Chapter 2

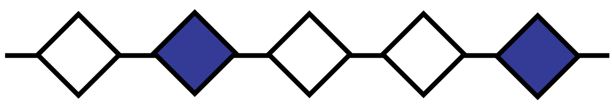
Combinatorics

Combinatorics is the art of counting. In this chapter we will count many things: colorings of bead chains, words of the Mumbo language, ice cream cones, walkways in Nowhere York city, friendly handshakes, and more. Somehow the answers to our exciting counting problems turn out to be boringly the same: either 10 or 45. This made one of our students remark:[1] "It looks like 45 is now the Answer to the Ultimate Question of Life, the Universe, and Everything." Hopefully, our discussions teach one to *foresee* why two different counting questions lead to the same result. Namely, the children can often establish a connection between different problems which guarantees the same answer even before the counting. Such a connection is called an *isomorphism*, a fancy term which the children love to use when they understand its meaning.

Teacher The familiarity with triangular numbers (in Chapter 1, "Numbers as Geometric Shapes") and with "off by one" situations (warmup Problems 6.4–6.9) comes in handy for the following material.

Coloring Beads

Problem 2.1. Coloring Two out of Five Beads. How many different 5-bead chains can you make by coloring two out of five beads and leaving the other three beads white?



For this problem we give the class handouts with 12 copies of uncolored beads. After some trial and error many children managed to find the correct answer: 10. Others made inadvertent omissions or repetitions. Even the students who got 10 different chains were unable to explain why all the possibilities are exhausted. We pointed out this difficulty but postponed the actual discussion, giving the kids a chance to discover for themselves a systematic way of listing all choices. ■

[1] Apparently quoting "The Hitchhiker's Guide to the Galaxy" by D. Adams.

Problem 2.2. Coloring Three out of Five Beads. How many different 5-bead chains can you make by coloring three out of five beads blue?

Our students solved this problem almost immediately. They pointed out that particular colors do not matter: one can exchange blue and white beads in the previous problem. The answer is still the same, 10. ■

Math Context. The equality of the answers to the last two problems is a particular case of the equality $\binom{n}{k} = \binom{n}{n-k}$. Here $\binom{n}{k}$ (read "n choose k") denotes the number of subsets of k elements in a set of n objects. This equality may be explained by replacing a subset of k elements by its complement consisting of $n - k$ elements. For example, instead of selecting seven objects out of nine, one can select the remaining two out of nine.

Teacher Our experience that the children realize right away that particular colors don't matter differs from Zwonkin's observations [**1**]. The most probable explanation is that he was working with much younger children.

Mumbo Language

Problem 2.3. Mumbo 5-Letter Words. The alphabet of the Mumbo people from Nowhere Land has only two letters, A and B. Every combination of these letters is a word. For example, BBB and ABA are Mumbo words. Find the number of Mumbo words with two As and three Bs.

Our students had the same difficulties as in the previous problem: repetitions, omissions, and no explanation why their answers are correct.

We suggested they come up with a system which would help to avoid repetitions and omissions. A few students proposed the list shown in the table below.

They started with placing the first letter A in the first spot and moving the second A through the remaining spots from left to right; then, placing the first letter A in the second spot, etc.

A	A	B	B	B
A	B	A	B	B
A	B	B	A	B
A	B	B	B	A
B	A	A	B	B
B	A	B	A	B
B	A	B	B	A
B	B	A	A	B
B	B	A	B	A
B	B	B	A	A

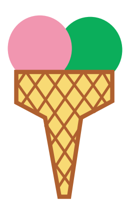
Now, the children were able to explain why the list has no repetitions or omissions: all the possible places for A are used. A couple of students commented that the words are listed in the same order as in a dictionary. They were excited to hear that they had discovered the method of ordering words which mathematicians call lexicographic or dictionary order, and which is indeed used in dictionaries. ■

In English only some arrangements of the 26 letters of the alphabet create meaningful words. For example, "hmett" is not a word. "Can one imagine a language with only two letters, and where any arrangement of these two letters makes sense?" It turns out that a language with a 2-letter alphabet is used in computers. The letters of this language are called "bits". We write them using digits 0 and 1. The name "bit" comes from "BInary digiT". Any word in this language can be interpreted as a number.

