Calculus and Prime Numbers

Shahriar Shahriari

Department of Mathematics
Pomona College
Claremont, CA 91711

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The sequence of primes

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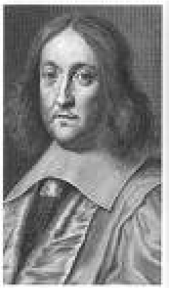
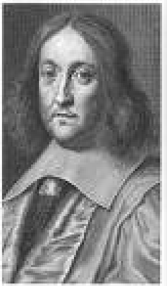
What can we say about them, and do techniques from calculus help at all?

Question

Are there any nice formulas for finding primes?



Fermat (1601–1661)



Fermat (1601–1661)

Definition (Fermat Numbers)

$$F_n = 2^{(2^n)} + 1 \quad \text{for} \quad n = 0, 1, \ldots$$

Example

The first few Fermat numbers $(2^{(2^n)} + 1)$ are:

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537,$$

$$F_5 = 4294967297, \quad F_6 = 18446744073709551617.$$

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Conjecture (Fermat)

Fermat: F_0, F_1, F_2, F_3, and F_4 are all prime numbers and so F_n should always be prime.

Unfortunately, for Fermat, *no* other Fermat numbers have been found to be primes.

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Example (Euler)

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All are open problems.

Latest results

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On November 20, 2010 Alexey V. Komkov found the following new factor of the Fermat number F_{299}:

$$272392805475 \times 2^{304} + 1.$$

Definition (Mersenne Numbers)

Father Marin Mersenne (1588–1648) suggested that the numbers

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Through December 2010, 47 Mersenne primes have been found. M_p is a prime for the following values of p:

2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609.

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See http://www.mersenne.org/ to join the hunt and to see the latest records.

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Theorem (Lucas-Lehmer Test)

p a prime greater than two. Construct the sequence:

$$4, 14, 194, 37634, 1416317954, 2005956546822746114, \ldots$$

where the first term is $r_1 = 4$ *and* $r_n = r_{n-1}^2 - 2$.

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where the first term is $r_1 = 4$ *and* $r_n = r_{n-1}^2 - 2$.
Now $M_p = 2^p - 1$ *is a prime if and only if* M_p *is a divisor of* r_{p-1}, *the* $(p-1)$*st term of the above sequence.*

$$
\begin{aligned}
&F(a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z) = \\
&\quad [k+2][1-(wz+h+j-q)^2-(2n+p+q+z-e)^2 \\
&\quad -(y^2(a^2-1)+1-x^2)^2-((e^4+2e^3)(a+1)^2+1-o^2)^2 \\
&\quad -(16(k+1)^3(k+2)(n+1)^2+1-f^2)^2 \\
&\quad -\left(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2\right)^2 \\
&\quad -(ai+k+1-l-i)^2-(16r^2y^4(a^2-1)+1-u^2)^2 \\
&\quad -((g(k+2)+k+1)(h+j)+h-z)^2 \\
&\quad -(p-m+l(a-n-1)+b(2a(n+1)-n(n+2)-2))^2 \\
&\quad -(z-pm+pl(a-p)+t(2ap-p^2-1))^2 \\
&\quad -(q-x+y(a-p-1)+s(2a(p+1)-p(p+2)-2))^2 \\
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\end{aligned}
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Theorem

p is a prime if and only if p is a positive value of the polynomial F.

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Prime Deserts

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So, there are prime deserts of arbitrary length.
Does it seem as if the primes are further and further apart as we go forward on the number line?

Twin Primes

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Conjecture (Schinzel's Conjecture)

There exists an infinite number of positive integers n such that each of the numbers $n+1$, $n+3$, $n+7$, $n+9$, and $n+13$ is a prime.

Definition

$\pi(x) =$ the number of primes between up to and including x.

