li, Xiao Pei [Li, Xiao Pei²] (PRC-ZJN; Zhanjiang)

Set-valued solutions of an iterative equation. (Chinese. English and Chinese summaries)

Acta Math. Sci. Ser. A Chin. Ed. **31** (2011), *no.* 4, 970–977.

In this paper, the authors discuss the existence and uniqueness of the set-valued solutions of the following iterative functional equation for f:

$$g(x) - \lambda_2 f^2(x) = \lambda_1 f(x),$$

where g is a known function. The dependence of f on g is also discussed. *Dilian Yang*

MR2772039 39B22

Eshaghi Gordji, M. [Eshaghi Gordji, Madjid] (IR-SEMN-M; Semnan);
Ramezani, M. [Ramezani, Maryam] (IR-SEMN-M; Semnan)

Erdös problem and quadratic equation. (English summary)

Ann. Funct. Anal. **1** (2010), *no.* 2, 64–67.

P. Erdős [Problem 310, Colloq. Math. **7** (1959/1960), no. 2, 311] stated the following problem: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for almost all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Does there exist an additive function $F\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = F(x)$ for almost all $x \in \mathbb{R}$?

The authors solve an analogous problem for quadratic mappings on $\mathbb{R}$.

The reader may also wish to see the paper by R. Ger [Colloq. Math. **24** (1971/72), 95–101; MR0306755 (46 #5877)]. *Justyna Sikorska*

MR2855066 39B52 20B30

Lê, Công-Trình; Thái, Trung-Hiêu

Jensen's functional equation on the symmetric group S_n. (English summary)

Aequationes Math. **82** (2011), *no.* 3, 269–276.

Summary: "Two natural extensions of Jensen's functional equation on the real line are the equations $f(xy) + f(xy^{-1}) = 2f(x)$ and $f(xy) + f(y^{-1}x) = 2f(x)$, where f is a map from a multiplicative group G into an abelian additive group H. In a series of papers [Aequationes Math. **39** (1990), no. 1, 85–99; MR1044167 (91c:39014); Aequationes Math. **58** (1999), no. 3, 311–320; MR1715402 (2000j:39022); Aequationes Math. **62** (2001), no. 1-2, 143–159; MR1849146 (2002e:39082)], C. T. Ng solved these functional equations for the case where G is a free group and the linear group $GL_n(R)$, $R = \mathbb{Z}, \mathbb{R}$, is a quadratically closed field or a finite field. He also mentioned, without a detailed proof, in the above papers and in [Aequationes Math. **70** (2005), no. 1-2, 131–153; MR2167991 (2006c:39042)] that when G is the symmetric group S_n, the group of all solutions of these functional equations coincides with the group of all homomorphisms from $(S_n, \cdot)$ to $(H, +)$. The aim of this paper is to give an elementary and direct proof of this fact."

MR2832087 39B62 39B72

Fechner, Włodzimierz (PL-SILS-IM; Katowice)

A note on alienation for functional inequalities. (English summary)

J. Math. Anal. Appl. **385** (2012), *no.* 1, 202–207.

The author generalizes a result of C. Hammer [Aequationes Math. **45** (1993), no. 2-3, 297–299; MR1212394 (94a:39021)] concerning the alienation phenomenon of the inequality

$$f(x+y)+f(xy)\geq f(x)+f(y)+f(x)f(y).$$

The main result of the paper is described in Theorem 1 and can be read as follows: Assume that $f\colon\mathbb{R}\to\mathbb{R}$ is differentiable at 0 and continuous and $b,c\in\mathbb{R}$ are arbitrary nonzero constants. Then f satisfies $f(x+y)+bf(xy)\geq f(x)+f(y)+cf(x)f(y)$, $x,y\in\mathbb{R}$, jointly with $f(0)=0$ if and only if $f=0$ or $f(x)=\frac{ac-b}{ac^2}(e^{acx}-1)+\frac{b}{c}x$, $x\in\mathbb{R}$, where $a=f'(0)$ and moreover $ac>0$ and $(ac-b)bc\geq 0$.

Special cases (when $c=0$ or $b=0$) are treated separately. *Liviu Cădariu*

MR2856028 39B82 39B12 39B52

Chudziak, J. (PL-RZSZ; Rzeszów)

Approximate dynamical systems on interval. (English summary)

Appl. Math. Lett. **25** (2012), *no.* 3, 532–537.

Let I be an open interval. The main result of the paper states that if $H\colon I\times\mathbb{R}\to I$ satisfies

$$|H(H(x_0,s),t)-H(x_0,s+t)|\leq\delta$$

for all $s,t\in\mathbb{R}$, $\delta>0$, and an $x_0\in I$ such that $H(x_0,\cdot)$ is a continuous surjection of $\mathbb{R}$ onto I, then there exists a function $F\colon I\times\mathbb{R}\to I$ such that $F(x,0)=x$, $x\in I$,

$$F(F(x,s),t)=F(x,s+t),\quad x\in I,\ s,t\in\mathbb{R},$$

and $|H(x,t)-F(x,t)|\leq 9\delta$, $x\in I$, $t\in\mathbb{R}$.

Under some additional assumptions, it is shown that the translation equation above is stable in the Hyers–Ulam sense, which can be viewed as a result on the existence of a dynamical system which is uniformly close to a given function H. *Zygfryd Kominek*

MR2906480 39B82 39B52 46F99

Chung, Jaeyoung (KR-KUN; Kunsan); **Chang, Jeongwook**

On the stability of the Pexider equation in Schwartz distributions via heat kernel. (English summary)

Honam Math. J. **33** (2011), *no.* 4, 467–485.

Summary: "We consider the Hyers-Ulam-Rassias stability problem

$$\|u\circ A-v\circ P_1-w\circ P_2\|\leq\varepsilon(|x|^p+|y|^p)$$

for the Schwartz distributions u,v,w, which is a distributional version of the Pexider generalization of the Hyers-Ulam-Rassias stability problem

$$|f(x+y)-g(x)-h(y)|\leq\varepsilon(|x|^p+|y|^p),\ x,y\in\mathbb{R}^n,$$

for the functions $f,g,h\colon\mathbb{R}^n\to\mathbb{C}$."

MR2907288 39B82 16W25 17A42 39B52

Gordji, M. Eshaghi [**Eshaghi Gordji, Madjid**] (IR-SEMN-M; Semnan);
Rassias, J. M. [**Rassias, John Michael**] (GR-UATH-MIP; Aghia Paraskevi);
Ghaemi, M. B. [**Ghaemi, Mohammad Bagher**] (IR-IUST; Tehran);
Alizadeh, B. [**Alizadeh, Badrkhan**] (IR-TCT; Tabriz)

Approximate quaternary Jordan derivations on Banach quaternary algebras. (English summary)

Bull. Inst. Math. Acad. Sin. (*N.S.*) **6** (2011), *no.* 3, 361–375.

Summary: "We show that a quaternary Jordan derivation on a quaternary Banach algebra associated with the equation

$$f\left(\frac{x+y+z}{4}\right)+f\left(\frac{3x-y-4z}{4}\right)+f\left(\frac{4x+3z}{4}\right)=2f(x).$$

is satisfied in generalized Hyers-Ulam stability."

MR2906417 39B82 39B52 46S40

Jang, Sun Young (KR-ULS; Ulsan);
Park, Choonkil [Park, Chun-Gil] (KR-HAN; Seoul)

Fuzzy stability of a functional equation related to inner product spaces. (English summary)

Hacet. J. Math. Stat. **40** (2011), *no.* 5, 711–723.

Summary: "The fuzzy stability problems for the Cauchy quadratic functional equation and the Jensen quadratic functional equation in fuzzy Banach spaces have been investigated by Moslehian *et al.* Th. M. Rassias introduced the following equality

$$\sum_{i,j=1}^{m}\|x_i-x_j\|^2=2m\sum_{i=1}^{m}\|x_i\|^2,\quad \sum_{i=1}^{m}x_i=0,$$

for a fixed integer $m\geq 3$. By the above equality, we define the following functional equation

$$\sum_{i,j=1}^{m}f(x_i-x_j)=2m\sum_{i=1}^{m}f(x_i),\quad \sum_{i=1}^{m}x_i=0. \tag{0.1}$$

In this paper, we prove the generalized Hyers-Ulam stability of the functional equation (0.1) in fuzzy Banach spaces."

MR2855773 39B82 39B22 47B39

Popa, Dorian (R-TUCN; Cluj-Napoca); **Raşa, Ioan** (R-TUCN; Cluj-Napoca)

The Fréchet functional equation with application to the stability of certain operators. (English summary)

J. Approx. Theory **164** (2012), *no.* 1, 138–144.

Summary: "We present a new approach to the classical Fréchet functional equation. The results are applied to the study of Hyers-Ulam stability of Bernstein-Schnabl operators."

Belaid Bouikhalene

MR2870891 39B82 26A18 39B12

Przebieracz, Barbara (PL-SILS-IM; Katowice)

On the stability of the translation equation and dynamical systems. (English summary)

Nonlinear Anal. **75** (2012), *no.* 4, 1980–1988.

Summary: "In this paper we prove the stability of the functional equation $F(s,F(t,x))=F(s+t,x)$ in the class of functions $F:\mathbb{R}\times I\to I$, which are continuous with respect to each variable, and where $I\subset\mathbb{R}$ is a real interval. We also discuss the stability in the sense of Hyers-Ulam of dynamical systems on I. We show some properties of δ-approximate solutions of the translation equation on a real interval." *Andrzej Mach*

40 SEQUENCES, SERIES, SUMMABILITY

40A Convergence and divergence of infinite limiting processes

MR2884892 40A05 26D15 40G05 44A60
Bennett, Grahame (1-IN; Bloomington, IN)
Mercer's inequality and totally monotonic sequences. (English summary)
Math. Inequal. Appl. **14** (2011), *no.* 4, 747–775.
The elegant Mercer inequality

$$\frac{1}{2^n}\sum_{k=0}^{n} C_n^k x_k \le \frac{1}{n+1}\sum_{k=0}^{n} x_k \quad (n=0,1,\dots)$$

for all convex sequences $\{x_k\}$ is extended in the case of comparison of arbitrary Hausdorff means; namely, necessary and sufficient conditions are indicated to compare these means.
Vitaliy A. Andrienko

MR2906514 40A30 40G05
Leindler, László (H-SZEG-B; Szeged)
Slight extensions of four celebrated Tandori theorems. (English summary)
Acta Sci. Math. (*Szeged*) **77** (2011), *no.* 3-4, 445–450.
In this paper some extensions of Tandori theorems on the convergence and summability of orthogonal series are given. Tandori proved four theorems on convergence and C_1-summability (C_1 is the first-order summability method of Cesàro) of orthogonal series $\sum_{n=0}^{\infty} c_n\varphi_n(x)$, where $\{\varphi_n(x)\}$ is an orthonormal system and (c_n) is a sequence of positive numbers. In these theorems, besides other assumptions, Tandori supposed that (c_n) satisfies some monotonicity condition. In this paper the author proves that all assertions of all above-mentioned Tandori theorems remain valid if the monotonicity condition for (c_n) is replaced by a weaker condition, called the rest bounded variation condition, defined as follows: a sequence $c := (c_n)$ of positive numbers tending to zero is said to be of rest bounded variation, or briefly $c \in \mathrm{RBVS}$, if there exists a constant $K(c)$ only depending on c such that $\sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(c)c_m$ for all m. *Ants Aasma*

MR2861502 40A35 54C35
Caserta, Agata (I-NAPL2; Caserta)**; Di Maio, Giuseppe** (I-NAPL2; Caserta)**;**
Kočinac, Ljubiša D. R. (SE-NISSM-NDM; Niš)
Statistical convergence in function spaces. (English summary)
Abstr. Appl. Anal. **2011,** *Art. ID* 420419, 11 *pp.*
What must be added to pointwise convergence to the limit of a sequence of functions in order to preserve the continuity of the limit function? Several papers have been published about this question. This paper covers this idea with more general notions of convergence than the usual one, including statistical Alexandrov convergence and statistical Arzelà convergence. The authors state a new notion, *statistically strong Arzelà convergence.* They end with a similar study on statistical exhaustiveness. *Pedro Garrancho*

MR2871670 40A35 26E50 46S40
Debnath, Pradip (6-NITS-M; Silchar)
Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces. (English summary)
Comput. Math. Appl. **63** (2012), *no.* 3, 708–715.
Let $\theta = (k_r)$ be an increasing sequence of positive integers such that $k_0 = 0$ and $h_r :=$

$(k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Then $\theta = (k_r)$ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$. A number sequence (x_k) is lacunary statistically convergent to L provided that for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\big|\{k \in I_r \colon |x_k - L| \geq \varepsilon\}\big| = 0,$$

where the vertical bars indicate the number of elements of the enclosed set. Lacunary statistical convergence has already been studied in some detail by J. A. Fridy and the reviewer [Pacific J. Math. **160** (1993), no. 1, 43–51; MR1227502 (94j:40014); J. Math. Anal. Appl. **173** (1993), no. 2, 497–504; MR1209334 (95f:40004)]. In the paper under review lacunary ideal convergence in intuitionistic fuzzy normed linear spaces is studied. Lacunary I-limit points and lacunary I-cluster points are also considered.

Cihan Orhan

MR2856015 40A35 26E50

Hazarika, Bipan

Fuzzy real valued lacunary I-convergent sequences. (English summary)

Appl. Math. Lett. **25** (2012), *no.* 3, 466–470.

In this paper the concept of a lacunary I-convergent sequence of fuzzy real numbers is introduced and some basic properties of this concept are studied. *İbrahim Çanak*

MR2859625 40A35 46A70

Mursaleen, M. [Mursaleen, Mohammad] (6-ALIG; Aligarh)**;**
Alotaibi, Abdullah [Alotaibi, Abdullah M.] (SAR-ABD; Jeddah)

On $\mathcal{I}$-convergence in random 2-normed spaces. (English summary)

Math. Slovaca **61** (2011), *no.* 6, 933–940.

In the paper under review, the authors define and study the notion of ideal convergence in a random 2-normed space. Let I be a nontrivial ideal of $\mathbb{N}$, let X be a linear space of dimension greater than one and let $(X, F, *)$ be a random 2-normed space. A sequence $x = (x_k)$ is said to be I-convergent to ξ in $(X, F, *)$ if for every $\varepsilon > 0$, $\theta \in (0, 1)$ and nonzero $z \in X$ one has $\{k \in \mathbb{N}\colon F(x_k - \xi, z; \varepsilon) \leq 1 - \theta\} \in I$ or equivalently $\{k \in \mathbb{N}\colon F(x_k - \xi, z; \varepsilon) > \theta\} \in F(I)$. *Kamil Demirci*

40C General summability methods

MR2895962 40C05 40B05 46A45

Natarajan, P. N.

The Schur and Steinhaus theorems for 4-dimensional matrices in ultrametric fields. (English summary)

Comment. Math. **51** (2011), *no.* 2, 203–209.

If every Cauchy double sequence of an ultrametric normed linear space X converges to an element of X, then X is said to be a double sequence complete (ds-complete) space. Let K be a ds-complete, nontrivially valued, ultrametric field with valuation $|\cdot|$. The author proves, in the paper under review, that a necessary and sufficient condition for a four-dimensional infinite matrix $A = (a_{m,n,k,l})$ to map double sequences in ds-complete bounded sequences into double sequences in ds-complete convergent sequences is that it satisfies

$$\lim_{k+l\to\infty} a_{m,n,k,l}=0,\quad m,n=0,1,2,\dots,$$
$$\lim_{m+n\to\infty}\sup_{k,l\ge 0}|a_{m+1,n,k,l}-a_{m,n,k,l}|=0,$$
$$\lim_{m+n\to\infty}\sup_{k,l\ge 0}|a_{m,n+1,k,l}-a_{m,n,k,l}|=0,$$

where the entries of double sequences and four-dimensional matrices are in K. It is also shown that a four-dimensional infinite matrix cannot be both a regular and a Schur matrix. *Cihan Orhan*

MR2858194 40C05

Savaş, E. **[Savaş, Ekrem]** (TR-ITICU-M; Istanbul)**;**
Patterson, R. F. **[Patterson, Richard F.]** (1-NFL-MS; Jacksonville, FL)

Some double lacunary sequence spaces defined by Orlicz functions. (English summary)

Southeast Asian Bull. Math. **35** (2011), *no.* 1, 103–110.

A lacunary $\theta=(k_r)$; $k=0,1,\dots$, where $k_0=0$, is an increasing sequence of non-negative integers with $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$.

A double sequence $\theta_{r,s}=\{(k_r,l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that $k_0=0$, $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$ and $l_0=0$, $\overline{h}_s=l_s-l_{s-1}\to\infty$ as $s\to\infty$.

By the convergence of a double sequence the authors mean the convergence in the Pringsheim sense, briefly denoted as P-convergence.

In the paper the authors introduce a new concept of lacunary strong P-convergence with respect to an Orlicz function which is compared to other summability methods. They also examine some properties of the resulting sequence space and show that if a sequence is lacunary strong P-convergent with respect to an Orlicz function, then it is $s_{\theta_{r,s}}$-convergent. Here, an Orlicz function $M\colon[0,\infty)\to[0,\infty)$ is a continuous, convex, nondecreasing function such that $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$ as $x\to\infty$.

Some notation used throughout the paper is as follows: $k_{r,s}=k_rl_s$, $h_{r,s}=h_r\overline{h}_s$, $q_r=k_r/k_{r-1}$, $\overline{q}_s=l_s/l_{s-1}$, $I_{r,s}=\{(k,l)\colon k_{r-1}<k\le k_r, l_{s-1}<l\le l_s\}$.

One of the main results of the paper reads as follows:

Theorem. Let $\theta_{r,s}=\{k_r,l_s\}$ be a double lacunary sequence with $\liminf_r q_r>1$ and $\liminf_s \overline{q}_s>1$. Then for any Orlicz function M, $[\sigma_{1,1},M,p]\subset[N_{\theta_{r,s}},M,p]$, where

$$[\sigma_{1,1},M,p]=\left\{x\colon\ P\text{-}\lim_{m,n}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left(M\left(\frac{|x_{k,l}-L|}{\rho}\right)\right)^{p_{k,l}}=0,\right.$$
$$\left.\text{for some } L \text{ and } \rho>0\right\}$$

and

$$[N_{\theta_{r,s}},M,p]=\left\{x\colon\ P\text{-}\lim_{r,s}\frac{1}{h_{rs}}\sum_{(k,l)\in I_{r,s}}\left(M\left(\frac{|x_{k,l}-L|}{\rho}\right)\right)^{p_{k,l}}=0,\right.$$
$$\left.\text{for some } L \text{ and } \rho>0\right\}.$$

Daniel Cárdenas-Morales

40D Direct theorems on summability

MR2862058 40D05 46A45
Hazarika, Bipan
Some lacunary difference sequence spaces defined by Musielak-Orlicz functions. (English summary)
Thai J. Math. **9** (2011), *no.* 3, 659–671.

For an increasing integer sequence ξ let N_ξ be the lacunary strongly convergent sequence space, M be an Orlicz function, $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions and $v = (v_k)$ be a sequence of nonzero complex numbers. Writing M_k and $v_k \Delta^m x_k$ instead of M and $\Delta^m x_k$, respectively, the author generalizes the sequence spaces of B. C. Tripathy, S. Mahanta and M. Et [Int. J. Math. Sci. **4** (2005), no. 2, 341–355; MR2334066 (2008e:46011)] to the linear spaces $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\nu$, $\nu \in \{0, 1, \infty\}$, defines a paranorm in $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ and studies various inclusion relations involving these spaces.
Enno Kolk

MR2855995 40D25 40F05
Aasma, Ants (ES-TALL-EC; Tallinn)
Some inclusion theorems for absolute summability. (English summary)
Appl. Math. Lett. **25** (2012), *no.* 3, 404–407.

In this paper the author studies the transforms of absolute summability domains of normal matrices under triangular factorable matrices and considers the Cesàro matrix as an application.
İbrahim Çanak

40E Inversion theorems

MR2859630 40E05 40G10
Çanak, İbrahim; Totur, Ümit; Dik, Mehmet (1-ROCKC; Rockford, IL)
On Tauberian theorems for (A, k) summability method. (English summary)
Math. Slovaca **61** (2011), *no.* 6, 993–1001.

Let $u := (u_n)$ and $\alpha := (\alpha_n)$ be sequences of real numbers. If $u_n = \alpha_n + \sum_{j=1}^n \alpha_j/j$, then the authors state that the sequence u is regularly generated by the sequence α and α is called a generator of u.

Define $\sigma_n^{(0)}(u) = u_n$ and for each integer $k \geq 1$ define $\sigma_n^{(k)}(u) = (n + 1)^{-1} \sum_{j=0}^n \sigma_j^{(k-1)}(u)$. A sequence u is said to be (A, k) summable to sum s, if

$$\lim_{x \to 1-} (1 - x) \sum_{n=0}^{\infty} x^n \sigma_n^{(k)}(u)$$

exists and equals s. Hence $(A, 0)$ is the Abel method. In this paper 9 Tauberian theorems (4 from these are below) for $A(k)$ summability are proved with Tauberian conditions in terms of regularly generated sequences.

Let $\omega_n^{(0)}(u) = n\Delta u_n$, where $\Delta u_n = u_n - u_{n-1}$ and $u_{=1} = 0$. For integer order $m \geq 1$ is defined inductively by

$$\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u)).$$

Let $\mu = (\mu_n)$ with $\mu_n = (n+1)\omega_n^{(m)}(u)$. Let $\Delta\mu = (\Delta\mu_n)$.

A sequence u is moderately oscillating if for $\lambda > 1$,

$$\overline{\lim}_n \max_{n+1 \leq j \leq [\lambda n]} |u_j - u_n| < \infty.$$

Theorem 3.1. Let u be (A,k) summable to s. If the generator of $\Delta\mu$ is moderately oscillating, then u converges to s.

Theorem 3.4. Let u be (A,k) summable to s. If the sequence whose generator is $(\Delta\mu_n)$ is moderately oscillating, then u converges to s.

Theorem 3.5. Let u be (A,k) summable to s. If the sequence whose generator is $(\sigma_n^{(1)}(\Delta\mu))$ is moderately oscillating, then u converges to s.

Theorem 3.6. Let u be (A,k) summable to s. If the sequence whose generator is $(\sigma_n^{(2)}(\Delta\mu))$ is moderately oscillating, then u is bounded. *S. Baron*

40G Special methods of summability

MR2869728 40G05 30B10

Estrada, Ricardo (1-LAS; Baton Rouge, LA); **Vindas, Jasson** (B-GHNT; Ghent)

Exterior Euler summability. (English summary)

J. Math. Anal. Appl. **388** (2012), *no.* 1, 48–60.

Summary: "We define and study a summability procedure that is similar to Euler summability but applied in the exterior of a disc, not in the interior. We show that the procedure is well defined and that it actually has many interesting properties. We use these ideas to study some problems of analytic continuation and to give a construction of the convex support of an analytic functional. We also use this exterior summability to study Mittag-Leffler developments." *İbrahim Çanak*

MR2867470 40G05 46S10

Natarajan, P. N.

Some properties of regular Nörlund methods in non-Archimedean fields. (English summary)

Indian J. Math. **53** (2011), *no.* 2, 287–299.

Let K be a complete nontrivially valued non-Archimedean field. For any sequence (p_n) from K the Nörlund method (N,p_n) is defined by the infinite matrix with entries p_{n-k}/P_n if $k \le n$ and 0 otherwise, where $P_n = \sum_{k=0}^n p_n$. The author assumes that $|p_j| < |p_0|$ for $j = 1, 2, \ldots$. He shows that (N,p_n) maps any null sequence from K to a null sequence if and only if $(p_n)_{n\ge 0} \in c_0$, which is also the characterizing condition for regularity of (N,p_n). Let N_{c_0} denote the set of all such Nörlund methods. After some inclusion theorems the author proves that N_{c_0} is an ordered abelian semigroup when using convolution of the (p_n) as the semigroup operation and the relation of being strictly c_0-weaker as the order relation. *Karl-Goswin Grosse-Erdmann*

41 APPROXIMATIONS AND EXPANSIONS

MR2894783 41-02 32A55 32E30 40A35 41A36

Anastassiou, George A. (1-MEMP; Memphis, TN);
Duman, Oktay (TR-ETU-M; Ankara)

★**Towards intelligent modeling: statistical approximation theory.**

Intelligent Systems Reference Library, 14.
Springer-Verlag, Berlin, 2011. *xvi*+234 *pp.* €106.95. *ISBN* 978-3-642-19825-0

The concept of statistical convergence was introduced by H. Fast [Colloquium Math. **2** (1951), 241–244 (1952); MR0048548 (14,29c)] about sixty years ago, but it has recently become an active area of research. Over the years this method of convergence has been examined in measure theory, approximation theory, fuzzy logic theory, summability theory, number theory, optimization and so on. In 2002, statistical convergence was first studied in the Korovkin-type approximation theory by A. D. Gadzhiev and the reviewer [Rocky Mountain J. Math. **32** (2002), no. 1, 129–138; MR1911352 (2003f:41041)]. This book mainly combines two directions: the statistical Korovkin theory and the fuzzy Korovkin theory. Recall that the latter topic was considered by the first author of this book in 2005 [Studia Univ. Babeş-Bolyai Math. **50** (2005), no. 4, 3–10; MR2247526]. This is the first monograph in statistical approximation theory and fuzziness, which contains mostly the recent joint works of the authors on the topics mentioned above. The book consists of eighteen chapters which are self-contained and include many significant applications. Several advanced courses can be taught out of the book.

Chapter 1 is devoted to an introduction and contains some historical comments, along with some notations, definitions, and important results on the concept of statistical convergence. In Chapter 2, the authors show that it is possible to approximate (in the statistical sense) a function by a sequence of bivariate smooth Picard singular integral operators which need not be positive in general. Chapter 3 deals with a similar problem on a sequence of bivariate smooth Gauss-Weierstrass singular operators. Chapters 4 and 5 present statistical L_p-approximation properties of the Picard and Gauss-Weierstrass operators. In Chapter 6, the authors use the notion of A-statistical convergence, where A is a non-negative regular summability matrix, to get some statistical variants of Baskakov's results in the Korovkin-type approximation theory. Chapter 7 deals with some statistical weighted approximation results concerning derivatives of functions. In Chapter 8, they obtain an approximation theorem which is a nontrivial generalization of Baskakov's result regarding the approximation to periodic functions by a general class of linear operators. In Chapter 9, the positivity condition of linear operators is weakened. Chapter 10 deals with strong Korovkin-type approximation theorems for stochastic processes, and their multivariate cases are discussed in Chapter 11. In Chapter 12, some statistical approximation theorems including fractional derivatives of functions are proved. In Chapter 13, some approximation results, regarding the fractional derivatives of trigonometric functions, are studied. Chapter 14 presents the statistical approximation results of fuzzy positive linear operators while Chapter 15 discusses the corresponding error estimations. In Chapter 16, in order to approximate (in the fuzzy sense) to high order continuously differentiable functions, the authors mainly use statistical convergence along with fuzzy-valued operators. In the last two chapters, they consider the complex-valued Picard and Gauss-Weierstrass integral operators and study their statistical approximation process and some geometric properties. A complete list of references and a useful index are presented at the end of the book.