Some children noticed that 10 was the answer to Problem 2.1, "Coloring Two out of Five Beads" as well. Is it a coincidence? We address this in Problem 2.6.

Ice Cream Cones

Problem 2.4. Four Ice Cream Flavors. Ann can put two scoops of ice cream in her cone side-by-side:



She must select two distinct flavors out of vanilla, chocolate, pistachio, and strawberry. How many different ice cream cones can she make?

Teacher Without the picture above many kids place the scoops one on top of the other, which makes chocolate-vanilla different from vanilla-chocolate.

We reminded our students that mathematicians are "lazy in a good way", and so they use shortcuts instead of writing long words, or drawing complicated pictures. The possible shortcuts for vanilla, chocolate, pistachio, and strawberry may be V, C, P, and S, respectively.

The children quickly realized that the choice CV (chocolate plus vanilla) is identical to VC (vanilla plus chocolate), and so on. Soon the children listed all possibilities making sure they are all different and none are omitted. Ann's choices were: VC, VS, VP, CS, CP, and SP. Thus, the answer is 6. ■

Problem 2.5. Five Ice Cream Flavors. What would happen if Ann had five flavors available? Now blueberry is added to the list of possible flavors.

This time a lot of students missed one or two combinations. Several children organized their solution in a systematic way, a few made a table. We added names for columns and rows in their table:

	V	C	S	P	B
V		VC	VS	VP	VB
C			CS	CP	CB
S				SP	SB
P					PB
B					

Students filled this table row by row. In the second row they left the cell CV empty since it is a duplicate of cell VC. Continuing this way one gets a triangular shape formed by 10 different ice cream combinations. The children immediately pointed out that 10 is a triangular number for side 4. A few students recalled that the answer to the previous problem was also a triangular number. Our children get very excited when they make such observations. ■

Math Context. The number $\binom{n}{2}$ ("n choose 2") is the number of pairs of objects in a set of n objects. Using the lexicographic ordering of pairs, it can be identified with the sum $(n-1)+(n-2)+\cdots+2+1$, which is triangular number $T(n-1)$ for side $(n-1)$:

$$\binom{n}{2} = T(n-1) = \frac{n(n-1)}{2}.$$

Problem 2.6. Introducing Isomorphism. The answers to the three previous problems, "Coloring Two out of Five Beads", "Mumbo 5-Letter Words", and "Five Ice Cream Flavors", are the same. Is there a reason for this? What is the connection between these three problems?

We started by reminding the class of the systematic way of listing the Mumbo words. Then we gave the children the handout with uncolored beads and asked them to redo Problem 2.1 using the similar system.

Comparing the picture with the systematically colored beads and the table of the Mumbo words, the students noticed right away that they look alike. In a few moments someone suggested to write the letter A over the blue beads and the letter B over the white ones. Now the children were screaming, "The problems are the same!"

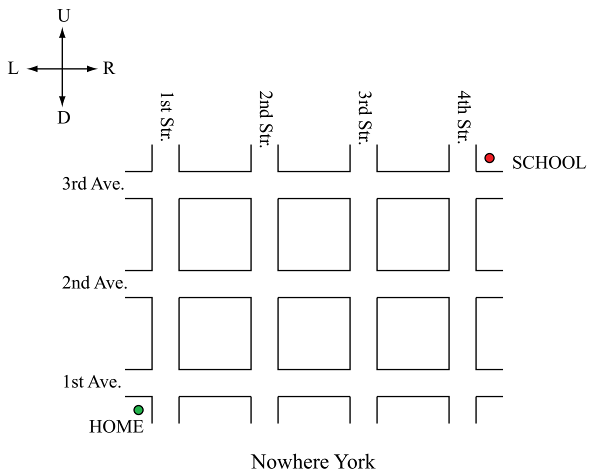
Comparing the problem of the colored beads with the problem of choosing two ice cream flavors turned out to be more difficult. Hint: "In the shop five containers with different flavors of ice cream stand in one line." Now, many students suggested that when Ann chooses two flavors, we should color the chosen containers blue. Then Ann's choices indeed look like colored beads! ■

We told the children that sometimes two problems are related so that a solution to one of them can be translated into a solution for the other, and vice versa. Such problems are called *isomorphic*. We just discovered that all three problems, "Coloring Two out of Five Beads", "Mumbo 5-Letter Words", and "Five Ice Cream Flavors" are isomorphic.