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Example

$$\pi(4) = 2, \qquad \pi(10) = 4$$
$$\pi(47) = 15, \qquad \pi(100) = 25$$

Question

$\pi(x)$ *is a discontinuous function. Can we find a smooth function that approximates* $\pi(x)$*?*

x	$\pi(x)$	$\pi(x)/x$	$x/\pi(x)$
10	4	.4	2.5
100	25	.25	4
1000	168	.168	5.95238
10000	1229	.1229	8.1367
100000	9592	.09592	10.4254
10^6	78498	.078498	12.7392
10^7	664579	.0664579	15.0471
10^8	5761455	.0576146	17.3567
10^9	50847534	.0508475	19.6666

Table: The number of primes up to 10^9

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$$f'(10^n) = \frac{1}{10^n},$$

for all positive integers n.

Bigger tables can convince you that, for any $a \in \mathbb{R}$, you want

$$f(ax) = f(x) + g(a),$$

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Definition

For all $x > 0$, we define $\ln(x)$ by

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$

$$\frac{x}{\pi(x)} \approx f(x) = \ln x$$

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Theorem (The Prime Number Theorem—PNT)

Let $\pi(x)$ denote the number of primes up to x then

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\ln(x)} = 1.$$

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Whenever he had a few spare minutes, Gauss would find the next so many primes. He had a list of primes up to 3 million. His list has only about 72 mistakes!

Carl Friedrich Gauss (1777–1855)

Corollary

$$\lim_{x \to \infty} \frac{\pi(2x)}{\pi(x)}$$

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$$\lim_{x\to\infty}\frac{\pi(2x)}{\pi(x)} = \lim_{x\to\infty}\frac{\pi(2x)}{2x/\ln(2x)}\frac{x/\ln(x)}{\pi(x)}\frac{2x/\ln(2x)}{x/\ln(x)}$$

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We have now argued that, for large enough x, there is roughly the same number of primes between x and $2x$ as there are between 1 and x.

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Heuristic: The "probability" that a large number x is prime is

$$\approx \frac{1}{\ln(x)}$$

Li(x)

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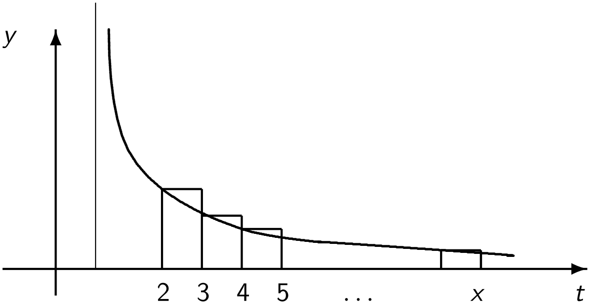

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Definition

$$\mathrm{Li}(x) = \text{Logarithmic integral} = \int_2^x \frac{1}{\ln(t)} dt$$

How good is Li(x)?

Gauss had found **6762** primes between $\mathbf{2,600,000}$ and $\mathbf{2,700,000}$.

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Maple gives the number of primes as 6765 and approximates the integral as 6761.3243

x	$\pi(x)$	$x/\ln x - \pi(x)$
10^2	25	-3
10^3	168	-23
10^4	1,229	-143
10^5	9,592	-906
10^6	78,498	$-6,116$
10^7	664,579	$-44,158$
10^8	5,761,455	$-332,774$
10^9	50,847,534	$-2,592,592$
10^{10}	455,052,511	$-20,758,029$

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10^5	9,592	-906	
10^6	78,498	$-6,116$	
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10^9	50,847,534	$-2,592,592$	
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x	$\pi(x)$	$x/\ln x - \pi(x)$	$\mathrm{Li}(x) - \pi(x)$
10^2	25	−3	5
10^3	168	−23	10
10^4	1,229	−143	
10^5	9,592	−906	
10^6	78,498	−6,116	
10^7	664,579	−44,158	
10^8	5,761,455	−332,774	
10^9	50,847,534	−2,592,592	
10^{10}	455,052,511	−20,758,029	