This monograph is recommended to graduate students and researchers, in both pure and applied mathematics, specializing in summability and approximation theories.

Cihan Orhan

MR2808753 41A05

Khudyakov, A. P. (BE-AOS; Minsk)

Interpolation polynomials of Hermite-Birkhoff type with respect to particular Chebyshev systems of functions. (Russian. English and Russian summaries)

Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz.-Mat. Navuk **2010,** *no.* 4, 29–36, 125.

An interpolation problem of Hermite-Birkoff type is considered in this paper. The author constructs trigonometric polynomials, rational and exponential functions satisfying the general interpolating Hermite-Birkoff conditions. *Zuhe Shen*

MR2805916 41A05 41A10 65D05

Palacios-Quiñonero, F. (E-UPB-A3M; Manresa);
Rubió-Díaz, P. [Rubió i Díaz, Pedro] (E-UPB-A3M; Manresa);
Díaz-Barrero, J. L. [Díaz-Barrero, José Luis] (E-UPB-A3M; Manresa);
Rossell, J. M. [Rossell i Garriga, Josep Maria] (E-UPB-A3M; Manresa)

Order regularity for Birkhoff interpolation with lacunary polynomials. (English summary)

Math. Æterna **1** (2011), *no.* 3-4, 129–135.

Using certain definitions and earlier results, the authors mainly prove the following result:

Theorem 1 (resp. 1′). Let E be an interpolation matrix that satisfies the Pólya K-condition and the upper (resp. lower) K-inclusive property. If E contains no odd upper (resp. lower) K-supported sequences, then E is order K-regular on $[0,\infty)$ (resp. $(-\infty,0]$).

The authors claim these two theorems are generalizations of Atkinson and Sharma's theorem (1969).

Based on these theorems, sufficient conditions for the order regularity problem in Birkhoff interpolation with lacunary polynomials $\sum_{j=1}^n a_j(x^{k_j}/k_j!)$ are established. The proofs are quite technical. The generalized Pólya condition is used to characterize conditionally regular interpolation matrices, which is simpler than Pólya's condition for algebraic Birkhoff interpolation and it is equivalent to Pólya's condition when $k_j = j-1$.

The reader should be warned about numerous typos. *Kewal Krishna Mathur*

MR2843040 41A05 41A30 41A63 47B06 65D05

Pazouki, Maryam (D-GTN-N; Göttingen); **Schaback, Robert** (D-GTN-N; Göttingen)

Bases for kernel-based spaces. (English summary)

J. Comput. Appl. Math. **236** (2011), *no.* 4, 575–588.

This paper provides a general framework for the discussion of alternate bases for computation on reproducing kernel Hilbert spaces with kernel K. It is well known that the standard basis consisting of "translates" $K(\cdot,x_j)$ of the kernel to a finite set of centers $\{x_1,\dots,x_N\}\subseteq\Omega$ often is unstable, i.e., it leads to an ill-conditioned system matrix $A=(K(x_j,x_k))_{1\le j,k\le N}$. On the other hand, the function space spanned by the standard basis has perfectly good approximation properties. The situation of piecewise polynomial splines, for which truncated power functions are a much less stable basis than B-splines, may serve as a well-known illustration of this phenomenon.

The kernel bases discussed in this paper are all interpreted in terms of standard matrix factorizations of the system matrix A, such as Cholesky (leading to a "Newton" basis), QR and SVD. In contrast to these data-dependent bases the authors also discuss the data-independent basis of eigenfunctions of the kernel. An interpretation of the SVD basis as a discretized (via numerical integration techniques) version of the eigenfunction basis is also provided.

The "Newton" basis receives special attention. Two iterative center selection algorithms are discussed for it. The first is a generic version that does not take into account the given data. This algorithm is akin to a column-pivoted Cholesky factorization and leads to a center distribution (design) that is relatively uniform and uses the power function as a point selection criterion. The second algorithm is a version of orthogonal matching pursuit and takes into account the given function values and therefore adapts to special features in the function. For example, centers will be clustered along derivative singularities as illustrated numerically in the paper.

Other theoretical aspects of the various bases, such as duality or orthonormality, are also easily accessible through the framework provided here. For example, the standard basis of kernel translates and the Lagrange basis are shown to be dual to each other, or orthonormal bases (in the reproducing kernel Hilbert space sense) are shown to be self-dual. The "Newton" basis is shown to be such a self-dual orthonormal basis.

Gregory E. Fasshauer

MR2853345 41A05 41A28

Xiulian, Wang (PRC-TINU; Tianjin); **Jingrui, Ning** (PRC-TINU; Tianjin)

Mean convergence rate of derivatives by Lagrange interpolation on Chebyshev grids. (English summary)

Discrete Dyn. Nat. Soc. **2011,** *Art. ID* 503561, 22 *pp.*

Mean convergence of Lagrange interpolation based on the zeros of orthogonal polynomials has been studied for at least 70 years (see the papers by P. G. Nevai and Y. Xu [J. Approx. Theory **77** (1994), no. 3, 282–304; MR1279277 (95d:41004)], P. Pottinger [J. Approx. Theory **23** (1978), no. 3, 267–273; MR0505750 (80b:41003)], P. Vértesi and Xu [Acta Math. Hungar. **69** (1995), no. 3, 185–210; MR1353012 (96j:41009)], G. Mastroianni and Nevai [J. Comput. Appl. Math. **34** (1991), no. 3, 385–396; MR1102592 (92b:41007)], J. Szabados and A. K. Varma [in *A tribute to Paul Erdős*, 397–404, Cambridge Univ. Press, Cambridge, 1990; MR1117033 (92g:41008)], Szabados and Vértesi [J. Comput. Appl. Math. **43** (1992), no. 1-2, 3–18; MR1193291 (93j:65016)] and many others). The authors consider the rate of mean convergence of derivatives by Lagrange interpolation operators based on Chebyshev nodes. They obtain some interesting estimates of the derivatives in terms of the error of best approximation by polynomials (in Theorem 1.1 and Theorem 1.2), and in this way give nice generalizations to an earlier result due to Y. F. Du and G. Q. Xu in [Numer. Math. J. Chinese Univ. **32** (2010), no. 1, 47–55; MR2682232 (2011f:41012)]. *Dan A. Bărbosu*

MR2869036 41A10 41A25 41A63 65N15 65N35

Chernov, Alexey (D-BONN-CM; Bonn)

Optimal convergence estimates for the trace of the polynomial L^2-projection operator on a simplex. (English summary)

Math. Comp. **81** (2012), *no.* 278, 765–787.

The author studies convergence of the L^2-projections onto the spaces of polynomials up to degree p defined on a simplex in $\mathbb{R}^d$, $d \geq 2$, as $p \to \infty$. Based on the collapsed coordinate transform and the expansion into various polynomial bases involving Jacobi polynomials, the optimal error estimates are established in the case of Sobolev regularity.

Mao Dong Ye

MR2875913 41A15 65D05 65D07

Delgado-Gonzalo, R. [Delgado-Gonzalo, Ricard] (CH-LSNP-BI; Lausanne);
Thévenaz, P. [Thévenaz, Philippe] (CH-LSNP-BI; Lausanne);
Unser, M. [Unser, Michael A.] (CH-LSNP-BI; Lausanne)

Exponential splines and minimal-support bases for curve representation. (English summary)

Comput. Aided Geom. Design **29** (2012), *no.* 2, 109–128.

Summary: "Our interest is to characterize the spline-like integer-shift-invariant bases capable of reproducing exponential polynomial curves. We prove that any compact-support function that reproduces a subspace of the exponential polynomials can be expressed as the convolution of an exponential B-spline with a compact-support distribution. As a direct consequence of this factorization theorem, we show that the minimal-support basis functions of that subspace are linear combinations of deriva-

tives of exponential B-splines. These minimal-support basis functions form a natural multiscale hierarchy, which we utilize to design fast multiresolution algorithms and subdivision schemes for the representation of closed geometric curves. This makes them attractive from a computational point of view. Finally, we illustrate our scheme by constructing minimal-support bases that reproduce ellipses and higher-order harmonic curves."

MR2894255 41A15 65D07 70E55

Jakubiak, Janusz (PL-WROCT-CEN; Wrocław)

Path planning for a double pendulum using natural splines on the torus. (English summary)

Mathematical papers in honour of Fátima Silva Leite, 31–41, *Textos Mat. Sér. B*, 43, *Univ. Coimbra*, *Coimbra*, 2011.

Summary: "In this paper we present an algorithm to generate splines on a torus. In the task solved it is assumed that the splines connect points with given velocities in boundary positions. Results are adapted to a double pendulum robotic arm, illustrated by computer simulations and compared with regular cubic interpolation and X-splines. This paper is an extended version of the work presented at Controlo 2010."

{For the entire collection see MR2894252 (2012i:70004).}

MR2853516 41A15 14Q10 65D07

Lai, Yisheng (PRC-HGSU-ICS; Hangzhou); **Wang, Renhong** (PRC-DUT-IM; Dalian); **Wu, Jinming** (PRC-HGSU-ICS; Hangzhou)

Solving parametric piecewise polynomial systems. (English summary)

J. Comput. Appl. Math. **236** (2011), *no.* 5, 924–936.

Piecewise polynomial systems have many applications in various academic and industrial domains, such as CAD, CAM, CAE and image processing. Many problems, in both practice and theory (for example, the construction of explicit interpolation schemes for spline spaces on a given partition, blending curves and surfaces and computer graphics), can be reduced to problems of solving parametric piecewise polynomial systems. It is obvious that the parametric piecewise polynomial system is also a kind of generalization of the parametric semi-algebraic system.

Lazard and Rouillier recently proposed a new framework for studying the basic constructible set and the basic semi-algebraic set, using a discriminant variety of the basic constructible set.

Based on this discriminant variety method, the authors show that solving a parametric piecewise polynomial system $\mathcal{Z}(f_1,\dots,f_n)$ is reduced to the computation of the discriminant variety of $\mathcal{Z}$. The variety can then be used to solve the parametric piecewise polynomial system. In this context, the authors present theoretical and algorithmic results. More precisely, this paper proposes a general method to classify the parameters of $\mathcal{Z}(f_1,\dots,f_n)$ and an algorithm that answers the following question:

Given a parametric piecewise polynomial system $\mathcal{Z}(f_1,\dots,f_n)$ and integers $N_1,\dots,N_m$, does there exist an open set $\mathcal{O}$ in the parameter space such that for all $p_0\in\mathcal{O}$, the zero-dimensional non-parametric piecewise polynomial system $\mathcal{Z}_{p_0}(f_1,\dots,f_n)$ obtained by specializing at point p_0 has exactly $N_1,\dots,N_m$ distinct torsion-free real zeros in the n-dimensional cells $\sigma_1,\dots,\sigma_m$, respectively?

In the affirmative case, the authors give explicitly a point $a\in\mathcal{O}$. Several experimental results are included to illustrate the theoretical and algorithmic results. *Juana Sendra*

MR2895593 41A15 34A45

Wesołowski, Krzysztof (PL-JAGL-DM; Kraków)

Approximation of solutions of nonlinear initial-value problems by B-spline functions. (English summary)

Function spaces IX, 391–398, *Banach Center Publ.*, 92, *Polish Acad. Sci. Inst. Math.*, *Warsaw*, 2011.

Summary: "This note is motivated by [D. Gámez, A. I. Garralda Guillem and M. Ruiz Galán, Nonlinear Anal. **63** (2005), no. 1, 97–105; MR2167318 (2006e:34022)], where an algorithm finding functions close to solutions of a given initial value-problem has been proposed (this algorithm has been recalled in Theorem 2.2). In this paper we present a commonly used definition and basic facts concerning B-spline functions and use them to improve the mentioned algorithm. This leads us to a better estimate of the Cauchy problem solution under some additional assumption on f appearing in the Cauchy problem. We also estimate the accuracy of the method (Theorem 2.6)."

{For the entire collection see MR2884443 (2012i:00023).}

MR2868366 41A17 41A27 42A10

Li, Ĭong Ping [Liu, Yong Ping] (PRC-BJN-NDM; Beijing);

Su, Chun Meĭ (PRC-BJN-NDM; Beijing);

Ivanov, V. I. [Ivanov, Valerii I.] (RS-TSU; Tula)

Some problems of approximation theory in the spaces L_p on the line with power weight. (Russian. Russian summary)

Mat. Zametki **90** (2011), *no.* 3, 362–383; *translation in Math. Notes* **90** (2011), *no.* 3-4, 344–364.

Let $L_{p,\alpha}$ be the L_p-space on $\mathbb{R}$ equipped with the norm

$$\|f\|_{p,\alpha} := \left(\int_{\mathbb{R}} |f(x)|^p \, |x|^{2\alpha+1} \, dx\right)^{1/p}, \quad \alpha > -1/2, \ 1 \le p < \infty.$$

In the paper under review the authors study approximation of functions in the space $L_{p,\alpha}$ by entire functions of exponential type. Using the Dunkl difference-differential operator and the Dunkl transform they define the generalized shift operator, the modulus of smoothness, and the K-functional. The authors prove direct and inverse theorems of Jackson–Stechkin type and of Bernstein type. Also, they establish the equivalence between the modulus of smoothness and the K-functional. *Sergei S. Platonov*

MR2852296 41A20 26C15 30E10 41A21

Blatt, Hans-Peter (D-EICH-LM; Eichstätt);

Kovacheva, Ralitza K. [Kovacheva, Ralitza Krumova] (BG-AOS-IMI; Sofia)

Growth behavior and zero distribution of rational approximants. (English summary)

Constr. Approx. **34** (2011), *no.* 3, 393–420.

This paper gives important results on the rate of convergence and distribution of zeros of rational uniform approximations. The authors consider meromorphic functions f on a compact set E, connected and with regular connected complement, and they study the behavior of rational approximants $r_{n,m_n}(z) = p_n(z)/q_{m_n}(z)$, with numerator of degree n and denominator of degree m_n, such that $m_n = o(n/\log n)$ $(n \to \infty)$. They generalize known results for the case where the degree of denominator is constant $(m_n = m)$.

They show that the geometric convergence of $\{r_{n,m_n}\}_n$ on the boundary of E to the function f implies the geometric convergence of this sequence m_1-almost uniformly inside the Green domain E_τ for some $\tau > 1$. They also show that if there exists a singularity of multivalued character (like branch points) on the boundary of the maximal

Green domain for f, $\partial E_{\rho(f)}$, then the sequence of rational approximants converges geometrically uniformly on ∂E. Concerning the behavior of zeros of the numerator, in the case where the sequence $\{r_{n,m_n}\}_n$ is an exact m_1-maximally convergent sequence to f, and taking a particular normalization of the denominator, they prove that the normalized zero counting measures ν_n of the numerators of r_{n,m_n} converge weakly to the equilibrium distribution of $\overline{E}_{\rho(f)}$, at least for a subsequence.

In the last section of the paper, the authors apply their results to classical Padé approximation and rational Chebyshev approximation of functions on the interval $[-1, 1]$. They show that the conditions of their theorems are satisfied and obtain new results for the distribution of zeros of these approximants. *Ana Cristina Matos*

MR2855439 41A20 41A50

Van Deun, Joris (B-UA-CS; Wilrijk); **Trefethen, Lloyd N.** (4-OX; Oxford)

A robust implementation of the Carathéodory-Fejér method for rational approximation. (English summary)

BIT **51** (2011), *no.* 4, 1039–1050.

Summary: "Best rational approximations are notoriously difficult to compute. However, the difference between the best rational approximation to a function and its Carathéodory-Fejér (CF) approximation is often so small as to be negligible in practice, while CF approximations are far easier to compute. We present a robust and fast implementation of this method in the Chebfun software system and illustrate its use with several examples. Our implementation handles both polynomial and rational approximation and substantially improves upon earlier published software." *Vanja Hadzijski*

MR2838979 41A21

Danchenko, V. I. (RS-VLSU; Vladimir); **Chunaev, P. V.** (RS-VLSU; Vladimir)

Approximation by simple partial fractions and their generalizations. (English summary)

Problems in mathematical analysis. No. 58.
J. Math. Sci. (*N. Y.*) **176** (2011), *no.* 6, 844–859.

The authors study Padé approximation of functions analytic near the origin by simple partial fractions of the form $\rho_n(z) = \sum_{k=1}^n (z - z_k)^{-1}$. They give a method to compute $\rho_n(z)$, based on the Hermite formula, from which they can estimate the interpolation remainder. Then, they show how these approximants can be used to numerically calculate the values of polynomials and rational functions. Finally, they propose a method for extrapolating an analytic function h by sums of the form $\sum_k \lambda_k h(\lambda_k z)$ where the λ_k are some parameters conveniently chosen and independent of h. When $h(z) = (z - 1)^{-1}$ and $\lambda_k = z_k^{-1}$, one recovers the previous partial fractions. For h an entire function of finite order, and a given $r_0 < 1$, they derive uniform convergence of the extrapolating sums to h in the unit disk, with all nodes in the disk $|z| < r_0$. *Franck Wielonsky*

MR2894553 41A25 42C10

Chripkó, Ágnes (H-EOTVO-NA; Budapest)

On the weighted Lebesgue function of Fourier-Jacobi series. (English summary)

Ann. Univ. Sci. Budapest. Sect. Comput. **35** (2011), 51–81.

This nice paper gives a pointwise estimation of the weighted Lebesgue functions by Jacobi-Fourier series. The main theorem is a generalization of the classical result of S. A. Agakhanov and G. I. Natanson [Vestnik Leningrad. Univ. **23** (1968), no. 1, 11–23; MR0224879 (37 #478)]. As a corollary of this pointwise estimation, an asymptotic expansion in the uniform norm, similar to that of U. Luther and G. Mastroianni [in *Problems and methods in mathematical physics (Chemnitz, 1999)*, 327–351, Oper.

Theory Adv. Appl., 121, Birkhäuser, Basel, 2001; MR1847217 (2002h:42058)], is also proved. *Ágota P. Horváth*

MR2869718 41A25 40A35 41A36
Dirik, Fadime; Demirci, Kamil
Four-dimensional matrix transformation and rate of A-statistical convergence of Bögel-type continuous functions. (English summary)
Stud. Univ. Babeş-Bolyai Math. **56** (2011), *no.* 3, 95–104.

The A-statistical approximation and the rates of A-statistical convergence are studied for sequences of positive linear operators defined on the space of all real-valued Bögel-type continuous functions. The results are established with the aid of a Robison-Hamilton regular summability matrix and the mixed modulus of smoothness.
Zoltán Finta

MR2872340 41A25 41A45 41A65 68T05 90C48
Gnecco, Giorgio (I-GENO-CM; Genoa)
A comparison between fixed-basis and variable-basis schemes for function approximation and functional optimization. (English summary)
J. Appl. Math. **2012,** *Art. ID* 806945, 17 *pp.*

The author compares approximation problems with a fixed basis and a variable basis. Classes of approximation problems are investigated for variable basis schemes which perform better than fixed basis schemes, in terms of the minimum number of computations. Similar results are described for optimization problems. *Günther Nürnberger*

MR2872217 41A25 41A28 41A36
Gupta, Vijay [Gupta, Vijay[1]] (6-NSIT-SAS; New Delhi);
Yadav, Rani (6-NSIT-SAS; New Delhi)
Rate of convergence for generalized Baskakov operators. (English summary)
Arab J. Math. Sci. **18** (2012), *no.* 1, 39–50.

In this study, Gupta and Yadav construct a generalization of Baskakov-Beta operators as follows:

$V_{n,r}(f;x) =$

$$\frac{(n+r-1)!\,(n-r-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)\,f(t)\,dt,$$

where $n\in\mathbb{N}$, $r\in\mathbb{N}_0$, $n>r$,

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},$$

$$b_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}},$$

with

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(n+m-1)!}.$$

The authors investigate the order of approximation for functions with derivatives of bounded variation.

Let $B_q(0,\infty)$, $q>0$, be the class of functions f which are absolutely continuous on $(0,\infty)$, have derivative on the interval $(0,\infty)$ which coincides almost everywhere with a function that has bounded variation on every finite subinterval of $(0,\infty)$ and satisfy the growth condition $|f(t)| \leq C_1 t^q$ for every $t > 0$.

The main theorem of this paper is stated as:

Theorem. Let $f \in B_q(0,\infty)$, $q > 0$ and $x \in (0,\infty)$. Then for a $C > 0$ and all sufficiently large n, we have

$$\left|\frac{((n-1)!)^2}{(n+r-1)!\,(n-r-1)!}V_{n,r}(f;x) - f(x)\right| \leq$$

$$\frac{C(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)$$
$$+\frac{C(1+x)}{(n-r-2)\,x}\left(\left|f(2x) - f(x) - xf'\left(x^+\right)\right| + |f(x)|\right)$$
$$+O(n^{-q}) + \left|f'(x^+)\right|\frac{C(1+x)}{(n-r-2)}$$
$$+\frac{1}{2}\sqrt{\frac{Cx(1+x)}{n-r-2}}\left|f'(x^+) - f'(x^-)\right|$$
$$+\frac{1}{2}\left|f'(x^+) + f'(x^-)\right|\frac{(1+r)+x(1+2r)}{n-r-1},$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and the auxiliary function f_x is defined by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \leq t < x,\\ 0, & t = x,\\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

Furthermore, the authors present the following corollary as a consequence of the theorem:

Corollary. Let $f^{(s)} \in DB_q(0,\infty)$, $q > 0$ and $x \in (0,\infty)$. Then for a $C > 2$ and all sufficiently large n, we have

$$\left|\frac{((n-1)!)^2}{(n+r-1)!\,(n-r-1)!}D^sV_{n,r}(f;x) - f^{(s)}(x)\right| \leq$$

$$\frac{C(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}\left(\left(D^{s+1}f\right)_x\right)$$
$$+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}\left(\left(D^{s+1}f\right)_x\right) + \frac{C(1+x)}{(n-r-2)\,x}$$
$$\times\left(\left|D^sf(2x) - D^sf(x) - xD^{s+1}f(x^+)\right| + |D^sf(x)|\right)$$
$$+O(n^{-q}) + \frac{C(1+x)}{(n-r-2)}\left|D^{s+1}f(x^+)\right|$$
$$+\frac{1}{2}\sqrt{\frac{Cx(1+x)}{n}}\left|D^{s+1}f(x^+) - D^{s+1}f(x^-)\right|$$
$$+\frac{1}{2}\left|D^{s+1}f(x^+) + D^{s+1}f(x^-)\right|\frac{(1+r)+x(1+2r)}{n-r-1},$$

where $\bigvee_a^b f(x)$ denotes the total variation on $[a,b]$ of the function f_x defined by

$$\left(D^{s+1}f\right)_x(t)=\begin{cases} D^{s+1}f(t)-D^{s+1}f(x^-), & 0\le t<x,\\ 0, & t=x,\\ D^{s+1}f(t)-D^{s+1}f(x^+), & x<t<\infty.\end{cases}$$

Ertan İbikli

MR2844664 41A25 41A35

Karsli, Harun

On convergence of general gamma type operators. (English summary)

Anal. Theory Appl. **27** (2011), *no.* 3, 288–300.

The paper deals with a class of Gamma type operators introduced by L. Mao [J. Shangqiu Teach. Coll., **23** (2007), no. 12, 49–52; Zbl 1174.41324] and defined as

$$(M_{n,k}f)(x):=\int_0^\infty g_n(x,u)\,du\int_0^\infty g_{n-k}(u,t)f(t)\,dt \\ =\frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\int_0^\infty\frac{t^{n-k}}{(x+t)^{2n-k+2}}f(t)\,dt,\quad x>0,$$

where f is a function of bounded variation on $[0,+\infty)$ and

$$g_n(x,u)=\frac{x^{n+1}}{n!}e^{-xu}u^n$$

is the kernel function of the Gamma operator. Such operators generalize, for particular choices of k, some families of variants of the classical Gamma operators studied by S. M. Mazhar [Math. Balkanica (N.S.) **5** (1991), no. 2, 99–104; MR1145883 (93a:41042)], A. Izgi and I. Büyükyazici [Kastamonu Eğitim Dergisi Ekim **11** (2003), no. 2, 451–460; per bibl.] and H. Karsli [Math. Comput. Modelling **45** (2007), no. 5-6, 617–624; MR2287309 (2007h:41017)].

The main result of the paper is an estimate of the rate of pointwise convergence of the operators $M_{n,k}$. In particular, the author proves that if f is of bounded variation on every finite subinterval of $[0,+\infty)$ and satisfies the growth condition $|f(t)|\le Mt^\gamma$, for some $\gamma\ge 0$ and for an absolute constant $M>0$, then, for every $x>0$, for $r\in\mathbf{N}$ $(2r\ge\gamma)$ and n sufficiently large,

$$\left|(M_{n,k}f)(x)-\frac{f(x^+)+f(x^-)}{2}\right|\le\frac{6}{n-1}\sum_{l=1}^n\bigvee_{x-\frac{x}{\sqrt{l}}}^{x+\frac{x}{\sqrt{l}}}(f_x) \\ +\frac{6\sqrt{2}}{\sqrt{n-k+1}}\left|\frac{f(x^+)-f(x^-)}{2}\right|+M2^{2r}A(2r,k)\frac{x^{2r}}{n^r},$$

where $\bigvee_{x-\frac{x}{\sqrt{l}}}^{x+\frac{x}{\sqrt{l}}}(f_x)$ is the total variation of the function

$$f_x(t):=\begin{cases} f(t)-f(x^+), & x<t,\\ 0, & t=x,\\ f(t)-f(x^-), & 0\le t<x,\end{cases}$$

on $[x-\frac{x}{\sqrt{l}},x+\frac{x}{\sqrt{l}}]$ and $A(2r,k)$ is a constant depending on r and k. *Laura Angeloni*

MR2896608 41A25 41A36

Peng, Lian Yong (PRC-SW-NDM; Chongqing); **Wang, Jian Jun** [**Wang, Jian Jun^3**] (PRC-SW-NDM; Chongqing)

Equivalent approximation theorems with Jacobi weight for combinations and higher-order derivatives of Bernstein operators. (Chinese. English and Chinese summaries)

Math. Appl. (*Wuhan*) **24** (2011), *no.* 4, 791–797.

In this paper, the authors establish several equivalent theorems on approximation by combinations of Bernstein operators with Jacobi weight, by using the Ditzian-Totik moduli of smoothness. They also obtain a relation between higher derivatives of the operators and smoothness of functions. *Lai Yi Zhu*

MR2869040 41A25 33C45 41A58 65D05

Wang, Haiyong (PRC-CSU-AMS; Changsha); **Xiang, Shuhuang** [**Xiang, Shu Huang**] (PRC-CSU-AMS; Changsha)

On the convergence rates of Legendre approximation. (English summary)

Math. Comp. **81** (2012), *no.* 278, 861–877.

