Supplementary Remarks to Ch.V, §1: Representation of Compact Lie Groups.

The notation of the book will be kept.

Remark 1. On p. 498 middle it is stated that the set $\Lambda(\pi)$ is "clearly" invariant under the Weyl group W. In fact, let $\lambda \in \Lambda(\pi)$, $s \in W$ and select $u \in U$ with $\mathrm{Ad}\,(u^{-1})$ realizing s on $\mathfrak{t}$. Then if $H \in \mathfrak{t}$, $v \in V_\lambda$,

$$\begin{aligned}\pi(\exp H)\pi(u)v &= \pi(u)\pi(u^{-1})\pi(\exp H)\pi(u)v = \pi(u)\pi(\exp sH)v \\ &= \pi(u)e^{\lambda(sH)}v = e^{(s^{-1}\lambda)(H)}\pi(u)v\,.\end{aligned}$$

So $s^{-1}\lambda \in \Lambda(\pi)$ as stated.

Remark 2, p. 502. The function h in (17) is $\not\equiv 0$. In fact, the subsequent integral formula for $\int \widetilde{h}\bar{\chi}\,du$ shows that $|\widetilde{h}|^2$ has integral $\neq 0$.

Remark 3, p. 543. Exercise A1 stating

$$\langle \delta+\rho, \delta+\rho\rangle - \langle \rho,\rho\rangle = 1$$

has a hint on p. 390 that seems a bit short. For more details, let $X_1, \dots, X_n$ be a basis of $\mathfrak{u}$ orthonormal for the Killing form $\langle\ ,\ \rangle$. As shown in Exercises A1, A4 in Ch. II (solutions pp. 567–568) the Laplace-Beltrami operator L_U satisfies

$$L_U = -\sum_i \widetilde{X}_i\widetilde{X}_i\,.$$

The representation ad of $\mathfrak{u}$ extends to the universal enveloping algebra so

$$\mathrm{ad}\,(L_U) = -\sum_i \mathrm{ad}\ \widetilde{X}_i\,\mathrm{ad}\ \widetilde{X}_i$$

and each member in this formula is a linear transformation of $\mathfrak{u}$. By Lemma 1.6(i) $\mathrm{ad}\,(L_U) = cI$ and by Lemma 1.6(ii) $L_U\chi = c\chi$. On the other hand, the linear transformation $\mathrm{ad}\ \widetilde{X}_i$ is just $\mathrm{ad}\ X_i$ so taking trace of the above equation we get

$$c\,\dim\mathfrak{u} = -\,\mathrm{Tr}\left(\sum_i \mathrm{ad}\ X_i\,\mathrm{ad}\ X_i\right) = -\dim\mathfrak{u}\,.$$

Thus $c=-1$, $L_U\chi = -\chi$ so the result follows from (16), δ being the highest weight of ad.

Another proof of the formula is given in Freudenthal-de Vries, Section 4.3.3.

Remark 4. We now invoke the simply connected complex group G with Lie algebra $\mathfrak{g}$.

Let H, N and $\bar{N}$ denote the analytic subgroups corresponding to the subalgebras

$$\mathfrak{t}, \mathfrak{n} = \sum_{\alpha>0}\mathfrak{g}^\alpha\,,\ \bar{\mathfrak{n}} = \sum_{\alpha<0}\mathfrak{g}^\alpha\,.$$

Let B denote the group HN with Lie algebra $\mathfrak{b} = \mathfrak{t}+\mathfrak{n}$.

a) The space G/B is compact.

For this consider the orbit $U \cdot eB$ of U in G/B. It is a compact submanifold but since $\mathfrak{u} \cap \mathfrak{b} = \mathfrak{t}_0$ the dimension equals $\dim \mathfrak{u} - \dim \mathfrak{t}_0$ which equals $\dim_{\mathbf{C}} \mathfrak{g} - \dim_{\mathbf{C}} \mathfrak{t}$ which in turn equals $\dim_{\mathbf{R}} G/B$. Thus $U \cdot eB$ is all of G/B, which thus is compact.

b) Let $\lambda \in \Lambda$. Since $\lambda(H) \in 2\pi i\mathbf{Z}$ if $\exp H = e$ there exists a holomorphic homomorphism $\omega : H \to \mathbf{C}^{\times}$. We extend this to a homomorphism $\omega : B \to \mathbf{C}^{\times}$ by $\omega(hn) = \omega(h)$ and consider the vector space $V_\omega = \{F$ holomorphic on $G : F(gb) = \omega(b)F(g)$.

