

## Expository Suggestions

While the exposition of Lie group theory in this book is connected to some parts of Chapter I (including affine connections) we shall now indicate how the book can be used more directly as a text for a one-term course on Lie groups.

The idea is to introduce the *exponential mapping* for a Lie group at the beginning and use it to develop the correspondence between Lie subalgebras and Lie subgroups.

The prerequisites are linear algebra, point set topology and some familiarity with topological groups.

### Introduction to Lie Groups.

(I) Examples of Lie Groups.

Discussion of Example 1, p.98 and  $\mathbf{GL}(n, \mathbf{R})$ .

(II) Introduction of basic notions of differential geometry. Manifolds,  $\mathcal{C}^\infty$  functions, tangent spaces, vector fields,  $\mathcal{C}^\infty$  mappings. Pages 1–7, 9–10, 22–24 from the book.

(III) Definition of a Lie group and its Lie algebra. Abstract Lie algebra and its universal enveloping algebra. Pages 97–101 (including Corollary 1.3).

(IV) Integral curves of a vector field. Let  $Y$  be a vector field on a manifold  $X$ . In a local chart

$$(U, \varphi), \quad \varphi(q) = (x_1(q), \dots, x_m(q)).$$

$Y$  can be written (cf. p. 10)

$$Y = \sum_{i=1}^m Y^i \frac{\partial}{\partial x_i}.$$

Given a point  $p \in U$ , an *integral curve* to  $Y$  through  $p$  is a curve  $t \rightarrow \gamma(t)$ , ( $t \in I$ ) satisfying

$$(A) \quad \gamma(0) = p, Y_{\gamma(t)} = \dot{\gamma}(t) \text{ (tangent vector to } \gamma \text{ at } \gamma(t) \text{)}.$$

In the local chart above let

$$a_i = x_i(p), \quad x_i(t) = x_i(\gamma(t)), \quad (1 \leq i \leq m).$$

Then (A) takes the form

$$(B) \quad \frac{dx_i}{dt} = (Y^i \circ \varphi^{-1})(x_1(t), \dots, x_m(t)), \quad x_i(0) = a_i \quad (1 \leq i \leq m).$$

By standard theory (B) has a locally unique solution, smooth (analytic) in  $t$  and  $(a_1, \dots, a_m)$ .

(V) The exponential mapping (replacing pages 102–104).

Take  $X$  in (IV) to be a Lie group  $G$  (as defined on page 98). Let  $L_g$  and  $R_g$  denote the translations  $h \rightarrow gh$  and  $h \rightarrow hg$ , respectively, and  $(dL_g)_h, (dR_g)_h$  their differentials at  $h$ . For each  $Y \in G_e$  consider the vector field  $\tilde{Y}$  on  $G$  given by  $\tilde{Y}_g = (dL_g)_e(Y)$ . Using the result of (IV) let  $\Gamma(t)$  ( $0 \leq t \leq \epsilon$ ) be the local integral curve of  $\tilde{Y}$  through  $e$  with  $\Gamma(0) = e$ . Thus,

$$\dot{\Gamma}(t) = \tilde{Y}_{\Gamma(t)} \quad \text{for } 0 \leq t \leq \epsilon.$$

Defining successively for  $n \in \mathbf{Z}$

$$\Gamma(t) = \Gamma(n\epsilon)\Gamma(t-n\epsilon), \quad n\epsilon \leq t \leq (n+1)\epsilon,$$

$\Gamma$  extends to  $\mathbf{R}$ . On the interval  $n\epsilon \leq t \leq (n+1)\epsilon$  we have ( $L$  referring to translations both on  $G$  and on  $\mathbf{R}$ )

$$\Gamma = L_{\Gamma(n\epsilon)} \circ \Gamma \circ L_{-n\epsilon}$$

so

$$\begin{aligned} \dot{\Gamma}(t) &= d\Gamma \left( \frac{d}{dt} \right)_t = dL_{\Gamma(n\epsilon)} \circ d\Gamma \circ dL_{-n\epsilon} \left( \frac{d}{dt} \right)_t \\ &= dL_{\Gamma(n\epsilon)} \circ d\Gamma \left( \frac{d}{dt} \right)_{t-n\epsilon} = dL_{\Gamma(n\epsilon)} \tilde{Y}_{\Gamma(t-n\epsilon)} = \tilde{Y}_{\Gamma(t)}. \end{aligned}$$

Thus  $\dot{\Gamma}(t) = \tilde{Y}_{\Gamma(t)}$  for all  $t \in \mathbf{R}$ . Fix  $s \in \mathbf{R}$  and consider the curve

$$\Gamma_s : t \rightarrow \Gamma(s)\Gamma(t-s) = L_{\Gamma(s)} \circ \Gamma \circ L_{-s}(t).$$