For the class of functions $f,\ f',\dots,f^{(k-1)}$ absolutely continuous on $[-1,1]$ with $\|f^{(k)}\|_T$ finite (here $\|u\|_T=\int_{-1}^1(|u'(x)|/\sqrt{1-x^2})\,dx$), upper bounds on the coefficients of the Legendre series are obtained. Also, bounds of the same coefficients are given for functions analytic inside and on the Bernstein ellipse. For the latter class of functions the bounds are better than the bounds given in [P. J. Davis, *Interpolation and approximation*, Dover, New York, 1975; MR0380189 (52 #1089)]. The decay rates of the Legendre coefficients are shown to be comparable to those of the Chebyshev coefficients, although the latter are somewhat faster. These decay rates are then used to give error bound estimates for Legendre series approximations for the above classes of functions.

Finally, explicit values for the barycentric weights are given for barycentric Lagrange interpolation formulas using the Gauss-Legendre points. An error bound for this interpolant is also given for the concerned classes of functions. Several examples are also provided. *James R. Angelos*

MR2857813 41A27 41A50 46E15

Voloshin, G. A. (UKR-CHNV-NDM; Chernovtsy); **Maslyuchenko, V. K.** [**Maslyuchenko, Volodymyr**] (UKR-CHNV-NDM; Chernovtsy)

Comparison of some generalizations of the inverse Bernstein theorem for F-spaces. (Ukrainian. English and Russian summaries)

Mat. Stud. **35** (2011), *no.* 2, 165–171.

In 1938, S. N. Bernshteĭn [C. R. Acad. Sci. Paris **206** (1938), 1520–1523] initiated the study of inverse problems in approximation theory. He proved that for every decreasing sequence of integral numbers α_n which tends to zero, there exists a continuous function $f\colon[0,1]\to\mathbb{R}$ such that for every $n\in\mathbb{N}$, the quantity $E_n(f)$ of the best approximation of the function f by polynomials of degree $\le n$ is equal to α_n. In 1948, S. M. Nikolʹskiĭ [in *Matematika v SSSR za 30 let*, 288–318, Gostekhizdat, Moscow, 1948] noted without proof that this result can be extended to the Banach spaces. This extension of Bernshteĭn's result was proved in the monograph of A. F. Timan [*Theory of approximation of functions of a real variable* (Russian), Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960; MR0117478 (22 #8257)].

There are further generalizations of the Bernshteĭn inverse theorem in different directions. In particular, in [A. S. Shvedov, Akad. Nauk SSSR Inst. Prikl. Mat. Preprint **1982**, no. 55, 20 pp.; MR0716191 (84h:41051); in *Proceedings of the International Conference on the Theory of Approximation of Functions. Kiev, 1983*, 473–475, Moscow, Nauka, 1987; per. bibl.; A. I. Vasil′ev, Dokl. Akad. Nauk **365** (1999), no. 5, 583–585; MR1714216 (2000g:41031); G. A. Voloshin and V. K. Maslyuchenko, Mat. Visn. Nauk. Tov. Im. Shevchenka **6** (2009), 62–72], the result of Bernshteĭn was extended to the F-space. The authors of these papers used the original idea of Bernshteĭn, which required some additional conditions on the F-space. In [A. S. Shvedov, op. cit., 1982; op. cit., 1987; A. I. Vasil′ev, op. cit.; G. A. Voloshin and V. K. Maslyuchenko, op. cit.], different additional conditions were imposed, which led to different results. Therefore, a natural question is to establish relations among these generalizations and to determine which one is stronger.

In the given paper, the authors show that these generalizations are incomparable.

Andriy Lyubomyrovych Shidlich

MR2838275 41A30 41A17 68T05 92B20

Cao, Feilong (PRC-CJU; Hangzhou); **Wang, Huazhong** (PRC-HZNU-CSC; Hangzhou); **Lin, Shaobo** (PRC-XJU-IF; Xi'an)

The estimate for approximation error of spherical neural networks. (English summary)

Math. Methods Appl. Sci. **34** (2011), *no.* 15, 1888–1895.

Summary: "Compared with planar hyperplane, fitting data on the sphere has been an important and an active issue in geoscience, metrology, brain imaging, and so on. In this paper,with the help of the Jackson-type theorem of polynomial approximation on the sphere, we construct spherical feed-forward neural networks to approximate the continuous function defined on the sphere. As a metric, the modulus of smoothness of spherical function is used to measure the error of the approximation, and a Jackson-type theorem on the approximation is established."

MR2876482 41A30

Ismailov, Vugar E. (AZ-AOS; Baku)

Approximation by neural networks with weights varying on a finite set of directions. (English summary)

J. Math. Anal. Appl. **389** (2012), *no.* 1, 72–83.

Summary: "Approximation properties of the MLP (multilayer feedforward perceptron) model of neural networks have been investigated in a great deal of works over the last 30 years. It has been shown that for a large class of activation functions, a neural network can approximate arbitrarily well any given continuous function. The most significant result on this problem belongs to Leshno, Lin, Pinkus and Schocken. They proved that the necessary and sufficient condition for any single hidden layer network to have the u.a.p. (universal approximation property) is that its activation function not be a polynomial. Some authors (White, Stinchcombe, Ito, and others) showed that a single hidden layer perceptron with some bounded weights can also have the u.a.p. Thus the weights required for u.a.p. are not necessary to be of an arbitrarily large magnitude. But what if they are too restricted? How can one learn approximation properties of networks with arbitrarily restricted set of weights? The current paper makes a first step in solving this general problem. We consider neural networks with sets of weights consisting of a finite number of directions. Our purpose is to characterize compact sets X in the d-dimensional space such that the network can approximate any continuous function over X. In a special case, when weights vary only on two directions, we give a lower bound for the approximation error and find a sufficient condition for the network to be a best approximation."

MR2875252 41A30 41A05 65D10

Lee, Mun Bae (KR-KOKU; Seoul); **Lee, Yeon Ju** (KR-AIST2; Taejŏn);
Yoon, Jungho (KR-EWHA; Seoul)

Sobolev-type L_p-approximation orders of radial basis function interpolation to functions in fractional Sobolev spaces. (English summary)

IMA J. Numer. Anal. **32** (2012), *no.* 1, 279–293.

The authors give a Sobolev-type L_p-approximation estimation for some radial basis functions in a fractional Sobolev space. The results are valid for conditional positive (negative) radial basis functions (see chapter 8 in [H. Wendland, *Scattered data approximation*, Cambridge Monogr. Appl. Comput. Math., 17, Cambridge Univ. Press, Cambridge, 2005; MR2131724 (2006i:41002)] for a definition) on bounded domain Ω with a Lipschitz boundary.

Let $W^{k,p}(\Omega)$ be a Sobolev space on $\mathbb{R}^d$ associated with (semi-)norms

$$|f|^p_{W^{k,p}(\Omega)} := \sum_{|\alpha|_1=k} \|D^\alpha f\|^p_{L_p(\Omega)}$$

and

$$\|f\|^p_{W^{k,p}(\Omega)} := \sum_{|\alpha|_1\le k} \|D^\alpha f\|^p_{L_p(\Omega)},$$

where $\alpha=(\alpha_1,\alpha_1,\dots,\alpha_d)\in\mathbb{Z}^d_+$ is a multi-index, $|\alpha|_1=\sum_{j=1}^d \alpha_j$, and $D^\alpha f\in L_p(\Omega)$ are distributional (weak) derivatives. For the case $p=2$, and any $\tau>0$, one can define a norm of the Sobolev space $W^{\tau,p}(\mathbb{R}^d)$ as

$$\|f\|_{W^{\tau,2}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} (1+|\theta|^2)^\tau |\widehat{f}(\theta)|^2 d\theta\right)^{1/2}. \tag{1}$$

For a non-negative integer k, $0<\mu<1$ and $1\le p\le\infty$, the fractional Sobolev spaces $W^{k+\mu,p}(\Omega)$ consist of all the functions f with the following norm being finite:

$$\|f\|^p_{W^{k+\mu,p}(\Omega)} := \|f\|^p_{W^{k,p}(\Omega)} + \sum_{|\beta|_1=k} \int_\Omega \left\| \frac{D^\beta f(\cdot)-D^\beta f(x)}{|\cdot-x|^{d/p+\mu}} \right\|^p_{L_p(\Omega)} dx.$$

For $p=2$ and $\tau=k+\mu$, the latter norm is equivalent to the norm in (1).

Let ν be a real number satisfying $d/2\le\nu\le\tau$, where τ is a parameter depending on the related radial basis function, and let $S_X f$ be the radial basis function interpolant of a function f. The authors show that

$$\|f-S_X f\|_{W^{|\alpha|_1,p}(\Omega)} \le c h^{\nu-|\alpha|_1-\max\{d(1/2-1/p),0\}} \|f\|_{W^{\nu,2}(\Omega)},$$

for $|\alpha|_1\le\lceil\nu_0\rceil$ with $\nu_0=\nu-\max\{d(1/2-1/p),0\}$, where h is the so-called *fill distance*.

This result has a statement that is similar to Theorem 6.1 of [R. Arcangéli, M. C. López de Silanes and J. J. Torrens, Numer. Math. **107** (2007), no. 2, 181–211; MR2328845 (2008f:46039)] but generalizes that result to a larger fractional Sobolev space. The proof itself is also related to the sampling inequalities in Theorem 4.1 of [R. Arcangéli, M. C. López de Silanes and J. J. Torrens, op. cit.]. *Shengxin Zhu*

MR2871668 41A30 41A05 41A63 65D05

Luh, Lin-Tian (RC-PU-M; Taichung)

The shape parameter in the Gaussian function. (English summary)

Comput. Math. Appl. **63** (2012), *no.* 3, 687–694.

This paper discusses the optimal choice of the parameter $\beta>0$ in the Gaussian function $h(x)=e^{-\beta|x|^2}$ for radial basis function interpolations. Let $s(x)=\sum_{i=1}^N c_i h(x-x_i)$ be an h-spline interpolant of data (x_i, y_i), $i = 1, 2, \dots, N$. From the interpolation error bound, a MIN function is derived from which criteria to choose an optimal value of the parameter β are proposed. *Falai Chen*

MR2879182 41A35 41A25 47G10

Bardaro, Carlo (I-PERG-MI; Perugia); **Mantellini, Ilaria** (I-PERG-MI; Perugia)

The moments of the bivariate Mellin-Picard-type kernels and applications. (English summary)

Integral Transforms Spec. Funct. **23** (2012), *no.* 2, 135–148.

In this paper, the authors study two kinds of bivariate Mellin-Picard convolution operators given by the general formula

$$T_w(f)(x,y) = \int_{\mathbb{R}_+^2} K_w(tx^{-1}, vy^{-1}) f(t,v) \frac{dtdv}{vt},$$

where the kernel K_w is given by

$$K_w(x,y) = \frac{w^2}{2\pi} \exp\left(-w\sqrt{\log^2(x) + \log^2(y)}\right), \quad (x,y) \in \mathbb{R}_+^2,\ w > 1,$$

in which case T_w is denoted by P_w, or by

$$K_w(x,y) = \frac{w^2 xy}{(2 - e^{-w})^2} \exp(-w(|1-x| + |1-y|)), \quad (x,y) \in \mathbb{R}_+^2,\ w > 1,$$

in which case T_w is denoted by B_w.

The first main result is qualitative Voronovskaya asymptotic formulas for these operators, stated as follows.

Theorem. Let $f \in L^\infty$. If f is locally C^2 at (x,y), then

$$\lim_{w\to\infty} w^2(P_w(f)(x,y) - f(x,y)) = \frac{3}{2}\left(x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) + x^2\frac{\partial^2 f}{\partial x^2}(x,y) + y^2\frac{\partial^2 f}{\partial y^2}(x,y)\right),$$

and

$$\lim_{w\to\infty} w^2(B_w(f)(x,y) - f(x,y)) = x^2\frac{\partial^2 f}{\partial x^2}(x,y) + y^2\frac{\partial^2 f}{\partial y^2}(x,y).$$

Then, quantitative estimates of the above convergences are deduced and, finally, qualitative and quantitative Voronovskaya-type formulas, in terms of the Mellin derivatives for these two operators, are presented. *Sorin Gheorghe Gal*

MR2814341 41A35 94A12 94A20

Coroianu, Lucian (R-ORA-MI; Oradea);
Gal, Sorin G. [**Gal, Sorin Gheorghe**] (R-ORA-MI; Oradea)

Approximation by nonlinear generalized sampling operators of max-product kind. (English summary)

Sampl. Theory Signal Image Process. **9** (2010), *no.* 1-3, 59–75.

In this paper, the authors consider the max-product modification of the Whittaker cardinal series and some generalized sampling approximation linear operators in signal theory. In particular, they study the following max-product type sampling operator:

$$S_{W,\varphi}^{(M)}(f)(t) = \frac{\bigvee_{k=-\infty}^{\infty} \varphi(Wt-k) f\left(\frac{k}{W}\right)}{\bigvee_{k=-\infty}^{\infty} \varphi(Wt-k)}, \quad t \in \mathbb{R}.$$

In the case of the max-product Whittaker operator, the authors obtain a better order of approximation for positive functions. They also give Jackson-type approximation orders in terms of the moduli of smoothness. Finally, approximation results for nonlinear sampling operators are compared with their linear counterparts. *Naokant Deo*

MR2812528 41A35 41A55 41A63 42C25 65D15
Michel, Volker (D-SGN-GMG; Siegen)
Optimally localized approximate identities on the 2-sphere. (English summary)
Numer. Funct. Anal. Optim. **32** (2011), *no.* 8, 877–903.

The paper deals with the construction of optimally localized approximate identities on the three-dimensional sphere. The differential in the paper is how optimal localization is found in the optimization problem, with a functional that keeps a balance between bad localization and deviation from an approximate identity. The part that controls localization includes a weight function that can be conveniently chosen. For each choice, the author proves the existence and uniqueness of the optimal kernel and shows how to generate an approximate identity in the band limited case. The paper includes results of numerical tests. *Valdir A. Menegatto*

MR2855621 41A35 41A60
Ostrovska, Sofiya (TR-ATL-M; Ankara)**; Özban, Ahmet Yaşar** (TR-ATL-M; Ankara)
The norm estimates of the q-Bernstein operators for varying $q > 1$. (English summary)
Comput. Math. Appl. **62** (2011), *no.* 12, 4758–4771.

For $q > 0$ any non-negative integers k and n, denote by $[k]_q$ the q-integer and by $\begin{bmatrix} n \\ k \end{bmatrix}_q$ the q-binomial coefficient. The q-Bernstein operators $B_{n,q}$, $n = 1, 2, \ldots$, on $C[0,1]$ are defined by

$$B_{n,q}(f)(x) = \sum_{k=0}^{n} f\left(\tfrac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{i=0}^{n-k-1} (1 - xq^i), \quad x \in [0,1],$$

where $f \in C[0,1]$. It is well known that $\|B_{n,q}\| = 1$ for $0 < q \le 1$, and in the case $q > 1$ the norm $\|B_{n,q}\|$ increases as $q \to \infty$. In this paper, the authors study asymptotic estimates for the norms $\|B_{n,q}\|$ as $q \to \infty$ or $n \to \infty$. Among other results they prove that $\|B_{n,q}\| \sim C_n q^{n(n-1)/2}$ as $q \to \infty$ with $C_n = \frac{2}{n}(1 - \frac{1}{n})^{n-1}$ and $\|B_{n,q}\| \sim \frac{2q^{n(n-1)/2}}{ne}$ as $q \to \infty$, $n \to \infty$. The results obtained are illustrated by numerical examples. *Marek Beśka*

MR2861017 41A36
Bustamante, Jorge [Bustamante González, Jorge] (MEX-UAPP; Puebla)
An estimate for some positive linear operators in L_p spaces. (English summary)
Rend. Circ. Mat. Palermo (2) **60** (2011), *no.* 3, 351–356.

The paper generalizes a result of V. I. Ivanov and S. Pichugov [Mat. Zametki **42** (1987), no. 6, 776–785, 909; MR0934810 (89h:41048)] which concerns certain periodic convolution operators. Let K be any positive bounded measurable function of two variables which is 2π-periodic in each variable. Further assume $\int K(y, -y)dy = 1$. The author considers the operator

$$M(f,x) = \frac{1}{2}\int f(y)[K(y-x,x-y)+K(x-y,y-x)]dy$$

for $f \in L_p$. For $1 < p \le 2$ the author proves that

$$\|f - M(f)\|_p \le 2^{-1/q}\Big(\int \|\Delta_t f(\cdot)\|_p^p K(t,-t)dt\Big)^{1/p}$$

where q is conjugate to p. An analogous result holds for $2 \le p < \infty$. The author's proof proceeds by identifying the operator $I - M$ in terms of the operator Δ_t and its adjoint and then using a variant of the Riesz-Thorin interpolation theorem. In a concluding remark the author asks whether any periodic symmetric function P can be expressed as in his theorem, i.e.,

$$P(x,y) = \frac{1}{2}(K(y-x,x-y)+K(x-y,y-x)),$$

with a suitable choice of K. *A. S. Cavaretta*

MR2880414 41A36 40A35 41A25

Erkuş-Duman, Esra (TR-GAZIA; Ankara)

Statistical rates in approximation by positive linear operators. (English summary)

Miskolc Math. Notes **12** (2011), *no.* 2, 159–165.

Using two concepts of the rate of statistical convergence [see O. Duman, E. Erkuş-Duman and V. Gupta, Math. Comput. Modelling **44** (2006), no. 9-10, 763–770; MR2253745 (2007k:41060)], the author considers sequences of positive operators defined on $C^*(\mathbb{R}^m)$, the space of all real-valued continuous and 2π-periodic functions on the real m-dimensional Euclidean space $\mathbb{R}^m$. For each of the rates of statistical convergence she proves a Korovkin type theorem under the assumptions that the modulus of continuity satisfies some conditions relative to the behavior of the positive operators on the test functions, and that the rate of statistical convergence for the function $f \equiv 1$ is given. *Marek Beśka*

MR2905407 41A36 33D99

Finta, Zoltán (R-CLUJ; Cluj-Napoca)

Approximation by q-Bernstein type operators. (English summary)

Czechoslovak Math. J. **61(136)** (2011), *no.* 2, 329–336.

Summary: "Using the q-Bernstein basis, we construct a new sequence $\{L_n\}$ of positive linear operators in $C[0,1]$. We study its approximation properties and the rate of convergence in terms of modulus of continuity."

MR2867177 41A36 33D99 41A25

Finta, Zoltán (R-CLUJ; Cluj-Napoca)

Approximation by q-parametric operators. (English summary)

Publ. Math. Debrecen **78** (2011), *no.* 3-4, 543–556.

The main goal of this paper is to obtain sufficient conditions for the uniform convergence of a sequence of positive linear operators to its limit operator and estimate the rate of convergence by means of the second-order Ditzian-Totik modulus of smoothness. Notice that, after the definition of q-analogue of the classical Bernstein polynomials [A. Lupaş, in *Seminar on Numerical and Statistical Calculus (Cluj-Napoca, 1987)*, 85–92, Preprint, 87-9, Univ. "Babeş-Bolyai", Cluj-Napoca, 1987; MR0956939 (90b:41026)], many sequences of positive linear operators based on the q-integers were introduced. In the light of the remarkable main result of this study indicated as Theorem 1, rates of convergence of the q-analogue of the Bernstein polynomials introduced by Lupaş [op. cit.], q-Bernstein polynomials [G. M. Phillips, Ann. Numer. Math. **4** (1997), no. 1-4, 511–518; MR1422700 (97k:41013)], and q-Meyer-König and Zeller operators [T. Trif, Rev. Anal. Numér. Théor. Approx. **29** (2000), no. 2, 221–229 (2002); MR1929093 (2003g:41034)] are obtained by means of the second-order Ditzian-Totik modulus of smoothness. *Ogün Doğru*

MR2862166 41A36 41A35

Gairola, Asha R. **[Gairola, Asha Ram]** (6-GEU-CPA; Dehra Dun);
Dobhal, Girish (6-GEU-CPA; Dehra Dun);
Singh, Karunesh K. **[Singh, Karunesh Kumar]** (6-IITR; Roorkee)

On certain q-Baskakov-Durrmeyer operators. (English summary)

Matematiche (Catania) **66** (2011), *no.* 2, 61–76.

The authors introduce a q-variant, $0 < q < 1$, of Baskakov-Durrmeyer operators, of the form

$$\mathcal{L}_{n,q}(f,x)=\sum_{k=0}^{\infty}p_{n,k}(q;x)\int_0^{\infty/A}q^k b_{n,k}(q;u)f(u)d_qu,$$

where, with the usual symbols of q-calculus, we have

$$b_{n,k}(q;x)=\frac{q^{k(k-1)/2}x^k}{B_q(k+1,n)(1+x)_q^{n+k+1}},$$

$$p_{n,k}(q;x)=\begin{bmatrix}n+k-1\\ k\end{bmatrix}_q\frac{q^{k(k-1)/2}x^k}{(1+x)_q^{n+k}}.$$

These operators could be also viewed as a q-analog of the operators introduced by V. Gupta and A. Ahmad [İstanbul Üniv. Fen Fak. Mat. Derg. **54** (1995), 11–22 (1997); MR1608198 (99a:41026)]. The paper contains many results related to the moments of operators, the uniform convergence on compact sets, the Voronoskaya-type property, and the local estimate of degree of approximation expressed using diverse kind of moduli of continuity of first and second order, as well as the weighted approximations. As for similar q-type operators, the convergence results obtained are for $\mathcal{L}_{n,q_n}$, where $0 < q_n < 1$ and $q_n \to 1$. The calculi are interesting. There are some misprints in the paper. We note that Theorem 3.1 is not a consequence of the classical Bohmann-Korovkin theorem. *Radu Păltănea*

MR2869722 41A36 41A25 41A28

Holhoş, Adrian (R-TUCN-NDM; Cluj-Napoca)

Uniform weighted approximation by positive linear operators. (English summary)

Stud. Univ. Babeş-Bolyai Math. **56** (2011), *no.* 3, 135–146.

Let $I \subset \mathbb{R}$ be a noncompact interval and let $\rho\colon I \to [1,\infty)$ be an increasing and differentiable function called a weight function. Let $B_\rho(I)$ be the space of all functions $f\colon I \to \mathbb{R}$ such that $|f(x)| \le M\rho(x)$, for every $x \in I$, where $M > 0$ is a constant depending on f and ρ, but independent of x. The space $B_\rho(I)$ is called a weighted space and it is a Banach space endowed with the ρ-norm

$$\|f\|_\rho := \sup_{x\in I}\frac{|f(x)|}{\rho(x)}.$$

Let $C_\rho(I) = C(I) \cap B_\rho(I)$ be the subspace of $B_\rho(I)$ containing continuous functions.

Let $(A_n)_{n\geq 1}$ be a sequence of positive linear operators acting from $C_\rho(I)$ onto $B_\rho(I)$.

In the present paper the author characterizes the functions from the weighted space $C_\rho(I)$ that can be approximated by a sequence of positive linear operators, and obtains the range of the weights which can be used for uniform approximation. Moreover, an estimation for the remainder term is given in terms of the usual modulus of continuity.

In the final part of this study, some particular results related to the Szász-Mirak′yan and Baskakov operators are also presented. *Harun Karsli*

MR2885005 41A36 41A60

Karsli, Harun

Complete asymptotic expansions for the modified gamma operators. (English summary)

Adv. Stud. Contemp. Math. (*Kyungshang*) **21** (2011), *no.* 4, 413–428.

In this paper the author studies the local rate of convergence of the modified Gamma operators

$$(M_{n,k}f)(x) = \int_0^\infty K_{n,k}(x,t)f(t)dt, \quad n \in \mathbb{N},$$

where

$$K_{n,k}(x,t) = \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\frac{t^{n-k}}{(x+t)^{2n-k+2}}, \quad x>0,\ t>0.$$

In addition, if f is right-side continuous at $x=0$, one sets

$$(M_{n,k}f)(0) := f(0).$$

In the above $f \in W_\gamma[0,\infty)$, $\gamma \geq 0$, the space of all locally bounded and integrable functions defined on $[0,\infty)$ and satisfying the growth condition $|f(t)| \leq Mt^\gamma$, for $t>0$, with some constant $M>0$.

The author investigates the asymptotic behaviour of the above operators. Also, he succeeds in obtaining in an elegant manner the complete asymptotic expansions for this class of operators. *Octavian Agratini*

MR2872523 41A36 47B65

Vogt, Andreas [Vogt, Andreas²] (D-TRR-NDM; Trier)

Universal properties of approximation operators. (English summary)

J. Approx. Theory **164** (2012), *no.* 3, 367–370.

The paper under review is concerned with the universal properties of a sequence of operators acting from $C[0,1]$ into $C[0,1]$.

Let $L_n\colon H(\mathbb{C}) \to H(\mathbb{C})$, $L_nf(z) := f(z+n)$, where $H(\mathbb{C})$ is the space of all entire functions endowed with the topology of local uniform convergence. The theorem of G. D. Birkhoff [C. R. Acad. Sci. Paris **189** (1929), 473–475; JFM 55.0192.07] asserts the existence of a function f such that $\{L_nf\}_{n\in\mathbb{N}}$, is dense in $H(\mathbb{C})$. Such a function f is said to be $(L_n)_{n\in\mathbb{N}}$-*universal* for $H(\mathbb{C})$, which is written as $f \in U((L_n)_{n\in\mathbb{N}}, H(\mathbb{C}))$.

The author studies the existence of universal functions with respect to a sequence of operators $L_n\colon C[0,1] \to C[0,1]$ that are closely related to those exhibited in the famous theorem of Korovkin, which deals with the uniform convergence.

The main result of this paper is as follows: There is a sequence of positive linear operators $(L_n)_{n\in\mathbb{N}}$ such that for every polynomial P with $P(0)=0$ we have

$$L_nP \to P \text{ uniformly on } [0,1], \quad \text{as } n\to\infty. \tag{1}$$

Moreover, there is a set of functions, residual in $(C_0[0,1], \|.\|_\infty)$, such that $(L_nf)_{n\in\mathbb{N}}$ is dense in $(C[0,1], \|.\|_\infty)$. Here $C_0[0,1]$ denotes the set of all functions $f\in C[0,1]$ with

$f(0) = 0$.

The author also proves that there are sequences of linear (but not positive) and of positive (but not linear) operators $(L_n)_{n\in\mathbb{N}}$ that satisfy (1) and such that $U((L_n)_{n\in\mathbb{N}}, C[0,1]) \neq \varnothing$. *Harun Karsli*

MR2728619 41A44 26D10 42B25

Zernyshkina, E. A. [Zërnyshkina, E. A.]

The Wirtinger-Steklov inequality between the norm of a periodic function and the norm of the positive cutoff of its derivative. (English summary)

Proc. Steklov Inst. Math. **264** (2009), *suppl.* 1, S199–S213.

In this paper the author studies sharp constants in some spaces of measurable essentially bounded 2π-periodic and mean-zero functions. More precisely, let W^q (resp. W^q_+) denote the space of absolutely continuous functions y whose derivative y' (resp. positive cutoff $y'_+(x) := \max\{0, y'(x)\}$) belongs to $L^q(-\pi,\pi)$. Let $Q^j = Q^j(q)$ $(j \in \{1,2,3\})$ be the class of 2π-periodic functions that are absolutely continuous on the real line and whose restriction to the segment $[-\pi,\pi]$ belongs to the space W^q, and such that the following property holds:

$$\max y + \min y = 0 \quad (\text{for } j=1),$$
$$\exists\, x_0 \in [-\pi,\pi]\colon y(x_0) = 0 \quad (\text{for } j=2),$$
$$\int_{-\pi}^{+\pi} y(x)dx = 0 \quad (\text{for } j=3).$$

The classes $Q^j_+ = Q^j_+(q)$ $(j \in \{1,2,3\})$ are defined in a similar way.