Teacher Usually when we introduce mathematical terms in class we do not expect that the kids would remember them, and we refrain ourselves from using them in class. The word isomorphic is an exception: the students love this word and use it all the time.

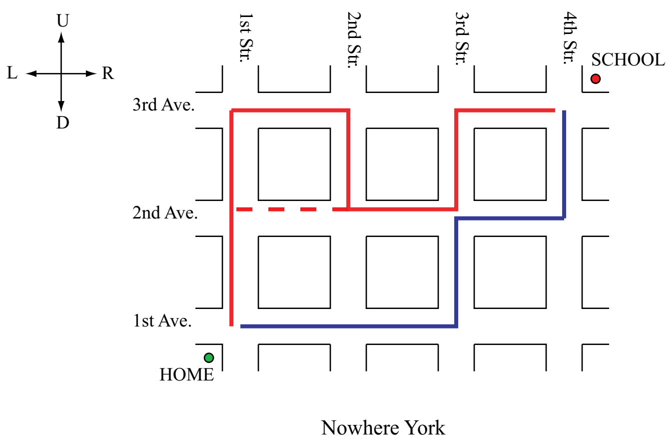
Nowhere York City

Problem 2.7. Walks in Nowhere York. Andy's home and school in Nowhere York are shown below. Andy walks from home to school always choosing the shortest possible routes. Determine the number of such routes.

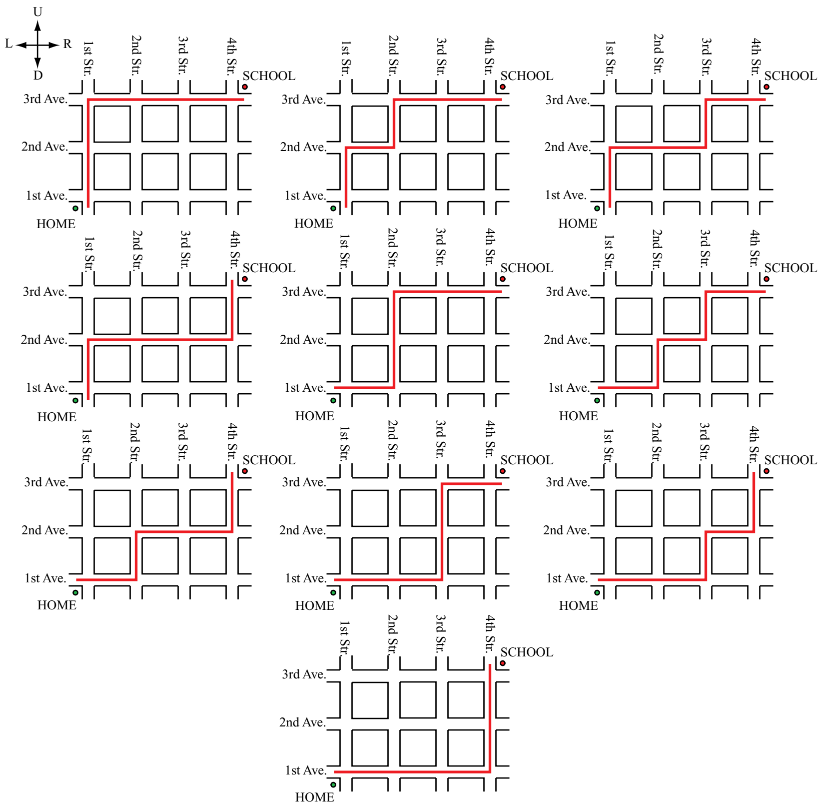


To help the children to distinguish between left and right we mark the directions on the top of the picture.