Again we have

$$\begin{aligned} d\Gamma_s \left( \frac{d}{dt} \right)_t &= dL_{\Gamma(s)} \circ d\Gamma \left( \left( \frac{d}{dt} \right)_{t-s} \right) = dL_{\Gamma(s)} \tilde{Y}_{\Gamma(t-s)} \\ &= \tilde{Y}_{\Gamma(s)\Gamma(t-s)} = \tilde{Y}_{\Gamma_s(t)} \end{aligned}$$

so

$$\dot{\Gamma}_s(t) = \tilde{Y}_{\Gamma_s(t)}.$$

Thus  $\Gamma_s$  is another integral curve of  $\tilde{Y}$ . Since it agrees with  $\Gamma$  at  $s = t$  we have  $\Gamma_s \equiv \Gamma$  so  $\Gamma$  is a one-parameter subgroup. We put

$$\exp Y = \Gamma(1).$$

For each  $s \in \mathbf{R}$ ,  $t \rightarrow \Gamma(st)$  is a one-parameter group with tangent vector  $sY$  at  $t = 0$ . Thus

$$\exp sY = \Gamma(s).$$

The map  $\exp : \mathfrak{g} \rightarrow G$  therefore satisfies

$$\exp(s+t)Y = \exp sY \exp tY .$$

The mapping  $\exp : \mathfrak{g} \rightarrow G$  is analytic: For  $Z \in \mathfrak{g}$  let  $\mathfrak{g}_Z$  denote the tangent space to  $\mathfrak{g}$  at  $Z$ . Consider the product group  $G \times \mathfrak{g}$  and the vector field  $Y$  on  $G \times \mathfrak{g}$  given by

$$Y(\sigma, Z) = (\tilde{Z}_\sigma, 0) \in G_\sigma \oplus \mathfrak{g}_Z .$$

Because of the formula p. 99,  $Y$  is analytic. By (IV) the integral curves of  $Y$  are analytic. The one through  $(\sigma, Z)$  is given by

$$\eta(t) = (\sigma \exp tZ, Z) ,$$

which we write as  $Y_t(\sigma, Z)$ . Writing  $\tau$  for the projection of  $G \times \mathfrak{g}$  on the first factor we have  $\exp Z = \tau \circ Y_1(e, Z)$  so  $\exp$  is analytic. Its differential at 0 is  $I$ . This then proves Proposition 1.6, p. 104 in the text.

(VI) Finally we give an independent proof of Theorem 1.7 in the text (p. 105).

In the statement,  $\frac{1-e^{-A}}{A}$  stands for  $\sum_0^\infty \frac{(-A)^n}{(n+1)!}$ . For the proof we use the Taylor formula (6) on p. 105. We have for  $f$  analytic near  $\exp X$ ,

$$\begin{aligned} (d \exp_X(Y)f)(\exp X) &= Y(f \circ \exp)(X) \\ &= \left\{ \frac{d}{dt} f(\exp(X+tY)) \right\}_{t=0} = \left\{ \frac{d}{dt} \sum_0^\infty \frac{1}{n!} (\tilde{X} + t\tilde{Y})^n f(e) \right\}_{t=0} \\ &= \sum_0^\infty \frac{1}{(n+1)!} (\tilde{Y}\tilde{X}^n + \tilde{X}\tilde{Y}\tilde{X}^{n-1} + \dots + \tilde{X}^n\tilde{Y})f(e) . \end{aligned}$$

Now  $(\text{ad } X(Y))^\sim = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}$  so we define

$$\text{ad } X(\tilde{Y}) = \tilde{X}\tilde{Y} - Y\tilde{X} = L_{\tilde{X}}\tilde{Y} - R_{\tilde{X}}\tilde{Y} .$$

Thus

$$R_{\tilde{X}} = L_{\tilde{X}} - \text{ad } X$$

and they commute. Thus

$$\begin{aligned} \tilde{Y}\tilde{X}^m &= (R_{\tilde{X}})^m(\tilde{Y}) = (L_{\tilde{X}} - \text{ad } X)^m(\tilde{Y}) \\ &= \sum_{p=0}^m (-1)^p \binom{m}{p} \tilde{X}^{m-p} (\text{ad } X)^p(\tilde{Y}) \end{aligned}$$

so

$$\begin{aligned}
\tilde{Y}\tilde{X}^n + \cdots + \tilde{X}^n\tilde{Y} &= \sum_{p=0}^n \tilde{X}^p\tilde{Y}\tilde{X}^{n-p} \\
&= \sum_{p=0}^n \tilde{X}^p \sum_{k=0}^{n-p} (-1)^k \binom{n-p}{k} \tilde{X}^{n-p-k} (\text{ad } X)^k (\tilde{Y}) \\
&= \sum_{p=0}^n \sum_{k=0}^{n-p} (-1)^k \binom{n-p}{k} \tilde{X}^{n-k} (\text{ad } X)^k (\tilde{Y}) \\
&= \sum_{k=0}^n \sum_{p=0}^{n-k} (-1)^k \binom{n-p}{k} \tilde{X}^{n-k} (\text{ad } X)^k (\tilde{Y}).
\end{aligned}$$