The author studies the sharp constants $K_{p,q}(Q^j_+)$ for the inequality

$$\|y\|_{L^p} < K_{p,q}(Q^3_+)\|y'\|_{L^q},$$

for $0 < p < 1$ and $1 < q < 1$, where $K_{p,q}(Q^3_+) = \sup\{\|y\|_{L^p}/\|y'\|_{L^q} : y \in Q^3_+,\ y \not\equiv 0\}$. This paper continues the research initiated in a previous paper by the author, namely [Izv. Ural. Gos. Univ. Mat. Mekh. No. 9(44) (2006), 76–88, 161–162; MR2907176], where similar results were obtained for functions in the classes Q^1_+ and Q^2_+. The author investigates the above inequality by bounding the constant $K_{p,q}(Q^3_+)$ from below for $0 < p < 1$ and from above for $1 < p < 1$ for $1 < q < 1$. Furthermore, the values of the sharp constant in the cases $p = 2$, $1 < q < 1$ and $p = 1$, $1 < q < 1$ are given in the form $K_{p,q}(Q^3_+) = 2K_{p,q}(Q^3)$. *Michele V. Bartuccelli*

MR2832720 41A46 42C40 65T60

Kutyniok, Gitta (D-OSNB-IM; Osnabrück)**; Lim, Wang-Q** (D-OSNB-IM; Osnabrück)

Compactly supported shearlets are optimally sparse. (English summary)

J. Approx. Theory **163** (2011), *no.* 11, 1564–1589.

The authors identify optimally sparse approximations of cartoon-like images, i.e., C^2 functions which are smooth apart from a C^2 discontinuity curve, up to a log-factor with the same exponent as in the curvelet-, contourlet-, and (band-limited) shearlet-approximation rates, by using a particular class of directional representation systems, consisting of compactly supported elements. This class is chosen as a subset of (non-tight) shearlet frames with shearlet generators having compact support and satisfying some weak directional vanishing moment conditions. This proof is different from previous ones since it extensively exploits the fact that the shearlet generators are compactly supported and the lack directional vanishing moments. The directional representation system of shearlets has recently gained attention because, in contrast to other such systems, shearlets provide a unified treatment of the continuum and digital worlds, similar to wavelets, due to the fact that the shearing operator, a means for deriving directionality, leaves the digital grid invariant. *Rémi Vaillancourt*

MR2908113 41A50 41A55 65D30

Perić, I. [Perić, Ivan] (CT-ZAGRFB; Zagreb)

Frequency variant of Euler type identities and the problem of sign-constancy of the kernel in associated quadrature formulas. (English summary)

J. Math. Inequal. **5** (2011), *no.* 4, 565–579.

Summary: "In the recent years many authors used extended Euler identities to obtain generalizations of some classical quadrature formulas with the best possible error estimates. The main step in obtaining the best possible error estimates was to control zeros of the kernel in the error term which consists of the affine combinations of the translates of periodic Bernoulli polynomials. This was done for some low degrees of Bernoulli polynomials. The main goal of this paper is to consider a general case. The frequency variant of extended Euler identities is found to be more tractable for this problem."

MR2905968 41A55 65D30

Boltaev, A. K.

On an extremal function for an optimal quadrature formula. (Russian. English, Russian and Uzbek summaries)

Vopr. Vychisl. Prikl. Mat. No. 125 (2010), 32–42, 173.

Summary: "In the present paper in the $W_2^{P_3}(0,1)$ Hilbert space the first part of the problem of construction of optimal quadrature formulas is solved, i.e. norm of the error functional of optimal quadrature formulas in the $W_2^{P_3}(0,1)$ space is calculated."

MR2905973 41A55 41A05 65D30

Mamatova, N. Kh.

Construction of quadrature formulas using an optimal interpolation formula. (Russian. English, Russian and Uzbek summaries)

Vopr. Vychisl. Prikl. Mat. No. 125 (2010), 83–91, 174.

Consider the space $\widetilde{L}_2^m(0,1)$ of 1-periodic functions $\varphi\colon\mathbb{R}\to\mathbb{R}$ endowed with the seminorm

$$\|\varphi\|_{\widetilde{L}_2^m(0,1)} := \Big(\int_0^1 (\varphi^{(m)}(x))^2 dx\Big)^{1/2}.$$

The author studies quadrature formulas of the form

$$\int_0^1 p(x)\varphi(x)dx \approx \sum_{k=1}^N c_k\varphi(\tfrac{k}{N}).$$

For $\varphi\in\widetilde{L}_2^m(0,1)$, let the map $\varphi\mapsto P(\varphi,x)$ be defined by

$$P(\varphi,x) := \sum_{k=1}^N c_k(x)\varphi(x_k).$$

The error functional at a point z can be represented by

$$\varphi(z)-P(\varphi,z) := \langle l_z,\varphi\rangle = \int_0^1 \Big[(\delta(x-z)-\sum_{k=1}^N c_k(z)\delta(x-x_k))*\phi_0(x)\Big]\varphi(x)dx,$$

where $\delta(x)$ is the delta function, $\phi_0(x) := \sum_{k\in\mathbb{Z}}\delta(x-k)$. Let $\widetilde{L}_2^{m*}(0,1)$ be the space of functionals l_z of the form above such that $\langle l_z, 1\rangle = 0$. An "optimal interpolation formula" is that whose error functional has the minimal L_2-norm over all choices of $c_k(z) \in \mathbb{R}$, with N and $x_k = k/N$ being fixed. Using optimal interpolation formulas the author establishes several quadrature formulas.

It is shown that if $p(x) \equiv 1$ then there is a unique quadrature formula of such type with the coefficients $c_k = N^{-1}$, $1 \le k \le N$. In the case $p(x) = e^{2\pi i p x}$, $p \in \mathbb{Z}$, it is shown that the coefficients are

$$c_k = \frac{N^{-1} e^{2\pi i p x}}{p^{2m} \sum_{k\in\mathbb{Z}} \frac{N^{-2m}}{(k-pN^{-1})^{2m}}}, \quad 1 \le k \le N,\ 1 \le p \le N-1.$$

If $p(x) = x^\alpha$, then

$$c_k = N^{-1}\left(\frac{1}{\alpha+1} - \sum_{s\in\mathbb{Z},\ sN^{-1}\notin\mathbb{Z}} \sum_{n=0}^{\alpha-1} \frac{\alpha! e^{2\pi i s k N^{-1}}}{(\alpha-n)! k^{2m} (2\pi i k)^{n+1} L(k)}\right),$$

where

$$L(k) = N^{-2m} \sum_s (s - kN^{-1}), \quad 1 \le k \le N,\ N = 2, 3, \ldots.$$

Alexander K. Kushpel′

MR2843703 41A55 33C45 42C05 65D30

Milovanović, Gradimir V.

Numerical quadratures and orthogonal polynomials. (English summary)

Stud. Univ. Babeş-Bolyai Math. **56** (2011), *no.* 2, 449–464.

The construction of quadrature formulae of the maximal, or nearly maximal, algebraic degree of exactness for integrals involving a positive measure $d\sigma$ is closely connected to orthogonal polynomials on the real line with respect to the inner product

$$(f, g) = (f, g)_{d\sigma} = \int_{\mathbb{R}} f(t) g(t) d\sigma(t) \quad (f, g \in L^2(d\sigma)).$$

This paper discusses different types of quadrature formulae such as Gauss-Christoffel quadratures, quadratures with multiple nodes, and Birkhoff-Young quadratures. For example, the Gauss-Christoffel quadrature formula is defined as follows: Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n and $d\sigma$ be a finite positive Borel measure on the real line $\mathbb{R}$ such that its support supp$(d\sigma)$ is an infinite set, and all its moments $\mu_k = \int_{\mathbb{R}} t^k d\sigma(t)$, $k = 0, 1, \ldots$, exist and are finite. The n-point quadrature formula

$$\int_{\mathbb{R}} f(t) d\sigma(t) = \sum_{k=1}^{n} \sigma_k f(\tau_k) + R_n(f),$$

which is exact on the set $\mathcal{P}_{2n-1}$, is known as the Gauss-Christoffel quadrature formula. It is a quadrature formula of the maximal algebraic degree of exactness, i.e. $R_n(\mathcal{P}_{d_{\max}}) = 0$, where $d_{\max} = 2n - 1$. In the same manner, other types of quadratures, such as quadratures with multiple nodes and Birkhoff-Young quadratures, are presented. This paper gives an account of some important connections between orthogonal polynomials and Gaussian quadratures, as well as several types of generalized orthogonal polynomials and corresponding types of quadratures with simple and multiple nodes.

The paper has 4 sections. The introduction is presented in Section 1. In Section 2 quadratures of Gaussian type (with maximal or nearly maximal degree of exactness) and quasi-orthogonal polynomials are considered. Section 3 is devoted to presenting a connection between s- and σ-orthogonal polynomials and quadratures with multiple nodes. Finally, in Section 4 multiple orthogonal polynomials and two applications are presented. *Mohammad R. Eslahchi*

MR2906461 41A58 49M15

Argyros, Ioannis K. (1-CMRN; Lawton, OK);
Ren, Hongmin (PRC-HZPT-CIE; Hangzhou)

A relationship between the Lipschitz constants appearing in Taylor's formula. (English summary)

J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math. **18** (2011), *no.* 4, 345–351.

In this paper the authors consider an m-times Fréchet-differentiable operator F such that $F^{(m)}$ is Lipschitz continuous. They obtain some Taylor formulas in terms of the Lipschitz constant. Moreover, the constant is used to relate the corresponding Lipschitz constants for the operators $F^{(i)},\ i=1,\dots,m$. Some applications are provided.

Cristian Tacelli

MR2855760 41A58 30K05 35A35 46E10 62E17

Nestoridis, Vassili (GR-UATH; Athens);
Schmutzhard, Sebastian (A-WIENM; Vienna);
Stefanopoulos, Vangelis [Stefanopoulos, Vagelis] (GR-AEG2; Karlovassi)

Universal series induced by approximate identities and some relevant applications. (English summary)

J. Approx. Theory **163** (2011), *no.* 12, 1783–1797.

The authors prove the existence of series $\sum a_n\psi_n$ having the property that their partial sums are dense in various function spaces, where the coefficient sequence (a_n) belongs to $\bigcap_{p>1}\ell^p$. Let us mention just one of the numerous results in this interesting paper.

Let $G(x,t)=(4\pi t)^{-d/2}e^{-|x|^2/4t}$ be the Gaussian heat kernel ($x\in\mathbb{R}^d, t>0$). If (ξ_n) is an enumeration of the rational vectors in $\mathbb{R}^d$, then there exists $(a_n)\in\bigcap_{p>1}\ell^p$ such that the series $\sum a_nG(x-\xi_n,t)$ has the following property: given any bounded solution u of the heat equation $u_t=\Delta_x u$ with a bounded and continuous initial data $u(x,0)=f$, there is an increasing sequence (ν_n) in $\mathbb{N}$ such that

$$\sum_{k=1}^{\nu_n} a_kG(x-\xi_n,t)\to u(x,t)$$

locally uniformly on $\mathbb{R}^d\times\mathbb{R}^+$ (together with all the associated higher-order derivatives in x and t). Moreover, the set of such universal sequences (a_n) is a G_δ-set in $\bigcap_{p>1}\ell^p$ and it contains a dense vector subspace of $\bigcap_{p>1}\ell^p$ (except the zero sequence).

Raymond Mortini

MR2844893 41A58 26E70 40A30

Pang, Lisha (PRC-WHIT-M; Weihai); **Wang, Ke [Wang, Ke[1]]** (PRC-WHIT-M; Weihai)

Function series theory of time scales. (English summary)

Comput. Math. Appl. **62** (2011), *no.* 9, 3427–3437.

Summary: "In this paper, we extend the concept of function series to time scales, and then we present two necessary and sufficient conditions and several criteria for uniform convergence. Moreover, we demonstrate several analytical properties of function series on general time scales. Furthermore, we illustrate our conclusions through examples, respectively."

MR2871785 41A60 41A65 65Y20

Weimar, Markus (D-FSU-MI; Jena)

Tractability results for weighted Banach spaces of smooth functions. (English summary)

J. Complexity **28** (2012), *no.* 1, 59–75.

Let $f\colon [0,1]^d \to \mathbb{R}$ be a bounded function belonging to a Banach function space $\mathcal{F}_d$, endowed with the L_∞-norm. The author considers the worst case error

$$e^{\rm wor}(A_{n,d};\mathcal{F}_d) = \sup_{\|f\|\le 1} \|f - A_{n,d}(f)\|_{L_\infty([0,1]^d)},$$

for the algorithms $A_{n,d} \in \mathcal{A}_n$. These algorithms use n pieces of information in d dimensions from a given class Λ of information. The author is interested in the nth minimal error

$$e(n,d;\mathcal{F}_d) = \inf\{e^{\rm wor}(A_{n,d};\mathcal{F}_d)\colon A_{n,d} \in \mathcal{A}_n\}.$$

If the quantity $n(\varepsilon,d;\mathcal{F}_d) = \min\{n \in \mathbb{N}_0\colon e(n,d;\mathcal{F}_d) \le \varepsilon\}$ (information complexity) depends exponentially on dimension d or ε^{-1} then the problem considered is called intractable. The problem is called weakly tractable if $\lim\limits_{\varepsilon^{-1}+d\to\infty} \frac{\ln(n(\varepsilon,d;\mathcal{F}_d))}{\varepsilon^{-1}+d} = 0$, and polynomially tractable if there exist the constants $c,p,q > 0$ such that $n(\varepsilon,d;\mathcal{F}_d) \ge c\varepsilon^{-1}d^q$, for all $d \in \mathbb{N}$, $\varepsilon > 0$.

The author gives necessary and sufficient conditions for several kinds of tractability for a whole scale of weighted Banach spaces of smooth d-variate functions. The equivalence of weak tractability with the fact that the problem does not suffer from the "curse of dimensionality" is also proved. *Costică Mustăţa*

MR2869746 41A65 46B99

Almira, J. M. [Almira Picazo, J. M.] (E-JAE; Jaén)**;**
Oikhberg, T. (1-CA3; Irvine, CA)

Shapiro's theorem for subspaces. (English summary)

J. Math. Anal. Appl. **388** (2012), *no.* 1, 282–302.

Shapiro's theorem of the title is the following [H. S. Shapiro, Michigan Math. J. **11** (1964), 211–217; MR0167769 (29 #5041)]: Let X be a Banach space, let $\{A_n\}$ be an increasing sequence of closed proper subspaces of X and let $\{\varepsilon_n\}$ be a null sequence of positive numbers. Then there exists $x \in X$ such that

$$E(x,A_n) = \inf_{a\in A_n} \|x-a\| \ne O(\varepsilon_n),$$

that is, the errors of approximation of an $x \in X$ by elements in the A_n's may decay arbitrarily slowly. This theorem is inspired by a classical result of Bernstein (see 2.5 of [A. F. Timan, *Theory of approximation of functions of a real variable*, Translated from the Russian by J. Berry. English translation edited and editorial preface by J. Cossar. International Series of Monographs in Pure and Applied Mathematics, Vol. 34, A Pergamon Press Book. The Macmillan Co., New York, 1963; MR0192238 (33 #465)] or Chapter II, Theorem 5.8. of [I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Translated from the Romanian by Radu Georgescu. Die Grundlehren der mathematischen Wissenschaften, Band 171, Publ. House Acad. SR Romania, Bucharest, 1970; MR0270044 (42 #4937)]).

In a previous paper [J. Approx. Theory **164** (2012), no. 5, 534–571, doi:10.1016/j.jat.2012.01.005], the authors studied the validity of Shapiro's theorem in quasi-Banach spaces when one considers certain sequences of subsets of X (the so-called approximation schemes) instead of just closed *subspaces* of X. In the paper under review they continue this study. Their aim now is to find out when the x's witnessing slow approximation can be taken in a previously fixed subspace Y of X. *José Mendoza*

MR2884661 41A65 41A50 46B20

Borodin, P. A. (RS-MOSCM-NDM; Moscow)

On the convexity of N-Chebyshev sets. (Russian. Russian summary)

Izv. Ross. Akad. Nauk Ser. Mat. **75** (2011), *no.* 5, 19–46; *translation in Izv. Math.* **75** (2011), *no.* 5, 889–914.

The paper deals with the problem of convexity of Chebyshev sets. A nonempty subset M of a Banach space X is called a Chebyshev set if, for each $x \in X$, its metric projection $P_M(x)$ contains exactly one element. M. M. Day [Bull. Amer. Math. Soc. **47** (1941), 313–317; MR0003446 (2,221b)] proved that every convex closed set in a strictly convex reflexive Banach space is Chebyshev. The converse—the problem of convexity of Chebyshev sets—is still open.

In the paper under review the author introduces N-Chebyshev sets. Given natural number N, a nonempty $M \subset X$, and $x_1, \ldots, x_N \in X$, let

$$\rho(x_1, \ldots, x_N; M) = \inf_{y \in M} \sum_{k=1}^{N} ||x_k - y||$$

and $\rho(x_1, \ldots, x_N) = \rho(x_1, \ldots, x_N; X)$. Now, the N-projection of the set $\{x_1, \ldots, x_N\}$ on M is

$$P_M(x_1, \ldots, x_N) = \left\{ y \in M\colon\ \rho(x_1, \ldots, x_N; M) = \sum_{k=1}^{N} ||x_k - y|| \right\}.$$

A set M is said to be N-Chebyshev if for all $x_1, \ldots, x_N \in X$ one of the following conditions is valid:

(1) $\rho(x_1, \ldots, x_N; M) > \rho(x_1, \ldots, x_N)$ and the corresponding N-projection contains exactly one element;

(2) $\rho(x_1, \ldots, x_N; M) = \rho(x_1, \ldots, x_N)$ and $P_M(x_1, \ldots, x_N) = \varnothing$.

This generalizes the 2-Chebyshev sets suggested by the author [Vestnik Moskov. Univ. Ser. I Mat. Mekh. **2008**, no. 3, 16–19, 71; MR2517003 (2010d:41039)], whereas $N = 1$ gives the usual Chebyshev sets. The main result is Theorem 3:

(1) Suppose N is even and X is a uniformly convex Banach space. Then a set M in X is N-Chebyshev if and only if it is convex and closed.

(2) Suppose $N \geq 3$ is odd and X is a uniformly convex and smooth Banach space. Then a set M in X is N-Chebyshev if and only if it is convex and closed.

Alexander P. Goncharov

MR2884461 41A65 41A44

Micek, Agnieszka (PL-JAGL-DM; Kraków)

Constants of strong uniqueness of minimal norm-one projections. (English summary)

Function spaces IX, 265–277, *Banach Center Publ.*, 92, *Polish Acad. Sci. Inst. Math.*, *Warsaw*, 2011.

Summary: "In this paper we calculate the constants of strong uniqueness of minimal norm-one projections on subspaces of codimension k in the space $l_\infty^{(n)}$. This generalizes a main result of W. P. Odyniec and M. P. Prophet [J. Approx. Theory **145** (2007), no. 1, 111–121; MR2300966 (2008f:41044)]. We applied in our proof Kolmogorov's type theorem [see A. K. Wójcik, in *Approximation and function spaces (Gdańsk, 1979)*, 854–866, North-Holland, Amsterdam, 1981; MR0649483 (83c:41040)] for strongly unique

best approximation."
{For the entire collection see MR2884443 (2012i:00023).}

MR2906831 41A65 41A50 46A55
Narang, T. D. (6-GNDU; Amritsar); **Sangeeta**
Some characterizations of suns in terms of fixed points in linear metric spaces. (English summary)
Math. Student **79** (2010), *no.* 1-4, 163–170 (2011).
Summary: "One of the outstanding open problems of approximation theory is: Whether every Chebyshev set in a Hilbert space is convex? While making an attempt to answer this question, N. V. Efimov and S. B. Stechkin [Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 17–19; MR0095445 (20 #1947)] introduced the notion of sun and L. P. Vlasov [Uspehi Mat. Nauk **28** (1973), no. 6(174), 3–66; MR0404963 (53 #8761)] introduced the concepts of α-, β-, γ- and δ-suns. B. Brosowski in [Mathematica (Cluj) **11 (34)** (1969), 195–220; MR0277979 (43 #3712)] gave some characterizations of α- and β-suns in terms of fixed points in normal linear spaces. In this paper we extend these results of Brosowski to linear metric spaces."

MR2883511 41A65 41A60
Temlyakov, V. N. [Temlyakov, Vladimir N.] (1-SC; Columbia, SC);
Yang, Mingrui (1-SC; Columbia, SC);
Ye, Peixin [Ye, Pei Xin] (PRC-NNK-SM; Tianjin)
Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases. (English summary)
East J. Approx. **17** (2011), *no.* 2, 203–214.
Let $(X, \|\cdot\|)$ be an infinite-dimensional separable Banach space and let $\Psi = \{\psi_k\}$ be a basis of X, with $\|\psi_k\| = 1$, $k \in \mathbb{N}$. For a given $f \in X$, the best m-term approximation with respect to Ψ is defined as follows:

$$\sigma_m(f) = \sigma_m(f, \Psi)_X = \inf_{b_k, \Lambda} \left\| f - \sum_{k\in\Lambda} b_k \psi_k \right\|$$

where the infimum is taken over coefficients b_k and the sets Λ of indices with cardinality $|\Lambda| = m$.

Consider the expansion of a given element $f \in X$ as

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

A permutation ρ, $\rho(j) = k_j$, $j = 1, 2, \ldots$, of the positive integers is called decreasing and $\rho \in D(f)$ if

$$|c_{k_1}(f)| \geq |c_{k_2}(f)| \geq \cdots.$$

The m-th greedy approximation of $f \in X$ with respect to the basis Ψ corresponding to a permutation $\rho \in D(f)$ is defined by

$$G_m(f) = G_m(f, \Psi, \rho) = \sum_{j=1}^{m} c_{K_j}(f)\psi_{k_j}.$$

The basis Ψ is called quasi-greedy if there exists a constant C independent of f such that

$$\sup_m \|G_m(f, \Psi)\| \leq C\|f\|.$$

The aim of this paper is to study greedy approximation with respect to quasi-greedy bases in separable Banach spaces. The main result is a theorem that states that if Ψ is a quasi-greedy basis of the L_p space, $1 < p < \infty$, $p \neq 2$, then for each $f \in L_p$, we have

$$\|f - G_m(f,\Psi)\|_p \le C(p,\Psi) m^{|\frac{1}{2}-\frac{1}{p}|}\sigma_m(f,\Psi)_p.$$

Further, the authors extend the above result to a separable Banach space X that has a quasi-greedy basis Ψ, which yields the following estimate, for any set of indices Λ and for each $f \in X$:

$$\Big\| \sum_{k\in\Lambda} c_k(f)\psi_k \Big\| \le w(|\Lambda|)\|f\|,$$

where $w(m)$ is an increasing function on m. They prove that

$$\|f - G_m(f)\| \le (1 + 2w(m) + (2K)^3 v(m)w(m))\sigma_m(f)$$

where $v(m) = v(m,\Psi)$ is an increasing function that, for any two sets of indices A and B, $|A| = |B| = m$, satisfies

$$\Big\| \sum_{k\in A} \psi_k \Big\| \le v(m) \Big\| \sum_{k\in B} \psi_k \Big\|.$$

In particular, if Ψ is a quasi-greedy basis of the L_p space, $1 < p < 2$, $2 < p < \infty$, then for any set of indices Λ,

$$\Big\| \sum_{k\in\Lambda} c_k(f)\psi_k \Big\|_p \le C(p)|\Lambda|^{|\frac{1}{p}-\frac{1}{2}|}\|f\|_p.$$

Finally, if $1 < p < \infty$, $p \neq 2$, Ψ is a quasi-greedy basis of the L_p space and $G^t_m(f) = \sum_{k\in\Lambda} c_k(f)\psi_k$ for fixed $t \in (0,1]$, then for each $f \in L_p$, we have

$$\|f - G^t_m(f)\|_p \le C(\Psi,p,t) m^{|\frac{1}{p}-\frac{1}{2}|}\sigma_m(f)_p.$$

Sundara M. Lalithambigai

Items with secondary classifications in Sections 39, 40, 41

MR2855062 06E30 39B22 91B16

Couceiro, Miguel (LUX-CULTC-MR; Luxembourg);
Marichal, Jean-Luc (LUX-CULTC-MR; Luxembourg)

Axiomatizations of quasi-Lovász extensions of pseudo-Boolean functions. (English summary)

Aequationes Math. **82** (2011), *no.* 3, 213–231.

Summary: "We introduce the concept of quasi-Lovász extension as being a mapping $f\colon I^n \to \mathbb{R}$ defined on a nonempty real interval I containing the origin and which can be factorized as $f(x_1,\ldots,x_n) = L(\varphi(x_1),\ldots,\varphi(x_n))$, where L is the Lovász extension of a pseudo-Boolean function $\psi\colon\{0,1\}^n \to \mathbb{R}$ (i.e., the function $L\colon\mathbb{R}^n \to \mathbb{R}$ whose restriction to each simplex of the standard triangulation of $[0,1]^n$ is the unique affine function which agrees with ψ at the vertices of this simplex) and $\varphi\colon I \to \mathbb{R}$ is a nondecreasing function vanishing at the origin. These functions appear naturally within the scope of

decision making under uncertainty since they subsume overall preference functionals associated with discrete Choquet integrals whose variables are transformed by a given utility function. To axiomatize the class of quasi-Lovász extensions, we propose generalizations of properties used to characterize Lovász extensions, including a comonotonic version of modularity and a natural relaxation of homogeneity. A variant of the latter property enables us to axiomatize also the class of symmetric quasi-Lovász extensions, which are compositions of symmetric Lovász extensions with 1-place nondecreasing odd functions."

MR2852960 06F15 40G10
Freedman, Walden (1-HUMB; Arcata, CA)
Abel's summation formula in partially ordered groups. (English summary)
Topology Appl. **159** (2012), *no.* 1, 175–182.
The author shows that certain results concerning infinite series of reals can be generalized for the case of infinite sums in the theory of partially ordered groups, when instead of reals we take elements of a partially ordered group. Namely, he deals with Abel's summation formula, the du Bois-Reymond test, Abel's test and some further theorems (as a reference book he uses the monograph [K. Knopp, *Theory and application of infinite series*, translated from the second German edition, Dover, New York, 1990]). For Abel's summation formula, the generalization is as follows: For each n, let $\alpha_n\colon G\to H$ be a homomorphism. Let (x_n) be a sequence in G, and let $X_k=\prod_{j=1}^k x_j$ for each $k\in\mathbb{N}$. Then for all $n\in\mathbb{N}$,

$$\prod_{k=1}^n \alpha_k(x_k)=\left[\prod_{k=1}^{n-1}(\alpha_k\alpha_{k+1}^*)(X_k)\right]\cdot\alpha_n(X_n),$$

where the term in brackets is understood to be the identity of H when $n=1$.