If non zero, V_ω is the space of sections of the line bundle over G/B defined by the homomorphism ω.

c) $\dim V_\omega < \infty$.

The space G/B is compact and the vector space V_ω (of holomorphic sections) becomes a Banach space when topologized by the uniform norm. Since a uniformly bounded sequence of holomorphic functions has a subsequence converging uniformly on compact subsets, V_ω is locally compact. Since a locally compact Banach space is finite-dimensional the statement follows.

d) The left action σ_ω of G on V_ω is irreducible.

By the semisimplicity of G, $V_\omega = \oplus_i V_i$, where G acts irreducibly on each V_i. Let $F \in V_i$ be a lowest weight vector. Then $F(\bar{n}g) \equiv F(g)$. Thus $F(\bar{n}hn) = F(hn) = \omega(h)F(e)$. Since $\bar{N}HN$ contains a neighborhood of e in G and since F is holomorphic, $\mathbf{C}F$ is the same for all i. This proves the irreducibility of σ_ω.

e) Let $\lambda \in \Lambda(+)$ and $\pi = \pi_\lambda$ the representation of G on V with highest weight λ. Then σ_ω in **d)** is equivalent to the contragredient of π operating on the dual space V':

$$\sigma_\omega \sim \check{\pi}\,,$$

and the highest weight is $-s\lambda$ where $s \in W$ maps $\mathfrak{t}^+$ into $-\mathfrak{t}^+$.

For this let $\mathbf{e}$ and $\mathbf{e}'$, respectively, denote highest weight vectors for π and $\check{\pi}$. Let $u \in U$ induce the Weyl group element s. Let ψ on G/N be defined by

$$\psi(gN) = \langle \pi(g^{-1})\mathbf{e}, \mathbf{e}' \rangle\,.$$

Then $\psi \not\equiv 0$ and the space V_ψ spanned by left translates of ψ is finite-dimensional. Since each $v \in V$ is a linear combination of translates $\pi(g_i^{-1})\mathbf{e}$ the mapping

$$v \longrightarrow \Psi_v\,, \qquad \Psi_v(gN) = \langle \pi(g^{-1})v, e' \rangle$$

maps V into V_ψ and satisfies

$$\Psi_{\pi(x)e} = \psi^{\tau(x)}\,,$$

setting up an equivalence between π on V and the natural representation of G on V_ψ.

Similarly, the contragredient representation $\check{\pi}$ induces the function

$$\check{\psi}(gN) = \langle \check{\pi}(g^{-1})\mathbf{e}', \mathbf{e} \rangle = \langle \mathbf{e}', \pi(g)\mathbf{e} \rangle = \psi(g^{-1}N)\,.$$

For $H \in \mathfrak{t}$,

$$\psi(\exp HuN) = \check{\psi}(u^{-1}\exp(-H)N) = \check{\psi}(\exp(-sH)u^{-1}N)\,,$$

whence

$$e^{-\lambda(H)}\psi(uN) = e^{\mu(sH)}\check{\psi}(u^{-1}N)\,,$$

where μ is the highest weight of $\check{\pi}$. Thus $\mu = -s\lambda$.

Extend λ to the homomorphism, $\omega : H \to \mathbf{C}^\times$. For $v' \in V'$ the function

$$F_{v'}(g) = \langle \pi(g)\mathbf{e}, v' \rangle$$

then satisfies $F_{v'}(gb) = \omega(b)F_{v'}(g)$ so $F_{v'} \in V_\omega$. Also

$$(\sigma_\omega(z)F_{v'})(g) = F_{v'}(z^{-1}g) = \langle \pi(g)\mathbf{e}, \check{\pi}(z)v' \rangle = F_{\check{\pi}(z)v'}(g)$$

so by **d)** σ_ω is equivalent to $\check{\pi}$. This establishes the following geometric model of $\check{\pi}_\lambda$.

Theorem. *The representation $\check{\pi}_\lambda$ is realized as the action of G on the space of holomorphic sections of the line bundle of G over G/B defined by the homomorphism $\omega : B \to \mathbf{C}^\times$ given by $\omega(\exp Hn) = e^{\lambda H}(H)$.*

References for Theorem: Borel-Weil in Serre, Séminaire Bourbaki, Exposé 100, 1954, Tits [1955], p. 113 and Harish-Chandra Representations of semisimple Lie groups V (Theorem 1), Amer. J. Math. 77 (1955), 743-777. Parts **c)** and **d)** simplify the customary proofs considerably.