First, we need to understand how the "shortest possible routes" look like. After some debate our students came to the conclusion that Andy should walk only right and up. Otherwise, the route would be longer, and one can make a shortcut as shown by the dashed line for the "red" route:



Now, we can start drawing all possible routes from home to school. Initially, it was not obvious (at least to some of our younger students) that all the paths going right and up have the same length: all go three blocks right and two blocks up. Some made this discovery only after drawing a few paths. Also, one has to be careful not to repeat the same path, and it might be unclear why the drawings include all possible shortest routes, but, eventually, the students ended up with 10 different pictures.



The children were excited to get the answer 10 again, and were sure that the problem should be isomorphic to each of the previous ones: “Coloring Two out of Five Beads”, “Mumbo 5-Letter Words”, and “Five Ice Cream Flavors”. Some of the students were even able to find an explanation, the others needed a hint: “Record all possible shortest routes using the letter R for walking right and the letter U for walking up.”

The 10 shortest paths are encoded by 5-letter words consisting of two Us and three Rs:

UURRR URURR URRUR URRRU
RUURR RURUR RURRU
RRUUR RRURU
RRRUU

Now, it becomes obvious that this problem is identical to the “Mumbo 5-Letter Words” problem: use A instead of U, and B instead of R. ■

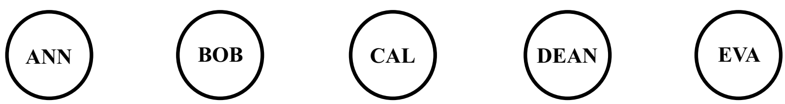
The Handshake Problem

Problem 2.8. Handshakes of Five Friends. Five friends, Ann, Bob, Cal, Dean, and Eva, shook hands with each other. Each person shook hands with every other person once. How many handshakes occurred?

The best way to clarify what happens (especially to the younger students) is to ask five children to come to the board and shake hands with each other. We would like the children to find a systematic way to do this. The question: "Who will shake hands first?" is a good way to initiate the due process. Our students suggested that first Ann should shake hands with Bob, Cal, Dean, and Eva (four handshakes). Bob should be the next to shake hands. Since Bob already shook Ann's hand, he needs to shake hands only with Cal, Dean and Eva (three handshakes). Now comes Cal who shakes hands with Dean and Eva (two handshakes). Finally, Dean shakes hands with Eva (one handshake). Thus, the total number of handshakes is $4 + 3 + 2 + 1 = 10$, or the fourth triangular number.

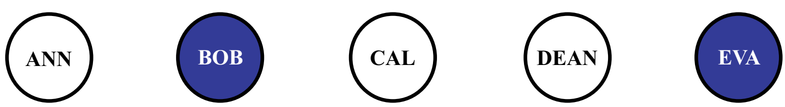
Once again the students claimed that this problem must be the same as the ones we solved before. How can we find a connection between this problem and coloring of the beads?

Hint 1: "Let Ann, Bob, Cal, Dean, and Eva line up in a single row."



Hint 2: "Find a way to represent the handshakes between the two kids."

Our students proposed to color blue two kids shaking hands, for example, Bob and Eva:



Now the kids play the role of the beads. Different handshakes match different colorings of the beads. So the problems are isomorphic. The children were so pleased with their own idea that it took us a couple of minutes to calm them down. ■

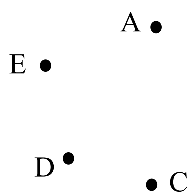
Problem 2.9. Handshakes of Six Friends. Six friends, Ann, Bob, Cal, Dean, Eva, and Fred, shook hands with each other. Each person shook hands with every other person once. How many handshakes occurred?

When we started discussing this problem the five kids who previously shook hands were still by the board. The kids suggested to invite one more child, Fred, to the board. They explained that the first five kids already shook hands with each other, so only Fred should shake hands with everybody else. Thus five more handshakes should be added to the previous answer; this may be written as $5+4+3+2+1=15$ handshakes, or the triangular number for side 5.

A couple of our students solved the handshake problems using a different approach. First, they counted how many times each of the six children shook hands. Each one shakes hands with everybody else, which means he or she shakes five hands. There are six children total, so the total number of handshakes is $6 \cdot 5 = 30$. "How come this answer is different (and wrong)?" Knowing the correct answer several students immediately realized that they had counted every handshake twice! For example, the Ann-Bob handshake was counted as Ann shaking Bob's hand and Bob shaking Ann's hand. To avoid double counting, the answer 30 should be divided by 2. ■

Sides and Diagonals

Problem 2.10. Connecting Dots. Five points A, B, C, D, and E are drawn in a circle. Tim connected every two points by a segment. How many segments did he draw?