J. Jakubík

MR2817651 11B30 11B13 11T23 41A46 94A12
Bourgain, Jean (1-IASP-SM; Princeton, NJ);
Dilworth, Stephen [Dilworth, Stephen J.] (1-SC; Columbia, SC);
Ford, Kevin [Ford, Kevin B.] (1-IL; Urbana, IL);
Konyagin, Sergei (RS-AOS; Moscow);
Kutzarova, Denka [Kutzarova, Denka N.] (BG-AOS; Sofia)
Explicit constructions of RIP matrices and related problems. (English summary)
Duke Math. J. **159** (2011), *no.* 1, 145–185.
Let $1\le k\le n\le N$ and $0<\delta<1$. A signal $\mathbf{x}=(x_j)_{j=1}^N\in\mathbb{C}^N$ is said to be k-sparse if $\mathbf{x}$ has at most k nonzero coordinates. An $n\times N$ matrix Φ is said to satisfy the Restricted Isometry Property (RIP) of order k with constant δ if, for all k-sparse vectors $\mathbf{x}$, we have $(1-\delta)\|\mathbf{x}\|_2^2\le\|\Phi\mathbf{x}\|_2^2\le(1+\delta)\|\mathbf{x}\|_2^2$. The matrices satisfying the RIP and its variants have many applications, in particular to sparse signal recovery. All previously known explicit examples of RIP matrices are based on constructions of systems of unit vectors with small coherence, which do not allow construction of RIP matrices of order larger than $\sqrt{n}$ and constant $\delta<1$. In the present paper the authors overcome the natural barrier $k=O(n^{1/2})$ using methods of additive combinatorics. They prove that there is an effective constant $\varepsilon_0>0$ and an explicit number n_0 such that, for any positive integers $n\ge n_0$ and $n\le N\le n^{1+\varepsilon_0}$, there is an explicit $n\times N$ RIP matrix of order $\lfloor n^{1/2+\varepsilon_0}\rfloor$ with constant $n^{-\varepsilon_0}$. The proof is extremely involved and uses a result on additive energy of sets, estimates for sizes of sumsets in product sets and bounds for exponential sums over products of sets possessing special additive structure. *Moubariz Z. Garaev*

MR2847501 11B65 11T22 39B52

Nathanson, Melvyn B. (1-CUNY7-DM; Bronx, NY)

Semidirect products and functional equations for quantum multiplication. (English summary)

J. Algebra Appl. **10** (2011), *no.* 5, 827–834.

A quantum integer $[n]_q$ is the polynomial $1+q+q^2+\cdots+q^{n-1}$, and the sequence of polynomials $\{[n]_q\}_{n=1}^{\infty}$ is a solution of the functional equation $f_{mn}(q)=f_m(q)f_n(q^m)$. This paper is a sequel to [J. Number Theory **103** (2003), no. 2, 214–233; MR2020269 (2005d:11023)], in which the author studied quantum integers. In this paper, he applies the concept of semidirect product of semigroups to produce families of functional equations that generalize the functional equation for quantum multiplication. A list of 4 open problems adds value to this paper. *Jau-Shyong Shiue*

MR2882184 11M06 41A58 41A60

Mozer, Ya. [Mozer, Ján] (SK-KMSK-NDM; Bratislava)

Jacobi's ladders and almost exact asymptotic representations of the Hardy-Littlewood integral. (Russian. Russian summary)

Mat. Zametki **88** (2010), *no.* 3, 446–455; *translation in Math. Notes* **88** (2010), *no.* 3-4, 414–422.

In the paper, the mean square

$$I(T)=\int_0^T Z^2(t)dt,$$

where $Z(t)$ is the Hardy-Littlewood function defined by

$$Z(t)=e^{i\theta(t)}\zeta(\tfrac{1}{2}+it)$$

with

$$\theta(t)=-\tfrac{1}{2}t\log\pi+\operatorname{Im}\log\Gamma(\tfrac{1}{4}+\tfrac{it}{2})$$

($\zeta(s)$ is the Riemann zeta-function), is considered, and a precise formula is obtained.

Let $\{\mu\}$ be the class of positive functions $\mu\in C^{\infty}([y_0,\infty])$ such that $\mu(y)$ increases to $+\infty$, and $\mu(y)\ge 7y\log y$. Then, for every $\mu\in\{\mu\}$, there exists the unique solution $\varphi(T)=\varphi_\mu(T)$, $T\in[T_0,\infty)$, $T_0=T_0(\varphi)$, $\varphi(T)\to\infty$ as $T\to\infty$, of the integral nonlinear equation

$$\int_0^{\mu(x(T))} Z^2(t)e^{-\frac{2t}{x(T)}}dt=\int_0^T Z^2(t)dt.$$

Moreover, the function φ has the following property. Suppose that $t=\gamma$ is a zero of order $n=n(\gamma)$ of the function $\zeta(\frac{1}{2}+it)$. Then $\varphi'(\gamma)=\cdots=\varphi^{(2n)}(\gamma)=0$ and $\varphi^{(2n+1)}(\gamma)\neq 0$. Denote the class of such solutions by $\{\varphi\}$. The author proves that, for all $\varphi\in\{\varphi\}$ and $T\ge T_0(\varphi)$, the formula

$$I(T)=F(\varphi(T))+r(\varphi(T)),$$

where

$$F(y)=\frac{y}{2}\log\frac{y}{2}+(c-\log 2\pi)\frac{y}{2}+c_0,$$
$$r(\varphi(T))=O\left(\frac{\log\varphi(T)}{\varphi(T)}\right)=O\left(\frac{\log T}{T}\right),$$

and c is the Euler constant, holds. Moreover, for any $\varphi_1,\varphi_2\in\{\varphi\}$ and $T\ge$

$\max(T_0(\varphi_1), T_0(\varphi_2))$,

$$\varphi_1(T) - \varphi_2(T) = O\left(\frac{1}{T}\right).$$

The above formula improves the estimate

$$I(T) = T \log T + (2c - 1 - \log 2\pi)T + O(T^{\frac{1}{3}+\varepsilon})$$

obtained by R. Balasubramanian in [Proc. London Math. Soc. (3) **36** (1978), no. 3, 540–576; MR0476664 (57 #16223)].

For the proof, the formula

$$\int_0^\infty Z^2(t)e^{-2\delta t}dt = \frac{c - \log(4\pi\delta)}{2\sin\delta} + \sum_{n=0}^{N} c_n\delta^n + O(\delta^{N+1}),$$

$\delta \to 0$, is applied. *Antanas Laurinčikas*

MR2882387 12H10 12J25 39A13 46S10

Boudjrida, Najet (DZ-UJ-LMA; Jijel)**;**
Boutabaa, Abdelbaki (F-CLEF2-LM; Aubière)**; Medjerab, Samia** (DZ-UJ-LMA; Jijel)

q-difference equations in ultrametric fields. (English summary)

Advances in non-Archimedean analysis, 39–49, *Contemp. Math.*, 551, *Amer. Math. Soc.*, *Providence*, *RI*, 2011.

In 1998, W. Bergweiler, K. Ishizaki and N. Yanagihara [Methods Appl. Anal. **5** (1998), no. 3, 248–258; MR1659151 (2000a:39025)] showed that transcendental meromorphic solutions $f(z)$ of the functional equation

$$\text{(E)} \qquad \sum_{j=0}^{n} a_j(z)f(c^j z) = Q(z),$$

where $0 < c < 1$ is a complex number and $a_j(z)$ $(j = 0, 1, \ldots, n)$ and $Q(z)$ are rational functions with $a_0(z) \equiv 0$ and $a_n(z) \equiv 1$, satisfy $T(r, f) = O((\log r)^2)$ and $(\log r)^2 = O(T(r, f))$. The equation has some similarity to the Schröder equation, but its study shows somewhat different aspects [E. Schröder, Math. Ann. **3** (1870), no. 2, 296–322; MR1509704]. The authors use Nevanlinna theory to solve the equation (E). Here, they obtain a generalization of some of these results in $\mathbb{C}$ to meromorphic functions in an ultrametric field. As in the complex case, the authors use a method based on Nevanlinna theory.

{For the entire collection see MR2883743 (2012i:46001).} *Fana Tangara*

MR2883824 13F25 40C15

Graev, M. I. [Graev, Mark Iosifovich] (RS-AOS-SS; Moscow)

Rings of formal power series and a multidimensional analogue of Lagrange's formula. (Russian)

Dokl. Akad. Nauk **439** (2011), *no.* 2, 151–155; *translation in Dokl. Math.* **84** (2011), *no.* 1, 459–463.

In the paper under review the author studies systems of n equations in n unknowns $y = (y_1, \ldots, y_n)$ of the following form: $y_i = \sum_{j=0}^{m_i} x_{ij} f_{ij}(y)$, where $f_{ij}(y)$ are formal power series in y with coefficients in a commutative ring with identity. By the implicit function theorem any such system has a unique solution $y(x) = (y_1(x), \ldots, y_m(x))$, where $x = (x_{ij})$.

The main result of the paper is as follows.

Theorem 5. Let $F(y)$ be a formal power series in $y = (y_1, \ldots, y_n)$ and let $F(y(x)) = \sum_\nu c(\nu)x^\nu$, where $x^\nu = \prod_{i,j} x_{ij}^{\nu_{ij}}$. Then $c(\nu) = \mathrm{Res}_x(F(y(x))\Phi_\nu(y(x)))$, where $\Phi_\nu(y)$ are

"Laurent power series" effectively computed by means of power series $f_{ij}(y)$.

In the case of one equation $y = x_1 f_1(y) + \cdots + x_n f_n(y)$ the author obtains the classical Lagrange formula [G. Belardinelli, *Fonctions hypergéométriques de plusieurs variables et résolution analytique des équations algébriques générales*, Mémor. Sci. Math., Fasc. 145, Gauthier-Villars, Paris, 1960; MR0121518 (22 #12256)]. *Arkadiusz Płoski*

MR2918509 26A06 40A05

Furdui, Ovidiu; Qin, Huizeng (PRC-SUT-IAM; Zibo)

When is the limit equal to the supremum norm of f? (English summary)

Creat. Math. Inform. **20** (2011), *no.* 2, 125–129.

Motivated by the fact that for a continuous function f on $[0,1]$, the limit of the L^p-norm is the sup norm, i.e.,

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 |f(x)|^n dx} = \|f\|_\infty = \sup_{0\le x\le 1} |f(x)|,$$

the authors investigate

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx}$$

for nonnegative f. The authors show that if f is continuous, nonnegative, and attains its maximum on $[0,1]$ at 0 or at 1, then

$$(1) \qquad \lim_{n\to\infty} \sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx} = \|f\|_\infty.$$

It is conjectured that (1) fails when f does not attain its maximum on $[0,1]$ at 0 or at 1, and an example is given where this is the case. *G. A. Heuer*

MR2837902 26D15 39A14

Feng, Qinghua (PRC-SUT-SSC; Zibo);
Meng, Fanwei [Meng, Fan Wei] (PRC-QTC-SM; Qufu);
Zhang, Yaoming (PRC-SUT-SSC; Zibo)

Generalized Gronwall-Bellman-type discrete inequalities and their applications. (English summary)

J. Inequal. Appl. **2011,** 2011:47, 12 *pp.*

Assume that the sequence $\{u(m,n)\}_{m,n\in\mathbb{Z}}$ satisfies the inequality

$$(1) \qquad \eta(u(m,n)) \le a(m,n) + k(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\varphi(u(s,t))$$

on $\mathbb{Z}^2$ ($\mathbb{Z}$ is the set of integers). Under some conditions on the functions η, a, k, b, φ, an upper bound for $\{u(m,n)\}_{m,n\in\mathbb{Z}}$ is obtained on some subset of $\mathbb{Z}^2$. The results obtained extend some previous results established by Q. H. Ma and W.-S. Cheung [J. Comput. Appl. Math. **202** (2007), no. 2, 339–351; MR2319961 (2008a:26024)]. Some applications of the authors' results are also considered. For example, they study the boundedness of solutions to the partial difference equation

$$\Delta_{12} \exp(u(m,n)) = F(m,n,u(m,n)).$$

Fozi M. Dannan

MR2855064 26E60 39B22

Matkowski, Janusz (PL-UZLGEM; Zielona Góra)

Quotient mean, its invariance with respect to a quasi-arithmetic mean-type mapping, and some applications. (English summary)

Aequationes Math. **82** (2011), *no.* 3, 247–253.

Let the real functions f and g be continuous, positive, and strictly monotonic in a real interval I, and let $p, r \in (0,1)$. Then the function $A_p^{[f]}\colon I^2 \to I$,

$$A_p^{[f]}(x,y) := f^{-1}(pf(x) + (1-p)f(y)), \quad x, y \in I,$$

is a mean called the weighted quasi-arithmetic mean, and if the functions f and g are of different types of monotonicity, the function $Q^{[f,g]}\colon I^2 \to \mathbb{R}$, defined by

$$Q^{[f,g]}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)}{g(y)}\right), \quad x, y \in I,$$

is also a mean, which is called a quotient mean. The author proves that $Q^{[f,g]}$ is $(A_p^{[f]}, A_r^{[g]})$-invariant, that is,

$$Q^{[f,g]} \circ (A_p^{[f]}, A_r^{[g]}) = Q^{[f,g]},$$

if and only if fg is a constant function and $p + r = 1$. An application is also given.

Silvia Toader

MR2833811 30B10 13F25 16W60 26E35 40A30 47B33

Gan, Xiao-Xiong (1-MRGS; Baltimore, MD);
Bugajewski, Dariusz (PL-POZN-MC; Poznań)

On formal Laurent series. (English summary)

Bull. Braz. Math. Soc. (*N.S.*) **42** (2011), *no.* 3, 415–437.

The purpose of this article is to introduce and study a number of operations for formal Laurent series. These operations are analogous to those on the space of formal power series. In particular, the authors define a multiplication for some classes of formal Laurent series. They also introduce a composition of a formal Laurent series with a formal power series and provide a necessary and sufficient condition for the existence of such compositions.

Jasson Vindas

MR2816945 30B40 30B30 40A25

Gharibyan, Tatevik L. (AR-AOS; Yerevan); **Luh, Wolfgang** (D-TRR; Trier)

Summability of elongated sequences. (English summary)

Comput. Methods Funct. Theory **11** (2011), *no.* 1, 59–70.

A sequence $\{s_n\} \subset \mathbb{C}$ is elongated with respect to a sequence $\{m_n\} \subset \mathbb{N}$ if the term s_n is listed m_n times and the resulting modified sequence is considered instead of $\{s_n\}$. The authors investigate partial sums of power series and Cesàro summability of elongated partial sums of power series outside the circle of convergence. As a main result they show that such an elongation exists if and only if the power series under consideration is overconvergent.

Piotr Stanislaw Kot

MR2905424 30D35 39B32
Liu, Kai [Liu, Kai[4]] (PRC-NCH; Nanchang)**;**
Yang, Lianzhong (PRC-SHAN-SM; Jinan)**;**
Liu, Xinling [Liu, Xin-Ling] (PRC-NCH; Nanchang)
Existence of entire solutions of nonlinear difference equations. (English summary)
Czechoslovak Math. J. **61(136)** (2011), *no.* 2, 565–576.
Summary: "In this paper we obtain that there are no transcendental entire solutions with finite order of some nonlinear difference equations of different forms."

MR2906170 30E05 41A05
Macía, Benxamín (E-VIGO2S; Orense)**; Tugores, Francesc** (E-VIGO2S; Orense)
Double interpolating sequences in Schwarz sense. (English summary)
Adv. Appl. Math. Sci. **10** (2011), *no.* 5, 543–547.
Summary: "This paper deals with a double interpolation problem in the unit disc $\mathbb{D}$ of the complex plane. We transfer a relationship between a bounded holomorphic function and its derivative to the sequences that are to be interpolated, as a restriction to the some ones."

MR2828544 30E10 30-02 30B10 30D30 41A21
López Lagomasino, Guillermo (E-CARL-M; Leganés)**;**
Vavilov, Valeri V. [Vavilov, V. V.] (RS-MOSC-EDC; Moscow)
On a theorem of Hadamard and the Cauchy-Hadamard formula. (Spanish. Spanish summary)
Gac. R. Soc. Mat. Esp. **14** (2011), *no.* 1, 85–96.
This work is an expository paper in which the authors present a short review of the theory of the rows of rational Padé approximants and its relation with some classical results on analyticity of complex power series.

In particular, they present a theorem of J. Hadamard, which improves the classical result of A. L. Cauchy on the radius of convergence of complex power series, and claim that this is the reason why this result is known as the Cauchy-Hadamard formula, although the first mathematician died before the second was born.

José A. Prado-Bassas

MR2760601 30E10 41A10
Martirosyan, V. A. (AR-YER; Yerevan)**; Mkrtchyan, S. E.** (AR-AOS; Yerevan)
Approximation in the mean by polynomials with gaps on non-Carathéodory domains. (Russian. English and Russian summaries)
Izv. Nats. Akad. Nauk Armenii Mat. **45** (2010), *no.* 3, 73–83; *translation in J. Contemp. Math. Anal.* **45** (2010), *no.* 3, 162–169.
A bounded simply connected domain Ω in $\mathbb{C}$ is called a Carathéodory domain if its boundary coincides with the boundary of the unbounded component of the complement of $\overline{\Omega}$. The topic of mean (with respect to area measure) approximation by polynomials on both Carathéodory and non-Carathéodory domains has been investigated by O. J. Farrell, A. I. Markushevich, M. V. Keldysh, A. L. Shahinyan, M. M. Djrbashyan, S. N. Mergelyan, S. O. Sinanjan, J. E. Brennan, and other mathematicians.

In the paper under review the authors consider the problem of (mean) polynomial approximation with gaps in the L_p-norm, $1 \leq p < \infty$, on some known subclasses of non-Carathéodory domains (i.e. domains with boundary cuts, traditional crescent, and generalized crescent). This interesting paper completes recent research of the same authors in which similar approximation on Carathéodory domains (sets) has been considered [see V. A. Martirosyan and S. E. Mkrtchyan, Izv. Nats. Akad. Nauk Armenii Mat. **43** (2008), no. 6, 82–88; MR2572280 (2011a:30091)]. *Arthur A. Danielyan*

MR2734198 31-02 30E10 31A15 41A20

Saff, E. B. [Saff, Edward B.] (1-VDB-CCA; Nashville, TN)

Logarithmic potential theory with applications to approximation theory. (English summary)

Surv. Approx. Theory **5** (2010), 165–200.

Back when I was a graduate student about to embark on research in rational approximation, one of the first articles my supervisor gave me to read was the survey article [E. B. Saff, in *Approximation theory (New Orleans, La., 1986)*, 21–49, Proc. Sympos. Appl. Math., 36, Amer. Math. Soc., Providence, RI, 1986; MR0864364 (88h:30061)] by this author. That article still stands as a good introduction to the field of complex approximation, but as potential theory is an essential tool in this field, and in view of recent developments related to potentials in the presence of an external field, an introduction to these methods should be a welcome addition and that is exactly what this survey provides.

The first sections and the major part of the survey is really a mini course on potential theory in the complex plane, covering topics such as Fekete points, logarithmic capacity, sub/superharmonic functions, potentials, equilibrium measure, balayage and Green functions, including fundamental theorems such as Frostman's theorem and the Riesz Decomposition theorem. While the presentation is condensed compared to a textbook with similar content, such as [T. J. Ransford, *Potential theory in the complex plane*, London Math. Soc. Stud. Texts, 28, Cambridge Univ. Press, Cambridge, 1995; MR1334766 (96e:31001)] or [E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Grundlehren Math. Wiss., 316, Springer, Berlin, 1997; MR1485778 (99h:31001)], care is still taken to provide the reader with proofs or ideas of proofs, as well as relevant exercises.

The last sections deal foremost with applications to polynomial and rational approximation of analytic functions. First, polynomial interpolation is discussed and the relevance of logarithmic potentials and capacity is explained in this simple situation. The generalization to rational intepolation using potentials of signed measures and condenser capacity follows and finally logarithmic potentials in the presence of an external field are introduced with a brief discussion of the support of the corresponding weighted equilibrium measure and its role in uniform approximation by weighted polynomials. *Anders Gustafsson*

MR2747884 32E30 41A10

Kumar, D. [Kumar, Devendra[1]]; Gupta, Deepti

Polynomial approximation and interpolation of entire functions of flow growth in several complex variables. (English summary)

Bull. Math. Anal. Appl. **2** (2010), *no.* 4, 25–34.

In this paper the authors give a characterization of the generalized order and type of entire functions of several complex variables by means of polynomial approximation and interpolation. They obtain necessary and sufficient conditions of generalized order and type of slow growth in certain Banach spaces ($B(p,q,m)$ spaces, Hardy spaces and Bergman spaces) introduced by S. B. Vakarchuk and S. Ī. Zhir [Ukraïn. Mat. Zh. **54** (2002), no. 9, 1155–1162; MR2016136 (2004i:30033)] and W. Rudin [*Function theory in polydiscs*, W. A. Benjamin, Inc., New York, 1969; MR0255841 (41 #501)].

Juan Carlos Fariña Gil

MR2869079 33C10 40A30

Baricz, Árpád (R-CLUJ-EC; Cluj-Napoca); **Jankov, Dragana** (CT-UOS-DM; Osijek); **Pogány, Tibor K.**

Integral representations for Neumann-type series of Bessel functions I_ν, Y_ν and K_ν. (English summary)

Proc. Amer. Math. Soc. **140** (2012), *no.* 3, 951–960.

Closed-form integral expressions for Neumann-type series of modified Bessel functions of the first kind I_ν, Bessel functions of the second kind Y_ν, K_ν, and Hankel functions $H_\nu^{(1)}$, $H_\nu^{(2)}$ are established. *Beyza Caliskan Aslan*

MR2847098 33C45 34A05 39A60 42C05

Branquinho, A. [Branquinho, Amílcar] (P-CMBR-CM; Coimbra);
Rebocho, M. N. (P-CMBR-CM; Coimbra)

Difference and differential equations for deformed Laguerre-Hahn orthogonal polynomials on the unit circle. (English summary)

J. Phys. A **44** (2011), *no.* 46, 465204, 12 *pp.*

The authors study sequences of orthogonal polynomials with respect to a positive Borel measure μ on the unit circle that belong to the Laguerre-Hahn class. These are the sequences for which the Carathéodory function $F(z)$ of μ satisfies a Riccati-type differential equation of the form $zAF' = BF^2 + CF + D$, where A, B, C, D are co-prime polynomials. The Laguerre-Hahn class on the unit circle was studied in [A. Branquinho and M. N. Rebocho, J. Comput. Anal. Appl. **10** (2008), no. 2, 229–242; MR2382757 (2008k:42076)], and also in [A. Cachafeiro and C. Pérez Iglesias, J. Comput. Anal. Appl. **6** (2004), no. 2, 107–123; MR2223289 (2007b:33025)].

In the paper under review the authors consider difference equations for the reflection parameters (constant terms) of Laguerre-Hahn monic orthogonal polynomials on the unit circle, and also study continuous equations that follow when deformations of Laguerre-Hahn Carathéodory functions occur through a dependence on a certain parameter t. The authors find matrix differential equations for the matrices that appear in the recurrence relations for the polynomial sequences and for the associated polynomials of the second kind. An example of a sequence in the Laguerre-Hahn class that is not in the semi-classical class is presented. *Luis Verde-Star*

MR2826264 33D15 30B70 40A15

Pathak, Maheshwar (6-MNNIT; Allahabad);
Srivastava, Pankaj [Srivastava, Pankaj Kumar] (6-MNNIT; Allahabad)

A note on continued fractions and ${}_3\psi_3$ series. (English summary)

Ital. J. Pure Appl. Math. No. 27 (2010), 191–200.

In this paper, the authors establish two continued fraction representations for the ratios of ${}_3\psi_3$-basic hypergeometric series. Proofs presented in the paper are essentially based on the transformation formula derived by S. N. Singh [Proc. Nat. Acad. Sci. India Sect. A **65** (1995), no. 3, 319–329; MR1471676 (98i:33024)]. *Chandrashekar Adiga*

MR2874976 33D15 11A55 30B70 40A15

Srivastava, Pankaj [Srivastava, Pankaj Kumar] (6-MNNIT; Allahabad)

Resonance of continued fractions related to ${}_2\psi_2$ basic bilateral hypergeometric series. (English summary)

Kyungpook Math. J. **51** (2011), *no.* 4, 419–427.

The author establishes 10 expansions in continued fractions for the ratios of basic bilateral hypergeometric series of the form

$$ {}_2\psi_2\left[\begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix}; q; z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n\,(a_2;q)_n\,z^n}{(b_1;q)_n\,(b_2;q)_n\,(q;q)_n}, $$

where

$$ (a;q)_n = \begin{cases} (1-a)(1-aq)\cdots(1-aq^n), & n \geq 1, \\ 1, & n = 0. \end{cases} $$

Anatoly P. Golub

MR2855280 33E10 26D15 33E20 40A25 65B15

Zastavnyĭ, V. P. (UKR-DONE; Donetsk)

Inequalities for Mathieu series and positive definiteness. (Russian. English and Russian summaries)

Anal. Math. **37** (2011), *no.* 4, 289–318.

Consider the series

$$ S_\mu(t,u) = {\sum_{k=1}^{\infty}}' \frac{2(k+u)}{((k+u)^2+t^2)^{\mu+1}}, $$

where $\mu > 0$, $u, t \in \mathbb{R}$. Prime means that for $-u \in \mathbb{N}$ the term for $k = -u$ in the sum is omitted. The results of the paper are certain inequalities for $s_\mu(t,u)$ valid for all $t > 0$. Specifically, estimates for $|S_\mu(t,u) - \frac{1}{\mu(p^2+t^2)^\mu}|$ and $(\mu S_\mu(t,u))^{-1/\mu} - t^2$ are given.

Aleksander Denisiuk

MR2862672 33E30 33C47 41A30 65N35

Zhang, Jing (SGP-NANT-MPM; Singapore);
Wang, Li-Lian (SGP-NANT-MPM; Singapore)

On spectral approximations by generalized Slepian functions. (English summary)

Numer. Math. Theory Methods Appl. **4** (2011), *no.* 2, 296–318.

In this paper, the authors explore a generalization of the prolate spheroidal wave (or Slepian, according to the authors) functions [D. Slepian, Bell System Tech. J. **43** (1964), 3009–3057; MR0181766 (31 #5993)]. These functions form a complete orthogonal system in $L^2_\omega(-1,1)$, where $\omega(x) = (1-x)^\alpha$, $\alpha > -1$.

The authors present various analytic properties and study spectral approximations by such functions. *Diego E. Dominici*

MR2875185 34A12 34A30 34A40 39B52

Jung, Soon-Mo (KR-HGIK2-MST; Chochiwon);
Kim, Byungbae (KR-HGIK2-MST; Chochiwon)

Simple harmonic oscillator equation and its Hyers-Ulam stability. (English summary)

J. Funct. Spaces Appl. **2012,** *Art. ID* 382932, 8 *pp.*

Recall that a function $y_{\rm h}$ is called a simple harmonic oscillator function if it satisfies the so-called simple harmonic oscillator equation:

$$ y''_{\rm h}(x) + \omega^2 y_{\rm h}(x) = 0 $$

for some positive ω.