Now what is  $\sum_{p=0}^{n-k} \binom{n-p}{k}$ ? Look at  $\sum_{p=0}^{n-k} (1+t)^{n-p}$  and collect coefficient to  $t^k$ .

$$\begin{aligned}
(1+t)^n + \cdots + (1+t)^k &= (1+t)^k (1 + \cdots + (1+t)^{n-k}) \\
&= (1+t)^k \frac{(1+t)^{n-k+1} - 1}{(1+t) - 1} = \frac{(1+t)^{n+1} - (1+t)^k}{t}
\end{aligned}$$

and the coefficient to  $t^k$  equals  $\binom{n+1}{k+1}$ . Thus

$$\sum_{p=0}^{n-k} \binom{n-p}{k} = \binom{n+1}{k+1},$$

so our sum is

$$\sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \tilde{X}^{n-k} (\text{ad } X)^k (\tilde{Y}).$$

Thus

$$\begin{aligned}
(d \exp_X(Y)f)(\exp X) &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} (\tilde{X}^{n-k} (\text{ad } X)^k (\tilde{Y})f)(e) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^k}{(k+1)!} \frac{1}{(n-k)!} (\tilde{X}^{n-k} (\text{ad } X)^k (\tilde{Y})f)(e) \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-1)^k}{(k+1)!} \frac{1}{(n-k)!} \tilde{X}^{n-k} (\text{ad } X)^k (\tilde{Y})f(e) \\
&= \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^k}{(k+1)!} \frac{1}{r!} \tilde{X}^r (\text{ad } X)^k (\tilde{Y})f(e) \\
&= \sum_{r=0}^{\infty} \frac{\tilde{X}^r}{r!} \sum_0^{\infty} \frac{(-1)^k}{(k+1)!} ((\text{ad } X)^k (\tilde{Y})f)(e)
\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{1 - e^{-\text{ad } X}}{\text{ad } X} (\tilde{Y}) f \right) (\exp X) \\
&= \left( \left( \frac{1 - e^{-\text{ad } X}}{\text{ad } X} (Y) \right) \sim f \right) (\exp X) \\
&= dL(\exp X) \circ \frac{1 - e^{-\text{ad } X}}{\text{ad } X} (Y) (f) (\exp X).
\end{aligned}$$

Thus

$$dL_{\exp(-tX)} \circ d\exp_{tX}(Y) = \frac{1 - e^{-\text{ad}(tX)}}{\text{ad}(tX)}(Y)$$

for all  $t$  in some interval  $|t| < \delta$ . Both sides of this equation are analytic in  $t$  with values in  $\mathfrak{g}$ . Since they agree for  $|t| < \delta$  they must agree for all  $t \in \mathbf{R}$ .

- (VII) After Theorem 1.7, p. 105, continue with Chapter II. The end of §6 (Theorem 6.9 and Proposition 6.10) uses Riemannian geometry results proved in Ch. I.

Alternative proof of Proposition 6.10 comes from Theorem 2.5 in Ch. VII (with  $T$  reduced to the identity element). Similarly, Theorem 6.9 (Ch. II) is contained in Theorem 6.1 in Ch. VII.

- (VIII) Section 7 on Invariant Differential Forms is rather optional for an introductory course. If included it would require the material on differential forms in Ch. I, that is pp. 20–25.

- (IX) Here §8 is of historical nature. Lie's theorem (pp. 144-145) was one of his original results. Since this theorem seems to have disappeared from recent books on ordinary differential equations we give here a sketch of a proof.

Suppose  $\varphi_t$  is a 1-parameter group leaving the equation

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)}$$

stable. If  $U(x, y) = c$  is a solution we have with  $\varphi_t(x, y) = (x_t, y_t)$ ,

$$U(x_t, y_t) = c(t) \quad (\text{all } t)$$

so

$$\frac{\partial U}{\partial x} \frac{dx_t}{dt} + \frac{\partial U}{\partial y} \frac{dy_t}{dt} = c'(t)$$

and by definition of  $\xi$  and  $\eta$

$$(1)$$

$$\frac{\partial U}{\partial x}\xi + \frac{\partial U}{\partial y}\eta = c'(0).$$

Secondly

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \frac{dy}{dx} = 0$$

so

(2)

$$\frac{\partial U}{\partial x}X + \frac{\partial U}{\partial y}Y = 0.$$

If  $c'(0) \neq 0$  we can normalize  $U$  such that  $c'(0) = 1$ . Then (1) and (2) imply

$$\frac{\partial U}{\partial x} = \frac{-Y}{X\eta - Y\xi}, \quad \frac{\partial U}{\partial y} = \frac{X}{X\eta - Y\xi}$$

so  $(X\eta - Y\xi)^{-1}$  is an integrating factor for  $Xdy - Ydx = 0$  as claimed.

On the other hand, if  $c'(0) = 0$ , (1)–(2) imply  $dy/dx = \eta/\xi$  so the integral curves are just the orbits of  $\varphi_t$ .