The students immediately identified the points with five friends shaking hands. The children explained that since the problems are "the same", the answers are also the same: there are 10 segments. ■

Same Problems with 10 Objects

Problem 2.11. Coloring Two out of 10 Beads. How many different 10-bead chains can you make by coloring two out of 10 beads blue?

The children unanimously agreed not to draw all the possibilities since there would be too many pictures for a 10-bead chain. Instead, they proposed to count the choices without drawing them.

First they looked at the 10-bead chains with the first bead colored blue. In this case there are nine choices for coloring the second bead blue: it can be any bead from the second through the tenth.

Then, if the first blue bead were in the second place, there would be only eight choices for coloring another bead blue. And so on.

The total number of different chains is $9+8+7+6+5+4+3+2+1=45$, the ninth triangular number. ■

Problem 2.12. Revisiting Mumbo Words and Handshakes. Invent a problem about Mumbo words that is the "same" as the "Coloring Two out of 10 Beads" problem.

Do the same for the handshake problem.

Teacher This exercise emphasizes the idea of two problems being the same (being isomorphic). We needed to remind a couple of students the solutions to the Mumbo 5-letter word problem.

The children immediately proposed replacing the blue beads with As and the white beads with Bs. The new Mumbo problem would read as: "How many different words in Mumbo language are there with exactly two As and eight Bs?"

It was more challenging to formulate the handshake problem. Eventually the students remembered how they solved the handshake problem: they thought of kids as beads with two kids painted blue shaking hands. Since there are 10 beads in the chain now, there should be 10 children in the handshake problem. The students came up with the following: "Compute the total number of handshakes if ten friends want to shake hands with each other." ■

Apples, Oranges, and More

Teacher The following problems are more challenging. They should be done only if the students are comfortable with the problems in the previous sections.

Problem 2.13. Eight Apples for Three Kids. Find the number of ways to distribute eight apples between three children, Ann, Bob, and Cal. It is OK if somebody gets no apples at all.

Many students solved this problem by systematically considering all possibilities. To help them we drew the first two rows of the following table (where Ann has eight or seven apples) on the board:

Ann	Bob + Cal	Number of choices
8	0+0	1
7	0+1, 1+0	2
6	0+2, 1+1, 2+0	3
…	…	…
0	0+8, 1+7, … 7+1, 8+0	9

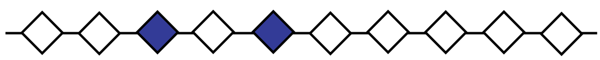
If Ann gets eight apples, Bob and Cal both have none — only one choice. If Ann gets seven apples, one apple remains. It can go either to Bob or to Cal — two choices.

After filling the third row a number of children discovered the pattern 1, 2, 3 and continued it numerically. We insisted on the explanation for this pattern. Many kids noticed that if Bob's numbers are listed starting with 0 and ending with 8, there are $8 + 1 = 9$ numbers in total. Other children simply filled the table row by row, listing all possibilities. Everyone found that the total number of ways to distribute eight apples between Ann, Bob, and Cal is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.

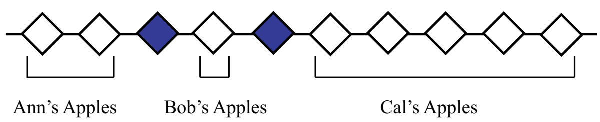
By now the class was on the lookout for the problems with the same answers. They immediately suspected that this problem is "the same" as coloring 10 beads.

Teacher This isomorphism is more complicated than those considered before. In some of our classes we had to explain it to the students on the board.

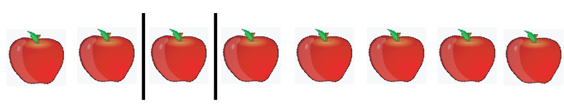
A lot of children noticed that there are eight white beads in the chain and associated them with eight apples in the problem. Only one or two kids were able to figure out what the blue beads represent. The rest received a hint: "Every chain represents a way to divide the apples between the kids. Given the chain below, how do we divide apples?"