The authors solve the inhomogeneous simple harmonic oscillator equation

$$ (1) \qquad y''(x) + \omega^2 y(x) = \sum_{m=0}^{\infty} a_m x^m $$

and obtain the following Hyers-Ulam stability result (Theorem 3.1):

Let $\omega,\varepsilon>0$ and $0<\rho<1$. If $\sum_{m=0}^\infty |a_m x^m|<\varepsilon$ on the interval $I=(-\rho,\rho)$, then there exists a constant $c>0$ such that for every solution y of (1) there exists a simple harmonic oscillator function $y_{\rm h}$ (for the same ω) so that

$$|y(x)-y_{\rm h}(x)|<c\varepsilon \quad \text{on } I.$$

Igor A. Vestfrid

MR2885141 34A33 34C14 39A12 39A70 68W30

Göktaş, Ünal (TR-TOU-CEN; Ankara);
Hereman, Willy [Hereman, Willy A.] (1-CSM-CS; Golden, CO)

Symbolic computation of conservation laws, generalized symmetries, and recursion operators for nonlinear differential-difference equations. (English summary)

Dynamical systems and methods, 153–168, *Springer*, *New York*, 2012.

Summary: "Algorithms for the symbolic computation of polynomial conservation laws, generalized symmetries, and recursion operators for systems of nonlinear differential-difference equations (DDEs) are presented. The algorithms can be used to test the complete integrability of nonlinear DDEs. The ubiquitous Toda lattice illustrates the steps of the algorithms, which have been implemented in *Mathematica*. The codes InvariantsSymmetries.m and DDERecursionOperator.m can aid researchers interested in properties of nonlinear DDEs."

{For the entire collection see MR2885134 (2012i:34002).}

MR2853554 34C25 41A50

Gasull, Armengol (E-BARA; Bellaterra (Barcelona));
Li, Chengzhi [Li, Cheng Zhi[1]] (PRC-BJ-MAM; Beijing);
Torregrosa, Joan (E-BARA; Bellaterra (Barcelona))

A new Chebyshev family with applications to Abel equations. (English summary)

J. Differential Equations **252** (2012), *no.* 2, 1635–1641.

Let $f_0,f_1,\dots,f_n$ be the functions defined on an open interval J in $\mathbb{R}$. One says that $(f_0,f_1,\dots,f_n)$ is an extended complete Chebyshev system on J if any non-trivial linear combination $a_0f_0(y)+a_1f_1(y)+\dots+a_kf_k(y)$, $k=0,1,\dots,n$, has at most k isolated zeros on J, counted with multiplicities.

In the present paper the authors introduce the family of analytic functions

$$I_{k,\alpha}(y)=\int_a^b \frac{g^k(t)}{(1-yg(t))^\alpha}dt,$$

for $k=0,1,\dots,n$. They prove that it is an extended Chebyshev system. A key point of the proof consists of reducing the study of certain Wronkians to the Gram determinants of a suitable set of new functions. The result is applied to study the number of isolated periodic solutions of some perturbed Abel equations; see Theorem B for details.

Yulin Zhao

MR2895583 34N05 26D15 26E70 34D20 39A12

DaCunha, Jeffrey J.

Dynamic inequalities, bounds, and stability of systems with linear and nonlinear perturbations. (English summary)

Nonlinear Dyn. Syst. Theory **9** (2009), *no.* 3, 239–248.

Some generalizations of Gronwall's inequality are introduced. Linear systems with linear and nonlinear perturbations and their stability characteristics versus the unperturbed system are investigated. *Yonghui Xia*

MR2843918 35J05 31B35 35A35 41A30 65N99

Moiola, A. [Moiola, Andrea] (CH-ETHZ-AM; Zürich); **Hiptmair, R. [Hiptmair, Ralf]** (CH-ETHZ-AM; Zürich); **Perugia, I.** (I-PAVI; Pavia)

Plane wave approximation of homogeneous Helmholtz solutions. (English summary)

Z. Angew. Math. Phys. **62** (2011), *no.* 5, 809–837.

The authors study the approximation of solutions of the homogeneous Helmholtz equation $\Delta u + \omega^2 u = 0$ in $\mathbb{R}^N$ with constant coefficients and wave number $\omega > 0$ by linear combinations of plane waves with different directions. They combine approximation estimates for homogeneous Helmholtz solutions by generalized harmonic polynomials with approximation estimates of generalized harmonic polynomials by plane waves. The first estimates are obtained by using Vekua's theory and the second ones by establishing best approximation error estimates in Sobolev norms, which are explicit in terms of the degree of the generalized polynomial to be approximated, the domain size, and the number of plane waves used in the approximations. *Michele Campiti*

MR2836561 35K57 33C05 35A35 35B40 35K51 35R35 41A60 80A30

Trevelyan, Philip M. J. (4-GLAM-MS; Pontypridd)

Approximating the large time asymptotic reaction zone solution for fractional order kinetics A^nB^m. (English summary)

Discrete Contin. Dyn. Syst. Ser. S **5** (2012), *no.* 1, 219–234.

A reaction-diffusion system corresponding to the reaction $nA + mB \to C$ with $0 \leq n, m \leq 1$ is studied in the paper under review. If the species are initially separated, then a reaction front starts to move; the main goal is to investigate the position and width of this front. In the case $n = 1 = m$, the position scales with $T^{1/2}$ for large time T and the width of the front scales with $T^{1/6}$. The position scales with the same order for other values of n and m, while the width scales with $T^{1/2-\sigma}$, where $\sigma = 1/(1 + n + m)$, if $n, m \geq 1$.

By using matched asymptotic expansion the solution of the reaction-diffusion system can be given separately in three regions; the middle one is the reaction zone where the fractional-order reaction term comes into play, while the left and right regions give purely diffusive solutions. The author investigates the reaction zone where a nonlinear boundary value problem is derived, with the help of which the concentrations of A and B can be determined. The boundary value problem turns out to be a free boundary problem when $n, m \geq 1$. Hence, the width of the reaction front can be obtained from the end points of the free boundary problem.

This free boundary problem is analyzed in three different ways in the paper. First, for certain values of n and m analytical solutions are given since the differential equation can be solved explicitly. Then an approximate solution is derived, based on the observation that the solution has a special form for the above special values of n and m. For this approximate solution explicit formulas can be derived for the width of the reaction front by using the Gamma function and the hypergeometric function. Finally, the free boundary problem is solved numerically and the numerical solution is compared with the approximate solution. They are in good agreement; hence the analytic formulas obtained from the approximate solution can be effectively used to estimate the width of the reaction front for fractional-order kinetics. *Peter L. Simon*

MR2754313 37B55 37C60 39A22 65L07
Pötzsche, Christian (D-MUTU-ZMG; Garching)
Nonautonomous continuation of bounded solutions. (English summary)
Commun. Pure Appl. Anal. **10** (2011), *no.* 3, 937–961.

In this paper the author proves the persistence of hyperbolic bounded solutions to nonautonomous difference and retarded functional differential equations under parameter perturbation, where hyperbolicity is expressed in terms of an exponential dichotomy in variation. The approach proposed by the author uses a formulation of the dynamical systems as operator equations in certain sequence or function spaces. The main results are illustrated in an interesting application concerning the behavior of hyperbolic solutions and stable manifolds for ODEs under numerical discretization with varying step sizes. *Bogdan Sasu*

MR2846783 37D10 37D25 39A22
Bento, António J. G. (P-UBI-M; Covilhã)**;**
Silva, César M. [Silva, C'esar Augusto Teixeira Marques da] (P-UBI-M; Covilhã)
Nonuniform (μ,ν)-dichotomies and local dynamics of difference equations. (English summary)
Nonlinear Anal. **75** (2012), *no.* 1, 78–90.

The authors obtain a local stable manifold theorem for perturbations of nonautonomous linear difference equations possessing a general type of nonuniform dichotomy, possibly with different growth rates in the uniform and nonuniform parts. They include situations where the Lyapunov exponents can be zero. This generalizes previous results in the same direction of Mañé, Barreira and Valls, and of Thieullen. *José L. Vieitez*

MR2847194 37E15 39A23
Al-Salman, Ahmad (OM-SUQA-MS; Muscat)**;**
AlSharawi, Ziyad (OM-SUQA-MS; Muscat)
A new characterization of periodic oscillations in periodic difference equations. (English summary)
Chaos Solitons Fractals **44** (2011), *no.* 11, 921–928.

The structure of periodic solutions of non-autonomous, p-periodic difference equations of the form

$$x_{n+1}=f_n(x_n),\quad n=0,1,2,\dots,$$

is studied, where f_n are self-maps of a compact interval and they form a periodic set, i.e., $f_{n+p}=f_n$ for every n and a fixed positive integer p. Two types of results are obtained: firstly, for solutions whose periods are multiples of p an extension of the Sharkovskiĭ ordering is established for the non-autonomous case; secondly, it is shown that an infinite number of periodic solutions exist whose periods are not multiples of p. This latter result is established by explicitly constructing the desired function sets on the interval $[0,1]$. *Hassan Sedaghat*

MR2861283 37E15 37E10 39A20 39A23
Cima, Anna (E-BARAS; Bellaterra (Barcelona))**;**
Mañosas, Francesc (E-BARAS; Bellaterra (Barcelona))
Real dynamics of integrable birational maps. (English summary)
Qual. Theory Dyn. Syst. **10** (2011), *no.* 2, 247–275.

In this paper the authors study the dynamics generated by iteration of birational maps in $\mathbb{R}^k$ with $k-1$ independent rational first integrals. A birational map is a map with rational components such that it has an inverse which is also rational, and a first integral of the dynamical system is a non-constant function which is constant on the orbits of the system. They prove that each level curve can be desingularized and compactified, being homeomorphic to a finite union of disjoint circles and open intervals. Furthermore, they show that the map can be extended homeomorphically in a natural way to this space. That is, the study of the system reduces to the study of some orientation-preserving homeomorphisms on circles and intervals. Then the authors consider the case in which the map has a rational invariant measure and they show that in most cases the orbit of a point is either periodic or is dense in some connected components of its corresponding level set. Several applications in dimension two and three are also presented.

Laura Gardini

MR2843905 37K05 35A30 35L65 39A10 58E30 58J70 70S10

Hydon, Peter E. (4-SUR; Guildford);
Mansfield, Elizabeth L. [**Mansfield, Elizabeth Louise**] (4-KENT-NDM; Canterbury)

Extensions of Noether's second theorem: from continuous to discrete systems. (English summary)

Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **467** (2011), *no.* 2135, 3206–3221.

Whereas Noether's first theorem, relating variational symmetries and conservation laws of Euler-Lagrange equations, has always received a lot of attention in the literature, this is not the case for Noether's second theorem. The latter applies to variational symmetries which depend on arbitrary local functions of the independent variables and the theorem states that such symmetries exist if and only if there exist certain differential relations between the given Euler-Lagrange equations.

In this paper, a simple local proof of Noether's second theorem is given, based on the construction of an extended variational problem with an augmented number of dependent variables. The case of a scalar charged particle with an electromagnetic field is considered as an example. The proof moreover gives rise to an immediate generalization of Noether's second theorem, applicable to any variational problem admitting symmetries depending on a number of free or partly constrained functions. It is shown that the approach presented here also extends to the case of finite-difference systems. Several well-known continuous and discrete systems are treated as illustrative examples.

Frans Cantrijn

MR2839089 37K60 39A12

Grammaticos, B. [**Grammaticos, Basile**] (F-PARIS7-IMN; Orsay);
Ramani, A. [**Ramani, Alfred**] (F-POLY-TP; Palaiseau);
Tamizhmani, K. M. (6-POND; Pondicherry); **Willox, R.** (J-TOKYOGM; Meguro)

On Quispel-Roberts-Thompson extensions and integrable correspondences. (English summary)

J. Math. Phys. **52** (2011), *no.* 5, 053508, 11 *pp.*

QRT mappings are intimately related to and help our understanding of integrable second-order discrete systems. Their invariants being given by a biquadratic homogeneous form, say $K(x,y)$, of bidegree $(2,2)$, the number of images of a given initial condition grows linearly with the number of iterations. Veselov's criterion of polynomial growth can therefore be applied, implying that the 2-to-2 correspondence associated to the QRT mapping is an integrable one. Let X denote the elliptic curve birational to the completion of the curve $\{K(x,y)=0\}$. The action of the mapping can be visualized as shifting in X, by a step d which depends on the integral conserved by the mapping. In the article under review, diverse extensions of the QRT mapping, with coefficients periodic functions of the autonomous variable, are introduced and studied from different points of view. The authors were inspired by results obtained for the discrete Painlevé equations. The latter give rise to new QRT-type integrable mappings. It is then shown, via an explicit calculation, that starting from an initial condition the number of distinct images grows, not linearly but, nevertheless, polynomially. Hence, Veselov's criterion still applies, implying that all associated 2-to-2 correspondences are integrable. *Armando K. Treibich*

MR2863806 42A10 41A25 41A50

Le, R. J. (PRC-NGBS-NDM; Ningbo)**; Zhou, S. P. [Zhou, Song Ping]**

L^1-approximation rate of certain trigonometric series. (English summary)

Acta Math. Hungar. **134** (2012), *no.* 1-2, 29–44.

The L^1 approximation of the trigonometric series

$$f(x)=\sum_{n=1}^{\infty}a_n\sin nx,\quad g(x)=\sum_{n=1}^{\infty}a_n\cos nx,\quad a_n\geq 0$$

and

$$h(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx}$$

is considered in this paper, by employing new techniques and tools, such as the following condition on the coefficients, raised by S. P. Zhou in [Sci. China Math. (Chinese Ed.) **40** (2010), 801–812; per bibl.]:

Let $\mathbf{C}=\{q_n\}_{n=1}^{\infty}$ be a complex bounded sequence, and N a positive integer. If

$$\sum_{k=n}^{\infty}\left|\triangle\frac{q_k}{\log^N k}\right|\leq M(\mathbf{C})\frac{|q_n|}{\log^N n}$$

holds for all natural numbers n, then $\mathbf{C}$ is called a Logarithm Rest Bounded Variation Sequence: $\mathbf{C}\in \mathrm{LRBVS}_N$.

The results obtained concern sufficient conditions for

$$\|\varphi-S_n(\varphi)\|_1=O(\psi_n),$$

related to the conditions

$$q_n\log n=O(\psi_n),\qquad \sum_{k=n}^{\infty}\frac{|q_k|}{\log^N k}=O(\psi_n),$$

where S_n denotes partial sums of trigonometric series with LRBVS_N coefficients $q_n=a_n$ and $q_n=c_n$, as required, converging to φ, while $\{\psi_n\}$ is a decreasing to zero sequence verifying $\psi_n=O(\psi_{2n})$.

One interesting novelty is that it is not required that f, g, or h belong to L^1. But, for functions on L^1, equivalent conditions are obtained. *Alfredo Lazaro González*

MR2868378 42A10 41A28

Telyakovskiĭ, S. A. [Telyakovskiĭ, Sergeĭ Aleksandrovich] (RS-AOS; Moscow)

Estimates for the simultaneous approximation of functions and their derivatives by Fourier sums. (Russian)

Mat. Zametki **90** (2011), *no.* 3, 478–480; *translation in Math. Notes* **90** (2011), *no.* 3-4, 464–466.

Let W^r be a class of real 2π-periodic functions f such that $|f^{(r)}_{(x)}|\leq 1$, $S_n(\varphi,x)$ is the n-th Fourier sum of φ, $0\leq k_1<k_2<\cdots<k_m\leq r$ is an arbitrary set of integers, and m_0, resp. m_1, denotes the number of even, resp. odd, numbers among $\{k_j\}$. We introduce a

characteristic

$$E_n^{(m)} = \sup_{f\in W^r} \sum_{j=1}^{m} \frac{1}{n^{k_j}} |f_{(0)}^{(k_j)} - S_n(f^{(k_j)}, 0)|.$$

A. I. Stepanets [Ukrain. Mat. Zh. **33** (1981), no. 3, 356–367; MR0621643 (82j:42002)] published a result concerning $E_n^{(m)}$: he asserted that

$$E_n^{(m)} = \frac{4}{\pi^2}\sqrt{m_0^2+m_1^2}\frac{\log n}{n^r} + O\left(\frac{1}{n^r}\right). \tag{1}$$

The author of the paper under review claims that instead of asymptotic equality (1) the proof of Stepanets gives only an inequality from above. So the author states his own equality,

$$E_n^{(2)} = \frac{4}{\pi^2}\frac{1}{n^r}\log n + O\left(\frac{1}{n^r}\log\frac{n}{\min(n,r+1)} + \frac{1}{n^r}\right),$$

in the case $m=2$, $k_1=0$, $k_2=r$. The author uses his paper [Mat. Zametki **4** (1968), 291–300; MR0236591 (38 #4886)]. *N. A. Shirokov*

MR2907177 42A16 41A27

Il′yasov, N. A.

Direct and inverse theorems in the theory of absolutely converging Fourier series of continuous periodic functions. (Russian. English and Russian summaries)

Izv. Ural. Gos. Univ. Mat. Mekh. No. 9(44) (2006), 89–112, 162.

In this paper, direct and inverse theorems regarding the interaction between the degree of smoothness and the convergence rate of absolutely converging Fourier series are proved.

Let $C(T)$ be the space of 2π-periodic continuous functions, let $\omega_l(f;\delta)$ be the l-order modulus of smoothness of f, and let $E_n(f)$ be the best uniform approximation of f by trigonometric polynomials of order at most n. Let

$$\rho_n(f) := \sum_{k=n+1}^{\infty} (|a_k\cos kx + b_k \sin kx|),$$

where a_k and b_k are the Fourier coefficients of the function f.

Theorem 1 (direct). Let $f\in C(T)$, $l\in\mathbb{N}$ and either the series $\sum_{n=1}^{\infty} n^{-1/2}E_{n-1}(f) < \infty$ converges or

$$\sum_{n=1}^{\infty} n^{-1/2}\omega_l(f;\pi/n) < \infty.$$

Then $\rho_0(f)<\infty$ and the following estimates hold:

$$\rho_0(f) \le C_1(l)\sum_{n=1}^{\infty} n^{-1/2}E_{n-1}(f) \le C_2(l)\sum_{n=1}^{\infty} n^{-1/2}\omega_l(f;\pi/n);$$

$$\rho_n(f) \le C_3(l)\Big\{n^{1/2}E_{n-1}(f) + \sum_{\nu=n+1}^{\infty} \nu^{-1/2}E_{\nu-1}(f)\Big\};$$

$$\rho_n(f) \le C_4(l)\sum_{\nu=n+1}^{\infty} \nu^{-1/2}\omega_l(f;\pi/\nu),$$

where $C_k(l)$ are constants depending only on l.

Theorem 2 (converse). Let $f \in C(T)$, $l \in \mathbb{N}$ and $\rho_0(f) < \infty$. Then the estimate

$$\omega_l(f; \pi/n) \leq C_5(l) n^{-l} \sum_{\nu=1}^{n} \nu^{l-1} \rho_{\nu-1}(f), \quad n \in \mathbb{N},$$

holds.

Let $\Omega_l(0, \pi]$ be the class of functions ω defined on $(0, \pi]$ and satisfying $0 < \omega(\delta) \downarrow 0$ $(\delta \downarrow 0)$ and $\delta^{-l}\omega(\delta) \downarrow (\delta \uparrow)$. Let M_0 be the class of all number sequences $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ satisfying $0 < \lambda_n \downarrow 0$ $(n \uparrow \infty)$. Let $\varepsilon = \{\varepsilon_n\} \in M_0$. Denote

$$H^l[\omega] = \{f \in C(T)\colon\ \omega_l(f; \delta) \leq \omega(\delta),\ \delta \in (0, \pi]\},$$

$$E[\varepsilon] = \{f \in C(T)\colon\ E_{n-1}(f) \leq \varepsilon_n,\ n \in \mathbb{N}\}.$$

Theorem 3. Let $l \in \mathbb{N}$, $\omega \in \Omega_l(0, \pi]$ and $\varepsilon \in M_0$.

(1) If the series $\sum_{\nu=1}^{\infty} \nu^{-1/2}\omega(\pi/\nu)$ converges, then

$$\sup\{\rho_n(f):\ f \in H^l[\omega]\} \asymp \sum_{\nu=n+1}^{\infty} \nu^{-1/2}\omega(\pi/\nu), \quad n \in \mathbb{N}.$$

(2) If the series $\sum_{n=1}^{\infty} n^{-1/2}\varepsilon_n$ converges and the sequence $\varepsilon \in M_0$ satisfies $n^{\beta}\varepsilon_n \uparrow$ $(n \uparrow)$ for some $\beta > 0$, then

$$\sup\{\rho_n(f):\ f \in E[\varepsilon]\} \asymp \sum_{\nu=n+1}^{\infty} \nu^{-1/2}\varepsilon_\nu, \quad n \in \mathbb{N}.$$

Theorem 4. Let $l \in \mathbb{N}$, $\lambda \in M_0$. Then

$$\sup\{\omega_l(f; \pi/n):\ f \in A[\lambda]\} \asymp n^{-l} \sum_{\nu=1}^{n} \nu^{l-1}\lambda_\nu, \quad n \in \mathbb{N}.$$

Vitaly E. Maĭorov

MR2838119 42B30 41A63 42C40

Dekel, Shai; Petrushev, Pencho (1-SC; Columbia, SC);
Weissblat, Tal (IL-TLAV; Tel Aviv)

Hardy spaces on $\mathbb{R}^n$ with pointwise variable anisotropy. (English summary)

J. Fourier Anal. Appl. **17** (2011), *no.* 5, 1066–1107.

In this paper a "continuous multilevel ellipsoid cover of $\mathbb{R}^n$" is considered. In other words, the authors consider a family $\{\vartheta(x; t)\}$ of ellipsoids (i.e. images of the unit ball of $\mathbb{R}^n$ under affine transforms) depending on $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ where the x's are the centers of these ellipsoids, $|\vartheta(x; t)| \asymp 2^{-t}$ and some additional conditions are required too. Via the mentioned ellipsoid cover a quasi-distance $\rho(x; y)\colon x, y \mapsto \mathbb{R}_+$ in the space $\mathbb{R}^n$ is introduced. In turn, via this quasi-distance "balls" $B(x; r)$ are introduced. As a consequence, corresponding Hardy-Littlewood type maximal functions appear in a natural way. In particular, a radial maximal function is introduced which makes it possible to define anisotropic Hardy spaces H^p $(0 < p \leq 1)$. It should be mentioned that the elements of these Hardy spaces are from the dual space S' where S is the Schwartz class of rapidly decreasing functions in $\mathbb{R}^n$. Then atomic Hardy spaces are introduced and an "atomic decomposition theorem" for H^p-spaces $(0 < p \leq 1)$ is established. The final main result asserts that two Hardy spaces are equivalent if and only if the ellipsoid covers (they are associated with) generate equivalent quasi-distances.

Arman H. Karapetyan

MR2870159 42C40 41A15 65D07 65T60

Dem′yanovich, Yu. K. (RS-STPT-NDM; St. Petersburg)

On nonsmooth spline-wavelet decompositions and their properties. (Russian. English and Russian summaries)

Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (*POMI*) **395** (2011), *Chislennye Metody i Voprosy Organizatsii Vychisleniĭ. XXIV*, 31–60, 173; *translation in J. Math. Sci.* (*N. Y.*) **182** (2012), *no.* 6, 761–778.

Wavelets are functions generated by translating and dilating a function. In this paper, the author gives wavelet decompositions for some nonsmooth and nonpolynomial splines. Moreover, the commutativity of the decomposition operators is established.

Paşc Găvruţa

MR2894352 45P05 34K40 39B22 42A85 44A35 45M10 47D06

Kukushkina, E. V.

Stability of stationary systems of functional-difference equations. (Russian. English and Russian summaries)

Izv. Ural. Gos. Univ. Mat. Mekh. No. 10(46) (2006), 107–118, 245.

The equation $x(t)=\int_{-r}^0 d\eta(\vartheta)\,x(t+\vartheta)$ is considered. It is assumed that η is a function of bounded variation. A semigroup $T(t)\colon \widetilde{C}([-r,0],\mathbf{R}^n)\to\widetilde{C}([-r,0],\mathbf{R}^n)$ is defined for the equation. Here, $\widetilde{C}([-r,0],\mathbf{R}^n)$ is the subspace of $C([-r,0],\mathbf{R}^n)$ consisting of functions z satisfying the compatibility condition $z(0)=\int_{-r}^0 d\eta(\vartheta)\,z(\vartheta)$. The infinitesimal generator A of $T(t)$ and its spectrum are found. Sufficient conditions for the stability of the equation with initial functions $\varphi\in D(A^2)$ are given.

V. G. Kurbatov

MR2849356 46A45 40C05

Dutta, H. [Dutta, Hemen]

Some vector valued multiplier difference sequence spaces defined by a sequence of Orlicz functions. (English and Russian summaries)

Vladikavkaz. Mat. Zh. **13** (2011), *no.* 2, 26–34.

In this article some multiplier difference sequence spaces are defined using 2-norms and a sequence of Orlicz functions.

Let $p=(p_k)$ be any bounded sequence of positive real numbers and $\Lambda=(\lambda_k)$ be a sequence of nonzero reals. Let m and n be non-negative integers. Then for a real linear 2-normed space $(X,\|\cdot,\cdot\|)$ and for a sequence $M=(M_k)$ of Orlicz functions the spaces $c_0(M,\|\cdot,\cdot\|,\Delta_m^n,\Lambda,p)$, $c(M,\|\cdot,\cdot\|,\Delta_m^n,\Lambda,p)$, and $\ell_\infty(M,\|\cdot,\cdot\|,\Delta_m^n,\Lambda,p)$ are defined. It is proved that these classes of sequences are linear spaces.

If $(X,\|\cdot,\cdot\|)$ is a 2-Banach space, then these spaces are complete paranormed spaces with paranorm

$$g(x)=\inf\left\{\rho^{p_k/H}\colon \sup_{k\ge1} M_k\left(\left\|\frac{\Delta_m^n\lambda_k x_k}{\rho},z\right\|\right)\le 1,\ z\in X\right\},$$

where $H=\max(1,\sup_{k\ge1}p_k)$.

Also, the author shows that these spaces are not monotone and therefore not solid in general. Likewise, these spaces are not symmetric, are not convergence free, and are not sequence algebras. All these assertions are demonstrated by means of suitable examples.

A. Pandiarani

MR2855761 46B15 41A10 46E15

Goncharov, A. P. (TR-BILK; Ankara); **Ozfidan, N.** (TR-BILK; Ankara)

Bases in Banach spaces of smooth functions on Cantor-type sets. (English summary)

J. Approx. Theory **163** (2011), *no.* 12, 1798–1805.

Let K be a Cantor-type compact subset of $\mathbb{R}$. The authors construct Schauder bases in the spaces $C^p(K)$ of all p-continuously differentiable functions on K and in the Whitney space $\mathcal{E}^p(K)$ of all functions $f\colon K\to\mathbb{R}$ admitting extensions in $C^p(\mathbb{R})$. The construction is based on local Taylor expansions of functions in these spaces. Similar results for the spaces $C^\infty(\mathbb{R})$ and $\mathcal{E}(K)$ were obtained by the first author [Constr. Approx. **23** (2006), no. 3, 351–360; MR2201471 (2006k:46036)]. *S. Cobzaş*

MR2852258 46B20 41A65

Azagra, D. [**Azagra, Daniel**] (E-MADCM-ICM; Madrid);
Fry, R. [**Fry, Robb**] (3-TRU; Kamloops, BC); **Keener, L.**

Real analytic approximation of Lipschitz functions on Hilbert space and other Banach spaces. (English summary)

J. Funct. Anal. **262** (2012), *no.* 1, 124–166.

The authors assert that on Hilbert spaces and separable Banach spaces which admit a separating polynomial (that is, a polynomial $Q\colon X\to\mathbb{R}$ such that $Q(0)=0<\inf\{Q(x)\colon \|x\|=1\}$), every real Lipschitz function can be uniformly approximated by an analytic Lipschitz function with a control over the Lipschitz constant.

The main results are the following:

Theorem 1. Let X be a separable Banach space which admits a separating polynomial. Then there exists a number $C\geq 1$ depending only on X, such that for every Lipschitz function $f\colon X\to\mathbb{R}$ and for every $\varepsilon>0$ there exists a Lipschitz, real analytic function $g\colon X\to\mathbb{R}$ such that $|f(x)-g(x)|\leq\varepsilon$ for all $x\in X$ and $\mathrm{Lip}(g)\leq C\,\mathrm{Lip}(f)$.

Theorem 2. Let X be a separable Hilbert space, and $f\colon X\to\mathbb{R}$ a Lipschitz function. Then for every $\varepsilon>0$ there exists a Lipschitz, real analytic function $g\colon X\to\mathbb{R}$ such that $|f(x)-g(x)|\leq\varepsilon$ for all $x\in X$, and $\mathrm{Lip}\,(g)\leq\mathrm{Lip}(f)+\varepsilon$.

As a matter of fact the proof of Theorem 1 works for any Banach space X having a separating function (that is, a function $Q\colon X\to[0,\infty)$ such that $Q(0)=0$ and there exist $M,m>0$ such that $Q(x)\geq m\|x\|$ whenever $\|x\|\geq M$) with a Lipschitz holomorphic extension to a uniformly wide neighborhood of X in the complexification $\tilde{X}$.

The proof of Theorem 2 is obtained by combining the Lasry-Lions sup-inf convolution regularization technique [J.-M. Lasry and P.-L. Lions, Israel J. Math. **55** (1986), no. 3, 257–266; MR0876394 (88b:41020)] with a simultaneous approximation result of the authors in [Bull. Lond. Math. Soc. **43** (2011), no. 5, 953–964; MR2854565] and with some of the techniques developed for the proof of Theorem 1. *Márcia Sayuri Kashimoto*

MR2855768 46E35 41A46 47B06

Triebel, Hans (D-FSUMI-IM; Jena)

Entropy and approximation numbers of limiting embeddings; an approach via Hardy inequalities and quadratic forms. (English summary)

J. Approx. Theory **164** (2012), *no.* 1, 31–46.

This paper deals with entropy numbers and approximation numbers for compact embeddings of weighted Sobolev spaces into Lebesgue spaces in limiting situations. In particular, the article treats the embedding

$$E^m_{p,\sigma}(E)\hookrightarrow L_p(B),\quad 1\leq p<\infty,\ \sigma>0,\ m\in\mathbb{N},$$

in terms of continuity and compactness. This treatment is based on Hardy's inequality and the spectral theory of some degenerate elliptic operators.

Mohammad Suboh Sababheh

MR2863811 46L57 39B82 46B03 46L89

O'Regan, D. [O'Regan, Donal] (IRL-GLWY; Galway);
Rassias, J. M. [Rassias, John Michael] (GR-UATH-MIP; Aghia Paraskevi);
Saadati, R. [Saadati, Reza] (IR-IAU-MSR; Tehran)

Approximations of ternary Jordan homomorphisms and derivations in multi-C^* ternary algebras. (English summary)

Acta Math. Hungar. **134** (2012), *no.* 1-2, 99–114.

Summary: "Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in multi-C^* ternary algebras and of derivations on multi-C^* ternary algebras for the additive functional equation

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1,\ j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right) \quad (m \in \mathbb{N},\ m \geqq 2).\text{"}$$

MR2895723 47A64 40A99

Slyusarchuk, V. Yu. [Slyusarchuk, V. E.] (UKR-RNTU; Rovno)

Operator analogues of Kummer's test. (Ukrainian. English and Russian summaries)

Mat. Stud. **36** (2011), *no.* 2, 188–196.

The work gives a proof of an analogue of Kummer's test (Theorems 4 and 5) for operator series with elements in a partially ordered Banach space with reproducing kernel. Examples of applications of results obtained are presented. The proved theorems allow one to obtain analogues of Raabe's test for corresponding operator series.

Vladimir A. Zolotarev

MR2858498 47B36 39A70 47B39 47B80

del Rio, Rafael [del Río Castillo, Rafael René] (MEX-NAM-NM; México);
Silva, Luis O. [Silva Pereyra, Luis Octavio] (MEX-NAM-NM; México)

Spectral measures of Jacobi operators with random potentials. (English summary)

Oper. Matrices **5** (2011), *no.* 3, 435–448.

The authors consider a random Jacobi operator H_ω on $l^2(\mathbb{Z})$, where randomness is introduced through the main diagonal of H_ω by a sequence of independent random variables $(\omega(n))_{n\in\mathbb{Z}}$ with continuous distribution. It is well known that for any ergodic Jacobi operator on $l^2(\mathbb{Z})$, a given $E \in \mathbb{R}$ can only be an eigenvalue with zero probability.

The main purpose of the present article is to extend this result to the random operator H_ω described above where E is allowed to depend on the sequence $(\omega(n))_{n\in\mathbb{Z}}$ except for its entries at two consecutive points (see Theorem 3.1 and 3.2). This in particular implies that eigenvalues of a Jacobi matrix do not coincide with eigenvalues, moments or entries of any its submatrices almost surely.

Christoph A. Marx

MR2848604 47B38 41A50 47G30 94A11
Dörfler, Monika (A-WIEN; Vienna);
Torrésani, Bruno [Torrésani, Bruno S.] (F-PROV-LAT; Marseille)
Representation of operators by sampling in the time-frequency domain. (English summary)
Sampl. Theory Signal Image Process. **10** (2011), *no.* 1-2, 171–190.
The authors introduce multiple and generalized Gabor multipliers for the approximation of time-variant systems. The basic idea of approximation in the spreading domain is explained. A general error estimate for the approximation of Hilbert-Schmidt operators by Gabor multipliers is derived, and several special cases are deduced thereof. As a noteworthy special case, the approximation of short-time Fourier multipliers by Gabor multipliers is considered. *Ke Cheng Zhou*

MR2861305 47D06 35J05 41A36
Altomare, Francesco (I-BARI; Bari); **Milella, Sabina** (I-BARI; Bari);
Musceo, Graziana (I-BARI; Bari)
Multiplicative perturbations of the Laplacian and related approximation problems. (English summary)
J. Evol. Equ. **11** (2011), *no.* 4, 771–792.
Summary: "Of concern are multiplicative perturbations of the Laplacian acting on weighted spaces of continuous functions on $\mathbb{R}^N$, $N \geq 1$. It is proved that such differential operators, defined on their maximal domains, are pre-generators of positive quasicontractive C_0-semigroups of operators that fulfill the Feller property. Accordingly, these semigroups are associated with a suitable probability transition function and hence with a Markov process on $\mathbb{R}^N$. An approximation formula for these semigroups is also stated in terms of iterates of integral operators that generalize the classical Gauss-Weierstrass operators. Some applications of such approximation formula are finally shown concerning both the semigroups and the associated Markov processes."

MR2870228 49Q20 41A17 60B10 76A02
Brasco, L. (F-PROV-LAT; Marseille)
A survey on dynamical transport distances. (English and Russian summaries)
Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (*POMI*) **390** (2011), *Teoriya Predstavleniĭ, Dinamicheskie Sistemy, Kombinatornye Metody. XX*, 5–51, 307;
reprinted in J. Math. Sci. (*N. Y.*) **181** (2012), *no.* 6, 755–781.
Summary: "In this paper we review some transport models based on the continuity equation, starting with the so-called *Benamou-Brenier* formula, which is nothing but a fluid mechanics reformulation of the Monge-Kantorovich problem with cost $c(x,y) = |x-y|^2$. We discuss some of its applications (gradient flows, sharp functional inequalities…), as well as some variants and generalizations to dynamical transport problems, where interaction effects among mass particles are considered."

MR2895548 52B15 39B12 39B42 52B12
Voĭnov, A. S. (RS-MOSCM-NDM; Moscow)
Self-affine polyhedra: applications to functional equations and matrix theory. (Russian. Russian summary)
Mat. Sb. **202** (2011), *no.* 10, 3–30; *translation in Sb. Math.* **202** (2011), *no.* 9-10, 1413–1439.
In this article properties of finite-dimensional convex self-affine compact sets are considered and a counterexample to the well-known conjecture about the structure of such bodies is constructed. A special class of self-affine compact convex bodies—the class of

fractional (that is, partitioned to zero diameter) bodies—is defined and studied. It is proved that the fractional body is a convex polytope with special properties.

As an application of the fractional bodies theory the affine self-similarity multidimensional equations are studied. A criterion for the existence and uniqueness of an L_p-solution is established. Some facts about the convergence of products of stochastic matrices are stated. The polynomial algorithm for verification of a finite family of stochastic matrices on the convergence is suggested. *George Kirillovich Kamenev*

MR2779376 53C07 39A12 70S15

Sushch, Volodymyr (PL-KUT; Koszalin)

Self-dual and anti-self-dual solutions of discrete Yang-Mills equations on a double complex. (English and Spanish summaries)

Cubo **12** (2010), *no.* 3, 99–120.

This paper continues the author's study of a discrete version of the Yang-Mills equations [Mat. Zametki **61** (1997), no. 5, 742–754; MR1620141 (2001b:53026); Cubo **6** (2004), no. 2, 35–50; MR2092042 (2005e:81144); Cubo **8** (2006), no. 3, 61–78; MR2287294 (2007k:70051)].

Section 2 of the paper reviews smooth Yang-Mills theory in $\mathbb{R}^4$. Section 3 reviews the double complex construction from the author's earlier paper [op. cit., 2006]. The construction involves a 1-dimensional chain complex $C = C^0 \oplus C^1$ which serves as a combinatorial model for the real line. The Euclidean space $\mathbb{R}^4$ is then modeled combinatorially by $C(4) = C \otimes C \otimes C \otimes C$, and the Hodge star operator and Lie algebra-valued 1-forms are defined for the combinatorial model.

Section 4 discusses discrete forms with quaternionic coefficients, discrete gauge transformations, a discrete analog of exterior covariant differentiation, and the discrete curvature 2-form. Section 5 discusses discrete instantons and anti-instantons. A combinatorial condition that determines whether or not a 2-form is su(2)-valued is derived and discrete instantons and anti-instantons are exhibited. *David E. Hurtubise*

MR2906166 54A05 03E72 41A65

Hosny, R. A. [Hosny, Rodyna A.] (SAR-TAIFS-MS; Ta'if)

Proximity structures on approximation spaces. (English summary)

Adv. Appl. Math. Sci. **10** (2011), *no.* 5, 491–507.

Summary: "Two types of proximity structures are initiated on the generalized approximation spaces (X, R) where R is a general binary relation. The suggested structures are used to get topologies on the universal set. Properties of approximation with respect to the topologies generated by relation are compared with their corresponding approximation with respect to proximity. A method for obtaining clopen topology from a general binary relations via proximity is obtained. The suggested structures and results will open the way for deep topological applications in information systems."

MR2728631 54F50 41A50 54F15

Koshcheev, V. A. (RS-AOSUR-A; Ekaterinburg)

Separators of hereditarily arcwise connected continua admitting Chebyshev systems of complex continuous functions. (English summary)

Proc. Steklov Inst. Math. **261** (2008), *suppl.* 1, *S*131–*S*137.

The first fact proved in the paper is purely continuum-theoretic: Each hereditarily arcwise connected continuum X without free arcs and θ-curves is not discoherent, i.e. it is the union of two proper subcontinua A and B whose intersection is connected. The main theorem concerns such X under the additional assumption that X admits a Chebyshev system of $n > 2$ complex continuous functions; then, in particular, there are proper subcontinua $A' \supseteq A$ and $B' \supseteq B$ of X with connected intersection such that A' is separated by an arc or a simple closed curve.

In general, the notion of a Chebyshev system on a compact space Y, which comes from approximation theory, is related to the embeddability of Y in the complex plane. Previously, the author considered similar problems [Proc. Steklov Inst. Math. **2004**, Topol. Math. Control Theory Differ. Equ. Approx. Theory, suppl. 1, S137–S146; MR2180445 (2007i:41023)]. *P. Krupski*

MR2847444 58E35 35A23 41A44 58E40

Cotsiolis, Athanase (GR-PATR; Patras); **Labropoulos, Nikos** (GR-PATR; Patras)

Sharp Nash inequalities on the unit sphere: the influence of symmetries. (English summary)

Nonlinear Anal. **75** (2012), *no.* 2, 612–624.

The authors study the so-called L^2- and L^1-Nash inequalities on the sphere S^n $(n \geq 3)$:

$$\Big(\int_{S^n} \varphi^2 ds\Big)^{1+\frac{2}{n}} \leq \Big[A \int_{S^n} |\nabla\varphi|^2 ds + B\Big(\int_{S^n} |\varphi|^p ds\Big)^{\frac{2}{p}}\Big]\Big(\int_{S^n} |\varphi| ds\Big)^{\frac{4}{n}}$$

for $\varphi \in H_1(S^n)$, $p = 2$ and $p = 1$, respectively. It is shown that the optimal constants A, B for the L^2-Nash inequality (the case $p = 2$) are given by

$$A = (A_0(n) =) \frac{(n+2)^{\frac{n+2}{n}}}{2^{\frac{2}{n}} n \lambda_{1,n} |\mathcal{B}^n|^{\frac{2}{n}}}, \quad B = \omega_n^{-\frac{2}{n}}$$

where $|\mathcal{B}^n|$, $\lambda_{1,n}$ and ω_n denote the volume of the unit ball $\mathcal{B}^n$ in $\mathbf{R}^n$, the first Neumann eigenvalue for the Laplacian for radial functions in $\mathcal{B}^n$ and the volume of the unit sphere in $\mathbf{R}^{n+1}$, respectively. Furthermore there exists an extremal function for the inequality. On the other hand, although the L^1-Nash inequality (the case $p = 1$) does not hold with the constants $A = A_0(n)$ and $B = \omega_n^{-1-\frac{2}{n}}$, it does hold with $A = A_0(n) + \varepsilon$ for any $\varepsilon > 0$. This shows that $A_0(n)$ and $\omega_n^{-1-\frac{2}{n}}$ are optimal and an extremal function of L^1-Nash inequality with those optimal constants does not exist.

The L^2-Nash inequality in S^2 on the function space $H_{1,G}(S^n)$ consisting of G-invariant functions with $G = \mathrm{O}(k) \times \mathrm{O}(m)$, $k + m = n + 1$, is also discussed. *Kimiaki Narukawa*

MR2854732 60E15 26D15 39B62

Zhang, Zhengliang [Zhang, Zheng Liang[2]] (PRC-WUHAN-DS; Wuhan);
Qian, Bin [Qian, Bin[1]] (PRC-CSIT-M; Changshu);
Ma, Yutao [Ma, Yu Tao] (PRC-BJN; Beijing)

Uniform logarithmic Sobolev inequality for Boltzmann measures with exterior magnetic field over spheres. (English summary)

Acta Appl. Math. **116** (2011), *no.* 3, 305–315.

Summary: "In this paper, we obtain the uniform logarithmic Sobolev inequality for the Boltzmann measures by reducing multi-dimensional measures to one-dimensional measures, and then applying the characterization on the constant of logarithmic Sobolev inequality for a probability measure on the real line."

MR2832914 60F05 33B20 41A60 60G50

van Leeuwaarden, J. S. H. [van Leeuwaarden, Johan S. H.] (NL-EIND; Eindhoven)**;**
Temme, N. M. [Temme, Nico M.] (NL-MATH; Amsterdam)

A uniform asymptotic expansion for weighted sums of exponentials. (English summary)

Statist. Probab. Lett. **81** (2011), *no.* 11, 1571–1579.

Summary: "We consider the random variable $Z_{n,\alpha} = Y_1 + 2^\alpha Y_2 + \cdots + n^\alpha Y_n$, with $\alpha \in \mathbb{R}$ and $Y_1, Y_2, \ldots$ independent and exponentially distributed random variables with mean one. The distribution of $Z_{n,\alpha}$ is in terms of a series with alternating signs, causing great numerical difficulties. Using an extended version of the saddle point method, we derive a uniform asymptotic expansion for $\mathbb{P}(Z_{n,\alpha} < x)$ that remains valid inside ($\alpha \geq -1/2$) and outside ($\alpha < -1/2$) the domain of attraction of the central limit theorem. We discuss several special cases, including $\alpha = 1$, for which we sharpen some of the results in [J. F. C. Kingman and S. E. Volkov, J. Theoret. Probab. **16** (2003), no. 1, 267–276; MR1956831 (2003m:60020)]." *José Antonio Adell*

MR2824564 60G50 31C20 39A45 52C20 60J67

Chelkak, Dmitry (RS-AOS2; St. Petersburg)**;**
Smirnov, Stanislav [Smirnov, Stanislav K.] (CH-GENV-SM; Geneva)

Discrete complex analysis on isoradial graphs. (English summary)

Adv. Math. **228** (2011), *no.* 3, 1590–1630.

This is the first in a series of papers dedicated to establishing universality of the critical Ising model with respect to a large class of graphs with neighboring vertices all equidistant from common centers. Applications of the results in this paper extend beyond the Ising model; universality of the 2d loop-erased random walk and natural approximations of harmonic and holomorphic functions on continuous subdomains of $\mathbb{C}$ also follow.

Section 2 and the Appendix provide a dense survey of definitions and results concerning discrete harmonic functions and discrete holomorphic functions defined on 2-d isoradial graphs; square lattices represent a special case of isoradial graphs with a history of intensive research. A comprehensive survey of established definitions and results in this field is included in Section 2, especially drawing from R. J. Duffin's ideas [Duke Math. J. **20** (1953), 233–251; MR0070031 (16,1119d)] and recent work of R. W. Kenyon [Invent. Math. **150** (2002), no. 2, 409–439; MR1933589 (2004c:31015)] and C. Mercat ["Discrete polynomials and discrete holomorphic approximation", preprint, arXiv:math-ph/0206041].

Section 2 allows the authors to establish consistent notation for the main results, but most calculations are omitted. An attentive reader must depend on a firm grasp of complex analysis and a keen awareness of the geometry of underlying rhombic lattices (isoradial graphs form rhombic tessellations when combined with their dual). The figures included are extremely helpful, and reference copies of Figures 1 (A) and (B) are useful while reading Section 2. Readers unfamiliar with the theory may get confused by some errors in the definitions; the latter equality in the definition of $\mathrm{Proj}[F;\psi]$ appears to be incorrect, and the vocabulary used to define prime ends in Subsection 3.2 is self-referential.

Section 3 is dedicated to establishing limits with respect to mesh width, δ, of families of pairs of holomorphic or harmonic functions with the subgraphs (of radius δ) they are defined on. The definitions of these subgraphs, limits of these subgraphs and the topology under which convergence is established require a series of definitions, and statements of the propositions are often as long as the proofs themselves. The reader is aided by recalling parallel results established in the context of precompact sequences of analytic functions in the classical context. The final subsection requires additional technicalities in order to establish convergence of discrete Poisson kernels to continuous counterparts on a more general class of discrete domains, a result particularly important in establishing convergence of interfaces arising from the critical models on isoradial graphs to Schramm-Loewner evolution. *Ben Dyhr*

MR2876823 62G08 41A60

Cao, Feilong (PRC-CJU-MTC; Hangzhou);
Lee, Joonwhoan (KR-CHNU-CEN; Chonju);
Zhang, Yongquan (PRC-CJU-MTC; Hangzhou)

Estimates of learning rates of regularized regression via polyline functions. (English summary)

Math. Methods Appl. Sci. **35** (2012), *no.* 2, 174–181.

Summary: "Based on the simplicity and calculability of polyline function, we consider, in this paper, the regularized regression learning algorithm associated with the least square loss and the set of polyline function $\mathcal{F}_{\rm d}$. The target is the error analysis for the regression problem. The approach presented in the paper yields satisfactory learning rates. The rates depend on the approximation property of $\mathcal{F}_{\rm d}$ and on the capacity of $\mathcal{F}_{\rm d}$ measured by covering numbers. Under some certain conditions, the rates achieve $m^{-4/5}\log m$."

MR2861915 65D05 41A05 65D17

Aslam, Muhammad [Aslam, Muhammad[2]];
Mustafa, Ghulam [Mustafa, Ghulam[2]] (PAK-ISLA-NDM; Bahawalpur);
Ghaffar, Abdul (PAK-ISLA-NDM; Bahawalpur)

$(2n-1)$-point ternary approximating and interpolating subdivision schemes. (English summary)

J. Appl. Math. **2011,** *Art. ID* 832630, 12 *pp.*

Subdivision is a technique used to generate smooth curves and surfaces approximated by a sequence of successively refined control polygons. In the paper under review the authors use Lagrange interpolation polynomials with integer nodes to obtain an explicit formula for the coefficients of the mask of a $(2n-1)$-point ternary scheme which contains free parameters and generalizes and unifies existing odd-point ternary interpolating and approximating subdivision schemes.

The authors compare the error bounds of odd-point and even-point ternary interpolating schemes and conclude that odd-point schemes are better than even-point schemes in the sense of computational cost, support and error bounds. *Luis Verde-Star*

MR2853514 65D05 41A05 65D17

Li, Baojun [Li, Bao Jun[1]] (PRC-DUTVM-IAE; Dalian);
Li, Bo (PRC-NCUA-CMI; Nanchang); **Liu, Xiuping** (PRC-DUT-SM; Dalian);
Su, Zhixun (PRC-DUT-SM; Dalian);
Yu, Bowen [Yu, Bo Wen] (PRC-DUT-SM; Dalian)

Exact evaluation of limits and tangents for interpolatory subdivision surfaces at rational points. (English summary)

J. Comput. Appl. Math. **236** (2011), *no.* 5, 906–915.

The paper gives an efficient method for exact evaluation in rational points of a surface generated by stationary, symmetric interpolatory subdivision schemes as well as the computation of the tangents at these points. The 1D ternary subdivision of M. F. Hassan et al. [Comput. Aided Geom. Design **19** (2002), no. 1, 1–18; MR1879678 (2002k:65019)] and the 2D quadmesh scheme of G. Q. Li and W. Y. Ma [Comput. Aided Geom. Design

23 (2006), no. 1, 45–77; MR2183817 (2006j:65033)] in 2D are used as examples. The basis function satisfies a dilation equation $\varphi(t)=\sum_{j=-N}^{N} a_j\varphi(Mt-j)$, $a_j\in\mathbb{R}$, $t\in\mathbb{R}^s$, $j,N\in\mathbb{Z}^s$, and M is an integer dilation matrix. This results in a scheme $\Phi(\frac{t+k}{m})=T_k\Phi(t)$ where Φ is a vector stacking values of φ and T_k, $k\in\mathbb{Z}^s$, is a refinement matrix. Because of the form $(t+k)/m$, some computational advantages can be obtained in the evaluation algorithm if the number t is expressed in a number system with basis m. The limiting vector Φ is a fixed point of a contractive operator T that corresponds to the cycle in the m-ary digits of t (T is a product of T_{d_i} where d_i are the m-ary digits in the cycle of t). An eigenvalue decomposition of T is used in the algorithm. Thus the limiting surface is obtained by attaching the limit function φ to the initial control points. Similar arguments hold for the evaluation of the limiting tangents. The method can be generalized to evaluate other stationary non-polynomial subdivision schemes. *A. Bultheel*

MR2882943 65D07 41A55

Sablonnière, P. (F-INSAR-CM; Rennes)

Some approximate methods for computing arc lengths based on quadratic and cubic spline interpolation or quasi-interpolation. (English summary)

Rend. Semin. Mat. Univ. Politec. Torino **69** (2011), *no.* 1, 1–20.

The paper presents a comparison of two families of methods for the computation of arc lengths. The first method is based on computing the exact length of a quadratic spline approximant of the original function or parametric curve. The author recalls a formula which gives the length of an arc of a parabola in Bernstein-Bézier form, and then he describes a list of C^1 quadratic spline approximants. Regarding the second method, the values of the first derivatives are approximated by those of cubic spline approximants, and then these estimates are used for the approximate computation of the arc length by means of Simpson's quadrature formula. Note that the overall convergence order of both classes of methods is $O(h^4)$. For each family of methods, several types of interpolants or quasi-interpolants are compared and numerous numerical examples illustrate the considered methods and procedures. *Alexandru Ioan Mitrea*

MR2870198 65D10 41A30

Le Gia, Q. T. [Le Gia, Quoc Thong] (5-NSW-SMS; Sydney)**;**
Tran, T. [Tran, Thanh] (5-NSW-SMS; Sydney)

Additive Schwarz preconditioners for interpolation of divergence-free vector fields on spheres. (English summary)

ANZIAM J. Electron. Suppl. **52** (2010), (C), C742–C758.

Summary: "The linear system arising from the interpolation problem of surface divergence-free vector fields using radial basis functions tends to be ill-conditioned when the separation radius of the scattered data is small. When the surface under consideration is the unit sphere, we introduce a preconditioner based on the additive Schwarz method to accelerate the solution process. Theoretical estimates for the condition number of the preconditioned matrix are given. Numerical experiments using scattered data from MAGSAT satellite show the effectiveness of our preconditioner."

MR2870183 65D30 41A55

Hegland, Markus (5-ANU-MT; Canberra)**; Leopardi, Paul C.** (5-ANU-MT; Canberra)

The rate of convergence of sparse grid quadrature on the torus. (English summary)

ANZIAM J. Electron. Suppl. **52** (2010), (C), C500–C517.

The rate of convergence of a sparse grid quadrature on a weighted Korobov space is considered. Integration over a torus Π^d is equivalent to integration of a periodic function over the d-dimensional unit cube. Given quadrature points x_i, $i = 1, \ldots, d$, the weights of the optimal quadrature method are obtained by solving a linear system with a matrix whose elements are the values of the reproducing kernel $K(x_i, x_j)$ of the reproducing kernel Hilbert space. The approach is compared with the weighted tensor product algorithm of G. W. Wasilkowski and H. Woźniakowski [J. Complexity **15** (1999), no. 3, 402–447; MR1716741 (2000h:65200)]. *G. A. Evans*

MR2847114 65M70 41A30

Kazem, S. (IR-IKIU-M; Qazvin);
Rad, J. A. [**Amani Rad, Jamal**] (IR-SHBHM-CS; Tehran);
Parand, K. [**Parand, Kourosh**] (IR-SHBHM-CS; Tehran)

Radial basis functions methods for solving Fokker-Planck equation. (English summary)

Eng. Anal. Bound. Elem. **36** (2012), *no.* 2, 181–189.