Soon the students came up with this explanation:



which may be translated into the "apple" language in the following way:



Several students had difficulties translating cases with two blue beads next to each other or with a blue bead at either end of the chain. In a few minutes the class found the answers. If the first bead is blue, Ann has no apples. If two blue beads are adjacent, Bob has no apples. If the last bead is blue, Cal has no apples.

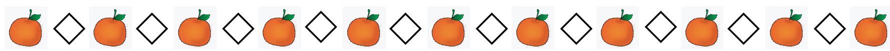
Going back from apples to beads it is crucial to have apples and separators on an equal footing: eight apples plus two separators equals 10 total, which we translated into 10 beads (eight white and two blue). ■

Problem 2.14. Eleven Oranges for Three Kids. In how many ways can 11 oranges be given to Ann, Bob, and Cal if each child should get at least one orange?

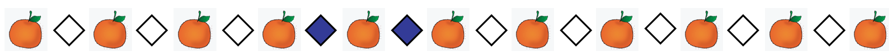
Some of our students successfully solved this problem by going through the table, similar to the first solution to the previous problem. Some children, to our great pleasure, were trying to reduce this problem to the previous one. They discovered that if we start with giving Ann, Bob, and Cal one orange each, the remaining eight oranges may be divided in exactly the same manner as eight apples in the previous problem (when zero counts were allowed). So, the answer is 45.

A couple of students had no idea how to begin the problem and needed a hint: “We have three more oranges than we had apples in the previous problem, but now everybody must get at least one orange. How would you distribute the oranges?”

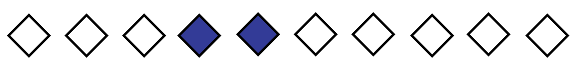
A couple of students solved the problem by constructing an unexpected direct isomorphism with the beads problem. They figured out how to fit 10 beads into the problem with 11 oranges. If the oranges are arranged in a line, there are 10 gaps between them and a bead can be placed in every gap:



The blue beads separate the oranges given to different kids, the remaining eight beads remain white. For example, if Ann gets four, Bob one, and Cal six oranges, the picture looks like this:



Even if blue beads were next to each other (or at the beginning, or at the end of the row), every child would get at least one orange. Now, removing the oranges, only a chain of beads remains:



So, each arrangement of two blue beads and eight white beads in a chain gives a way to divide oranges. This is the isomorphism! ∎

Problem 2.15. Nine Plums for Three Kids. In how many ways can nine plums be given to Ann, Bob, and Cal if Ann has to share one plum with her baby sister and must get one while the other kids don’t mind getting none?

This problem was not difficult for the students. They suggested to give one plum to Ann first. The remaining eight plums should be divided in the same way as apples. Thus, the answer is 45. ∎

Problems about Numbers

Teacher Some of our students solved the next two problems by direct counting or by constructing an elaborate isomorphism with the coloring of 10 beads. Here we do not explain either of these solutions. Instead, we discuss the simpler isomorphisms with the problems from the previous section discovered by our students.

Problem 2.16. Sum of Digits Is 9. How many three-digit numbers with the sum of all digits being 9 are there?

A number of students solved this problem right away. The rest needed some hints: "How many digits are there? What are the options for the first digit of any three-digit number? What about other digits?" After realizing that the first digit must be at least 1, the children concluded that the problem is identical to the previous one, nine plums for three kids. The first digit is the number of Ann's plums, the other two digits represent Bob's and Cal's plums, so the answer is 45. ■

Problem 2.17. Sum of Digits Is 11. How many three-digit numbers with all digits greater than zero and their sum being 11 are there?

Almost all children recognized that this problem is the same as 11 oranges for three kids: the first digit is Ann's oranges, the second is Bob's, and the third is Cal's oranges. Thus, the answer is 45 again. ■

Problem 2.18. Decreasing Digits. How many eight-digit numbers are there, such that every digit is greater than the one on its right? 87543210 is an example of such a number.