Summary: "In this paper two numerical meshless methods for solving the Fokker-Planck equation are considered. Two methods based on radial basis functions to approximate the solution of Fokker-Planck equation by using collocation method are applied. The first is based on the Kansa's approach and the other one is based on the Hermite interpolation. In addition, to conquer the ill-conditioning of the problem for big number of collocation nodes, two time domain Discretizing schemes are applied. Numerical examples are included to demonstrate the reliability and efficiency of these methods. Also root mean square and N_e errors are obtained to show the convergence of the methods. The errors show that the proposed Hermite collocation approach results obtained by the new time-Discretizing scheme are more accurate than the Kansa's approach."

MR2861706 65N21 35J25 35R30 41A21 78A46

Hanke, Martin (D-MNZ-IM; Mainz)

Locating several small inclusions in impedance tomography from backscatter data. (English summary)

SIAM J. Numer. Anal. **49** (2011), *no.* 5, 1991–2016.

Let B be the two-dimensional unit disk, $T = \partial B$, and let

$$\Omega = \bigcup_{j=1}^{J} \overline{\Omega}_j \subset B,$$

where $\Omega_1, \ldots, \Omega_J$ are simply connected regions with C^2-boundary which are disjoint and do not touch T. The problem treated in the paper is the location of the inclusions Ω_j using as data the measured backscatter function at some points on T. Recall that the potential on B is a solution of the equation $\nabla(\sigma\nabla u) = 0$ in B, $\frac{\partial}{\partial\nu} u = f$ on T, $\int_T u ds = 0$, where f is a current imposed on T with $\int_T f ds = 0$. The conductivity is assumed to be piecewise constant:

$$\sigma(x) = \begin{cases} \kappa_j, & x \in \Omega_j,\ \kappa_j \neq 1,\ \kappa_j \geq 0; \\ 1, & \text{elsewhere.} \end{cases}$$

The backscatter data $b(\vartheta)$ is the value of the induced voltage at the point $(x_1, x_2) \in T$ where a dipole-type current is imposed. Using potential theory it is shown that the backscatter function can be extended as a holomorphic function to the complex plane with the exception of the set $\overline{\Omega}$ and its reflection $\overline{\Omega}^*$ with respect to the unit circle, and we have $b(\zeta^*) = \overline{b(\zeta)}$ for $\zeta \in \overline{B} \setminus \overline{\Omega}$, where for a point (x_1, x_2) in the plane $\zeta = x_1 + ix_2$.

In case the inclusions are of the form

$$\Omega_j = x_j + \varepsilon O_j, \quad x_j \in B,\ 0 \in O_j$$

it is shown that $b_\varepsilon(\zeta) \searrow \varepsilon^2$ converges, as $\varepsilon \to 0$, to a function

$$F(\zeta) = \frac{\zeta^2}{4\pi^2} \sum_{j=1}^{J} \left(\frac{\delta_j}{(\zeta - \zeta_j)^4} + \frac{\overline{\delta}_j}{(\overline{\zeta}_j \zeta - 1)^4} + \frac{2\alpha_j}{(\zeta - \zeta_j)^2 (\overline{\zeta}_j \zeta - 1)^2} \right),$$

where $\zeta_j = x_{j1} + i x_{j2}$ corresponds to the point $x_j \in B$. This result means that in the limit $\varepsilon \to 0$ locations of the point inclusions are found.

The idea is that $b(\zeta)$ should provide approximations to the locations of inclusions in the case $\varepsilon = 1$.

Assume that the backscatter function is given as a Laurent series $b(\zeta) = \sum_{\nu=-\infty}^{\infty} \beta_\nu \zeta^\nu$ convergent in a neighbourhood of T. This function can be approximated by the Laurent-Padé $(m-2, m)$-approximation r_m, a rational function with numerator of degree $m-2$ and denominator of degree m, whose Laurent expansion coefficients match those of b for $\nu = -2m+2, \dots, 2m-2$. Notice that β_ν are just Fourier coefficients of $b(e^{i\vartheta})$ and that r_m has an expansion of the form

$$r_m(\zeta) = \lambda_0 + \sum_{k=1}^{m} \left(\frac{\lambda_k}{\zeta - \zeta_k} + \frac{\overline{\lambda}_k \zeta}{1 - \zeta \overline{\zeta}_k} \right).$$

Section 6 considers the question of how to choose m using the Fourier coefficients. In Section 7.1 determination of the poles using the exact data $b(\vartheta)$ for different m after discharging poles with small residues is discussed. In Section 7.2, instead of exact data on T, only exact data at say 32 or 64 equidistant points on T are used. Also, the influence of the values of κ_j's on the position of the poles is discussed.

In applications, measured values of $b(\vartheta)$ are corrupted by noise. As the problem of determining the location of the inclusion is extremely ill-posed, some regularization measures have to be used. Section 7.4 gives rules for how many Fourier coefficients of $b(\vartheta)$ should be used and which terms of the expansion of r_m should be discarded. The poles ζ_j obtained by the described procedure correspond to the position of the inclusions. The theory is illustrated by many numerical examples. *Anton Suhadolc*

MR2875245 65N25 41A58

Adcock, Ben (4-CAMB-CMA; Cambridge);
Iserles, Arieh (4-CAMB-CMA; Cambridge); **Nørsett, Syvert P.** (N-NUST; Trondheim)

From high oscillation to rapid approximation II: expansions in Birkhoff series. (English summary)

IMA J. Numer. Anal. **32** (2012), *no.* 1, 105–140.

Summary: "We consider the use of eigenfunctions of polyharmonic operators, equipped with homogeneous Neumann boundary conditions, to approximate nonperiodic functions in compact intervals. Such expansions feature a number of advantages in comparison with classical Fourier series, including uniform convergence and more rapid decay of expansion coefficients. Having derived an asymptotic formula for expansion coefficients, we describe a systematic means to find eigenfunctions and eigenvalues. Next we demonstrate uniform convergence of the expansion and give estimates for the rate of convergence. This is followed by the introduction and analysis of Filon-type quadrature techniques for rapid approximation of expansion coefficients. Finally, we consider special quadrature methods for eigenfunctions corresponding to a multiple zero eigenvalue."

{For Part I see [A. Iserles and S. P. Nørsett, IMA J. Numer. Anal. **28** (2008), no. 4, 862–887; MR2457350 (2010g:65253)].}

MR2847118 65N35 41A05 41A30

Cheng, A. H.-D. (1-MS-SEN; University, MS)

Multiquadric and its shape parameter—a numerical investigation of error estimate, condition number, and round-off error by arbitrary precision computation. (English summary)

Eng. Anal. Bound. Elem. **36** (2012), *no.* 2, 220–239.

Summary: "Hardy's multiquadric and its related interpolators have been found to be highly efficient for interpolating continuous, multivariate functions, as well as for the solution of partial differential equations. Particularly, the interpolation error can be dramatically reduced by varying the shape parameter to make the interpolator optimally flat. This improvement of accuracy is accomplished without reducing the fill distance of collocation points, that is, without the increase of computational cost. There exist a number of mathematical theories investigating the multiquadric family of radial basis functions. These theories are often not fully tested due to the computation difficulty associated with the ill-conditioning of the interpolation matrix. This paper overcomes this difficulty by utilizing arbitrary precision arithmetic in the computation. The issues investigated include conditional positive definiteness, error estimate, optimal shape parameter, traditional and effective condition numbers, round-off error, derivatives of interpolator, and the edge effect of interpolation."

MR2895495 65N38 41A55 65D30

Aimi, A. (I-PARM; Parma)**; Diligenti, M. [Diligenti, Mauro]** (I-PARM; Parma)**;**
Guardasoni, C. [Guardasoni, Chiara] (I-PARM; Parma)

Numerical integration schemes for applications of energetic Galerkin BEM to wave propagation problems. (English summary)

Riv. Math. Univ. Parma (N.S.) **2** (2011), *no.* 1, 147–187.

Summary: "Here we consider wave propagation problems with vanishing initial and mixed boundary condition reformulated as space-time boundary integral equations. The energetic Galerkin boundary element method used in the discretization phase, after a double analytic integration in time variables, has to deal with weakly singular, singular and hypersingular double integrals in space variables. Efficient numerical quadrature schemes for evaluation of these integrals are here proposed. Several numerical results are presented and discussed."

MR2847119 76D05 41A30 65N35

Bourantas, G. C.; Petsi, A. J.; Skouras, E. D.; Burganos, V. N.

Meshless point collocation for the numerical solution of Navier-Stokes flow equations inside an evaporating sessile droplet. (English summary)

Eng. Anal. Bound. Elem. **36** (2012), *no.* 2, 240–247.

Summary: "The Navier-Stokes flow inside an evaporating sessile droplet is studied in the present paper, using sophisticated meshfree numerical methods for the computation of the flow field. This problem relates to numerous modern technological applications, and has attracted several analytical and numerical investigations that expanded our knowledge on the internal microflow during droplet evaporation. Two meshless point collocation methods are applied here to this problem and used for flow computations and for comparison with analytical and more traditional numerical solutions. Particular emphasis is placed on the implementation of the velocity-correction method within the meshless procedure, ensuring the continuity equation with increased precision. The Moving Least Squares (MLS) and the Radial Basis Function (RBF) approximations are employed for the construction of the shape functions, in conjunction with the general framework of the Point Collocation Method (MPC). An augmented linear system for im-

posing the coupled boundary conditions that apply at the liquid-gas interface, especially the zero shear-stress boundary condition at the interface, is presented. Computations are obtained for regular, Type-I embedded nodal distributions, stressing the positivity conditions that make the matrix of the system stable and convergent. Low Reynolds number (Stokes regime), and elevated Reynolds number (Navier-Stokes regime) conditions have been studied and the solutions are compared to those of analytical and traditional CFD methods. The meshless implementation has shown a relative ease of application, compared to traditional mesh-based methods, and high convergence rate and accuracy."

MR2849034 81Q35 39A12 39A70 47B39 81Q10

Carvalho, S. L. (BR-SACA; São Carlos)**;**
de Oliveira, C. R. [de Oliveira, César R.] (BR-SACA; São Carlos)**;**
Prado, R. A. [Prado, Roberto A.] (BR-PAUL6-MSC; Presidente Prudente)

Sparse one-dimensional discrete Dirac operators II: spectral properties. (English summary)

J. Math. Phys. **52** (2011), *no.* 7, 073501, 21 *pp.*

A one-dimensional discrete-variable Dirac Hamiltonian is considered. It is of the form

$$H_D = \begin{pmatrix} mc^2 + V_n & cd^* \\ cd & -mc^2 + V_n \end{pmatrix},$$

where $m \geqslant 0$, $c > 0$, where d and d^* are, respectively, the forward and backward difference operators: $(d\psi)_n = \psi_{n+1} - \psi_n$, $(d^*\psi)_n = \psi_n - \psi_{n-1}$. The potential V_n is nonzero only at some sparse and randomly distributed positions. The Hamiltonian H_D acts on the space $l^2(\mathbb{N}_0, \mathbb{C}^2)$ of functions of the form $\Psi_n = \binom{\psi_{1,n}}{\psi_{2,n}}$. Spectral properties of the operator H_D are investigated in the case in which the components of Ψ_n are subjected to the boundary condition

$$\psi_{2,-1} \cos\varphi - \psi_{1,0} \sin\varphi = 0 \quad (\varphi \in [0, \pi]).$$

{For Part I see [R. A. Prado and C. R. de Oliveira, J. Math. Anal. Appl. **385** (2012), no. 2, 947–960; MR2834902].} *Radosław Szmytkowski*

MR2867149 81R12 37K35 39A12 39A70

Hasegawa, Koji (J-TOHOE; Sendai)

Quantizing the Bäcklund transformations of Painlevé equations and the quantum discrete Painlevé VI equation. (English summary)

Exploring new structures and natural constructions in mathematical physics, 275–288, *Adv. Stud. Pure Math.*, 61, *Math. Soc. Japan*, *Tokyo*, 2011.

The author proposes a formal quantization of the rational Weyl group representations which originate at the discrete symmetries of the classical Painlevé equations. The construction is based on the observation that the Weyl group action, $s_j(\alpha_k) = \alpha_k - a_{jk}\alpha_j$, where a_{ij} are the entries of the generalized Cartan matrix of affine type, is given by the adjoint action, $s_j(a_k) = \rho_j a_k \rho_j^{-1}$, of $\rho_j = e^{\sqrt{-1}\,\frac{\pi}{2}\alpha_j\partial_j}$ on $a_k := e^{\alpha_k}$, where ∂_j are "dual" letters such that $[\partial_j, \alpha_k] = a_{jk}$.

The author shows that it is possible to construct the affine Weyl group action of the form $s_j(\varphi) = S_j \varphi S_j^{-1}$ on a skew field $\mathbf{F}$ defined by the relations $F_k F_{k+1} = q^{-1} F_{k+1} F_k$, $F_k F_j = F_j F_k$, $j \neq k \pm 1$. The operators $S_j := \Psi_q(z, F_j)\rho_j$ involve the above variable ρ_j and particular multiplication operators,

$$\Psi_q(z, F_j) := \frac{(qF_j, q)_\infty (F_j^{-1}, q)_\infty}{(zqF_j, q)_\infty (zF_j^{-1}, q)_\infty},$$

with the standard notation $(x,q)_\infty := \prod_{m=0}^{\infty}(1+xq^m)$ and z, q as central letters. Since the discrete Painlevé equations can be understood as the (discrete) time Hamiltonian flows generated by a specially chosen Weyl group element commuting with the simple reflections in the orthogonal directions, the above operators S_j allow the author to introduce the noncommutative versions of the q-difference Painlevé III and Painlevé VI equations.

{For the entire collection see MR2867142 (2012g:00012).} *Andrei A. Kapaev*

MR2874699 91A40 39A60

Hou, Chengmin [Hou, Cheng Min] (PRC-YANB-M; Yanji);
Cheng, Sui Sun (RC-NTHU; Hsinchu)

Complete asymptotic analysis of a two-nation arms race model with piecewise constant nonlinearities. (English summary)

Discrete Dyn. Nat. Soc. **2012,** *Art. ID* 745697, 17 *pp.*

Summary: "A discrete time two-nation arms race model involving a piecewise constant nonlinear control function is formulated and studied. By elementary but novel arguments, we are able to give a complete analysis of its asymptotic behavior when the threshold parameter in the control function varies from 0^+ to ∞. We show that all solutions originated from positive initial values tend to limit one or two cycles. An implication is that when devastating weapons are involved, 'terror equilibrium' can be achieved and escalated race avoided. It is hoped that our analysis will provide motivation for further studying of discrete-time equations with piecewise smooth nonlinearities."

MR2836568 91B55 37N40 39A60 91B25

Belitsky, Vladimir (BR-SPL-IMS; São Paulo);
Pereira, Antonio Luiz [Pereira, Antônio Luiz] (BR-SPL-IMS; São Paulo);
Prado, Fernando Pigeard de Almeida [Pigeard de Almeida Prado, Fernando] (BR-SPL2Q-PM; Ribeirão Prêto)

Stability analysis with applications of a two-dimensional dynamical system arising from a stochastic model for an asset market. (English summary)

Stoch. Dyn. **11** (2011), *no.* 4, 715–752.

In this paper, the authors study a bidimensional dynamical system in discrete time whose orbits can be interpreted as price and excess demand dynamics for a risky asset traded by heterogeneous agents. After presenting the map, in the first part of the paper the principal properties of the model are established analytically. The map under consideration admits a unique equilibrium, whose local stability properties are fully analyzed in the parameters space. In addition, the authors study the appearance of a stable periodic orbit through a Neĭmark-Sacker bifurcation and the conditions for the global stability of the equilibrium. The remaining part of the paper is devoted to linking the proposed map to the stylized financial market under consideration, which belongs to the area of HIAMs (Heterogeneous Interacting Agent Models). The authors consider a market where a unique risky asset is traded by several agents (fundamentalists and chartists) with different beliefs on the fundamental value of the asset. Each agent can trade one share of the asset and this decision is a random variable depending on the asset price, the decision of the other players and the individual evaluation of the fundamental value of the asset. The price dynamic is then driven by excess demand. When the number of agents tends to infinity, the trajectories of prices and excess demand for the asset converge almost surely to the map analyzed in the first part of the paper.

Fabio Lamantia

MR2864345 92D25 39A30 39A60

Giordano, Gaël (3-OTTW-MS; Ottawa, ON);
Lutscher, Frithjof (3-OTTW-MS; Ottawa, ON)

Harvesting and predation of a sex- and age-structured population. (English summary)

J. Biol. Dyn. **5** (2011), *no.* 6, 600–618.

Summary: "We study a discrete-time system of equations for a structured ungulate population exploited by human harvesting or a dynamic predator. The population is divided into juveniles, and female and male adults. Harvesting is concentrated on adults (trophy hunting of males or population control measures on females), whereas predation occurs in juveniles. Though the model consists of four nonlinear equations, we find explicit expressions for the steady states. We use these explicit expressions to investigate harvesting rates that allow population persistence, rates that ensure population control, and optimal harvesting efforts. Several reductions of complexity allow for a detailed analysis of the dynamics of the model. Most notably, we find that even compensatory density dependence can lead to a period doubling bifurcation, that the model does not support consumer-resource cycles, and that an Allee effect can emerge from the interplay of stage-specific predation and density-dependent prey reproduction."

MR2869763 92D25 39A28 45J99

Robertson, Suzanne L. (1-AZ-AM; Tucson, AZ);
Cushing, J. M. [Cushing, Jim M.] (1-AZ-AM; Tucson, AZ)

A bifurcation analysis of stage-structured density dependent integrodifference equations. (English summary)

J. Math. Anal. Appl. **388** (2012), *no.* 1, 490–499.

Summary: "There is evidence for density dependent dispersal in many stage-structured species, including flour beetles of the genus *Tribolium*. We develop a bifurcation theory approach to the existence and stability of (non-extinction) equilibria for a general class of structured integrodifference equation models on finite spatial domains with density dependent kernels, allowing for non-dispersing stages as well as partial dispersal. We show that a continuum of such equilibria bifurcates from the extinction equilibrium when it loses stability as the net reproductive number n increases through 1. Furthermore, the stability of the non-extinction equilibria is determined by the direction of the bifurcation. We provide an example to illustrate the theory."

MR2834502 92D25 39A30

Sacker, Robert J. (1-SCA; Los Angeles, CA)

Global stability in a multi-species periodic Leslie-Gower model. (English summary)

J. Biol. Dyn. **5** (2011), *no.* 5, 549–562.

This paper studies a population model where d species interact (competitively) in a periodic environment. The system of equations is a set of coupled Beverton-Holt equations which, in the notation developed by the author, is written as

$$x(n+1)=f(x(n)),\quad f(x)=\frac{\mu Kx}{K+(\mu-I+C^0)x}$$

where I is the identity matrix, K a column vector of carrying capacities of the d species, μ a diagonal matrix of growth parameters of the d species and C^0 a $d\times d$ matrix of competition coefficients c_{ij} with zeros on its diagonal. Some of the vector operations in this notation are the usual ones but others are defined explicitly by the author.

The autonomous case (period $p=1$) is discussed first. It is shown that when c_{ij} are "sufficiently small" (to ensure that a certain mapping is a contraction), the system possesses a fixed point or equilibrium in the interior $\mathcal{C}^0$ of the positive cone and that this equilibrium is exponentially stable and a global attractor of all points in $\mathcal{C}^0$. This indicates that "competitive coexistence" is possible when competition is weak. The proof relies on a previous article of the author on "dynamic reduction", which basically separates out the coupling terms C^0 first to reduce the system to an uncoupled one via the introduction of a modified f as

$$\widehat{f}(x) = \frac{\mu K x}{K + (\mu - I + C^0)v}$$

where v is a constant column vector. The periodic case ($p > 1$) is discussed next for the special case where $d = 2$ and $p = 3$. No reasons beyond simplifying the presentation are offered for choosing these particular numbers. It is shown that a stable solution of period p exists that globally attracts all orbits in $\mathcal{C}^0$, again subject to the condition that coupling terms c_{ij} are sufficiently small. An extension of this proof to all values of d, p seems to be straightforward. Finally, numerical simulations are presented to examine the consequences of having some large coupling terms. Among other things, it is found here that when the numbers c_{ij} are nearly equal in magnitude, discrepancies in such factors as inherent growth rates (components of μ) or in individual carrying capacities (components of K) may drive a species to extinction, a situation that the analytical part of the article shows does not occur when c_{ij} are all very small. *Hassan Sedaghat*

MR2759576 92D30 39A30 39A60

Zhou, Yicang (PRC-XJU-MSC; Xi'an)**; Cao, Hui [Cao, Hui²]** (PRC-XJU-MSC; Xi'an)

Discrete tuberculosis models and their application. (English summary)

New perspectives in mathematical biology, 83–112, *Fields Inst. Commun.*, 57, *Amer. Math. Soc.*, *Providence*, *RI*, 2010.

In this paper the authors study the dynamics of four discrete models for tuberculosis (TB). The first model addressed is a discretization of the classical SEIR model. The other three models are obtained from this basic one by considering, respectively, immigration, seasonal variation of some parameters and, finally, age structure in the exposed compartment.

The most complete analytic results are presented for the first and fourth models. Namely, by using Lyapunov functions and comparison principles, the global stability of the trivial equilibrium is proved when $R_0 < 1$ as well as its instability and the existence of a nontrivial equilibrium when $R_0 > 1$. As usual, R_0 denotes the basic reproduction number, whose expression as a function of the parameters is provided. Also, in the first model it is proved that the nontrivial equilibrium is locally asymptotically stable when R_0 is close to 1. These results are complemented by numerical simulations.

The dynamic behaviors of the model with immigration and of the model with seasonality are explored mainly numerically, although for the seasonal model sufficient conditions for the stability or instability of the trivial equilibrium are given. The simulations of the seasonal model show the existence of a stable periodic orbit.

By estimating the parameters using statistical data, the model with immigration and the model with seasonality are used to address, respectively, the immigration influence on TB infection in Canada and the seasonal fluctuation of the reported TB cases in China.

{For the entire collection see MR2759571 (2012b:92002).} *Alessandro Margheri*

MR2864347 92D40 37N25 39A28 39A60 92D25

Luís, Rafael (P-TULT-CAN; Lisbon);
Elaydi, Saber [Elaydi, Saber N.] (1-TRI; San Antonio, TX);
Oliveira, Henrique [Oliveira, Henrique Manuel] (P-TULT; Lisbon)

Stability of a Ricker-type competition model and the competitive exclusion principle. (English summary)

J. Biol. Dyn. **5** (2011), *no.* 6, 636–660.

The "Ricker model" is one of a number of discrete analogues of the familiar Lotka-Volterra analyses of interspecific interaction between two species. The competitive exclusion principle applies to models which force the extinction of one or other of the species. One such model is subjected to detailed scrutiny. The focus is strictly mathematical; no biological conclusions are advanced. *Michael A. B. Deakin*

MR2894298 93D05 34K20 39A60 60G35 93C23 93D15

Sun, Jian-Qiao; Song, Bo

Analysis and control of deterministic and stochastic dynamical systems with time delay. (English summary)

Complex systems, 119–203, *Nonlinear Phys. Sci.*, *Higher Ed. Press*, *Beijing*, 2011.

Summary: "This chapter presents a comprehensive summary of recent advances in the analysis and control of time-delayed deterministic and stochastic systems. The studies of numerical methods for time-delayed systems in the mathematics literature are reviewed including a discussion of the abstract Cauchy problem for delayed differential equations. Several numerical methods for computing the response of and designing controls for time-delayed systems are presented. These include semidiscretization, continuous time approximation, lowpass filter based continuous time approximation, and continuous time approximation with Chebyshev nodes. A 1arge number of examples are presented including optimal feedback gain design, stability domains in the feedback gain space of linear time-invariant and periodic systems, optimal control, Lyapunov stability, supervisory control of systems with uncertain time delay, moment stability, Fokker-Planck-Kolmogorov equation and reliability formulation of stochastic systems."

{For the entire collection see MR2894294 (2012i:00014).}

MR2879459 93E12 39A60 62M20 65Q10 93B30

Medvedev, Alexander [Medvedev, Aleksandr Vladislavovich]
(S-UPPS-DIT; Uppsala);
Evestedt, Magnus (S-UPPS-DIT; Uppsala)

Elementwise decoupling and convergence of the Riccati equation in the SG algorithm. (English summary)

Automatica J. IFAC **45** (2009), *no.* 6, 1524–1529.

Summary: "It is shown that the difference Riccati equation of the Stenlund-Gustafsson (SG) algorithm for estimation of linear regression models can be solved elementwise. Convergence estimates for the elements of the solution to the Riccati equation are provided, directly relating convergence rate to the signal-to-noise ratio in the regression model. It is demonstrated that the elements of the solution lying in the direction of excitation exponentially converge to a stationary point while the other elements experience bounded excursions around their current values."

MR2884115 94A20 30E10 41A35
Xu, Yan Yan (PRC-XHU-SMC; Chengdu)**;**
Zhang, Ya Lan (PRC-XHU-SMC; Chengdu)**;**
Chen, Guang Gui (PRC-XHU-SMC; Chengdu)
Irregular sampling theorem of Hermite type and estimate of aliasing error. (Chinese. English and Chinese summaries)
Acta Math. Sci. Ser. A Chin. Ed. **31** (2011), *no.* 5, 1220–1229.

In this paper, the authors consider nonuniform sampling of Hermite type for signals living in $B_{\sigma,p}$, where $B_{\sigma,p}$, $1 \le p < \infty$, contains all p-integrable functions on the real line bandlimited to $[-\sigma, \sigma]$, $\sigma > 0$. Denote the derivative of a function g by g'. The main result of this paper is that any function $g \in B_{2\sigma,p}$ can be stably reconstructed from samples $g(\lambda_k)$ and $g'(\lambda_k)$, $k \in \mathbf{Z}$, if $\delta := \sup_{k\in\mathbf{Z}} |\lambda_k - k\pi/\sigma|$ is sufficiently small.

Qiyu Sun

MR2863568 94C12 39A60 90C20 90C31
Li, Wei [Li, Wei[14]] (PRC-HZDZ-ORC; Hangzhou)**;**
Tian, Xiaoli (PRC-HZDZ-ORC; Hangzhou)
Fault detection in discrete dynamic systems with uncertainty based on interval optimization. (English summary)
Math. Model. Anal. **16** (2011), *no.* 4, 549–557.

Summary: “The imprecision and the uncertainty of many systems can be expressed with interval models. This paper presents a method for fault detection in uncertain discrete dynamic systems. First, the discrete dynamic system with uncertain parameters is formulated as an interval optimization model. In this model, we also assume that there are some errors of observation values of the inputs/outputs. Then, M. Hladík’s newly proposed algorithm is exploited for this model. Some numerical examples are also included to illustrate the method efficiency.”