Teacher This is an example of a problem where constructing the isomorphism is significantly easier than direct counting. To invent the direct counting one needs a system that tracks the missing digits instead of those present, but noticing this immediately leads to an isomorphism. So, we won't discuss the direct counting approach, which was used by a couple of our students.

Some of our students solved this problem very quickly by making the following discovery: any eight-digit number in which every digit is greater than the one on its right is obtained from the sequence 9 8 7 6 5 4 3 2 1 0 by erasing two digits. Let's think of the erased digits as the blue beads, and the remaining digits as the white beads. Then any coloring of two out of 10 beads blue corresponds to one of the numbers we are looking for. Thus, the problems are the same and the answer is 45. A few students struggled with this problem for a long time, and finished it at home. Here is a hint for them: "Consider a number that fits the requirements. Which digits, from 9 to 0, are missing?" ■

Harder Problems

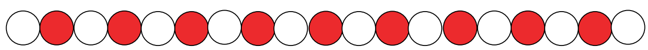
Teacher The following two problems are quite challenging and are intended for more advanced students.

Problem 2.19. Arranging Red and White Balls. In how many ways can nine red and eight white balls be arranged in a row with no two white balls next to each other?

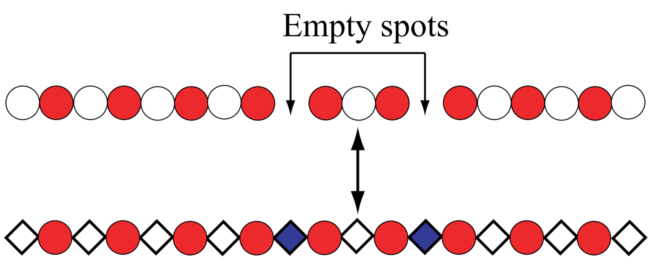
This time all of our students were trying to see whether this problem was the same as coloring of a 10-bead chain. Some found the solution quickly but others struggled.

Hint 1: "Draw the red balls first."

Hint 2: "What is the largest number of white balls (instead of 8) that can be arranged without touching?"



The kids pointed out that nine red balls create 10 possible spots for the white balls. Eight out of 10 spots should be occupied by white balls — place them there. The remaining two spots are left empty. Placing blue beads there creates a familiar picture: coloring two out of 10 beads. Therefore, there are 45 ways to create the required arrangement.



■

Problem 2.20. Ordering Ice Cream. An ice cream shop has nine flavors of ice cream. In how many ways can Sam order a two-scoop cone (see Problem 2.4) if he is allowed to choose the same flavor twice?

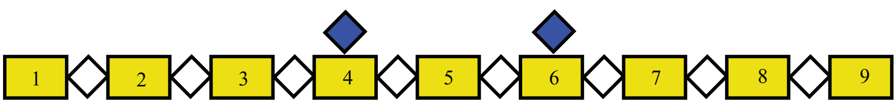
Many children solved this problem by directly counting all possible ways of choosing flavors. There are nine choices for the first of Sam's scoops, 1 through 9; let's list all of them in a table:

Flavor of the 1st scoop	Flavor of the 2nd scoop	Number of choices
1	1, 2, 3, . . . , 9	9
2	2, 3, . . . , 9	8
3	3, . . . , 9	7
. . .	. . .	. . .
9	9	1

The first row of the table lists all combinations containing the 1st flavor. In the second row the 1st flavor is not an option for the 2nd scoop to avoid duplication (all the combinations with the first flavor are already in the first row). Continuing with the rows, the students obtained the sum $9+8+7+6+5+4+3+2+1=45$.

It looks like this problem is "the same" as coloring "Two out of 10 Beads". Why? The explanation happened to be truly challenging, and we had to help the students. After we drew nine ice cream containers in a row, one of our students came up with a good hint. She noticed that $8=9-1$, and it is the number of gaps between the containers. So, she put eight white beads in the gaps.

Where should one put two blue beads? After some pondering the children decided that the blue beads should mark the scoops that Sam chose. Now, ignoring the containers, but looking only at the beads, we obtained a ten-bead chain with eight white and two blue beads. This means that the problems are the same and have the same answer, 45.



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