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10^2	25	-3	5
10^3	168	-23	10
10^4	1,229	-143	17
10^5	9,592	-906	38
10^6	78,498	$-6,116$	
10^7	664,579	$-44,158$	
10^8	5,761,455	$-332,774$	
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10^5	9,592	-906	38
10^6	78,498	$-6,116$	130
10^7	664,579	$-44,158$	339
10^8	5,761,455	$-332,774$	754
10^9	50,847,534	$-2,592,592$	
10^{10}	455,052,511	$-20,758,029$	

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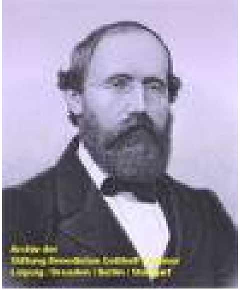
10^8	5,761,455	$-332,774$	754
10^9	50,847,534	$-2,592,592$	1,701
10^{10}	455,052,511	$-20,758,029$	3,104



Georg Bernhard Riemann (1826–1866)

Riemann in 1860

Gauss counted primes as 1 and composites as 0 and got

$$\mathrm{Li}(x) \approx \pi(x)$$

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Prob n prime	1	1	1/2	...	0	1
Prob n prime	$1/\ln(2)$	$1/\ln(3)$	$1/\ln(4)$	...	$1/\ln(46)$	$1/\ln(47)$

$$\Rightarrow \mathrm{Li}(x) \approx \pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \cdots$$

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Or equivalently

$$\pi(x) \approx \underbrace{\mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(\sqrt{x}) - \frac{1}{3}\mathrm{Li}(\sqrt[3]{x}) - \dots}_{R(x)}$$

Riemann found that

$$R(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n\zeta(n+1)} \frac{(\ln x)^n}{n!}$$

where

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

is the Riemann zeta function.

How good is $R(x)$?

x	$\pi(x)$
$100,000,000$	$5,761,455$
$200,000,000$	$11,078,937$
$300,000,000$	$16,252,323$
$400,000,000$	$21,336,326$
$500,000,000$	$26,355,867$
$600,000,000$	$31,324,703$
$700,000,000$	$36,252,931$
$800,000,000$	$41,146,179$
$900,000,000$	$46,009,215$
$1,000,000,000$	$50,847,534$

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How good is $R(x)$?

x	$\pi(x)$	$R(x)$
$100,000,000$	$5,761,455$	$5,761,552$
$200,000,000$	$11,078,937$	$11,079,090$
$300,000,000$	$16,252,323$	
$400,000,000$	$21,336,326$	
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$600,000,000$	$31,324,703$	
$700,000,000$	$36,252,931$	
$800,000,000$	$41,146,179$	
$900,000,000$	$46,009,215$	
$1,000,000,000$	$50,847,534$	

How good is $R(x)$?

x	$\pi(x)$	$R(x)$
100, 000, 000	5, 761, 455	5, 761, 552
200, 000, 000	11, 078, 937	11, 079, 090
300, 000, 000	16, 252, 323	16, 252, 355
400, 000, 000	21, 336, 326	21, 336, 185
500, 000, 000	26, 355, 867	
600, 000, 000	31, 324, 703	
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400, 000, 000	21, 336, 326	21, 336, 185
500, 000, 000	26, 355, 867	26, 355, 517
600, 000, 000	31, 324, 703	31, 324, 622
700, 000, 000	36, 252, 931	36, 252, 719
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600,000,000	31,324,703	31,324,622
700,000,000	36,252,931	36,252,719
800,000,000	41,146,179	41,146,248
900,000,000	46,009,215	46,009,949
1,000,000,000	50,847,534	

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200, 000, 000	11, 078, 937	11, 079, 090
300, 000, 000	16, 252, 323	16, 252, 355
400, 000, 000	21, 336, 326	21, 336, 185
500, 000, 000	26, 355, 867	26, 355, 517
600, 000, 000	31, 324, 703	31, 324, 622
700, 000, 000	36, 252, 931	36, 252, 719
800, 000, 000	41, 146, 179	41, 146, 248
900, 000, 000	46, 009, 215	46, 009, 949
1, 000, 000, 000	50, 847, 534	50, 847, 455

Riemann does even more!

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$$

is only defined for $z > 1$.

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Riemann finds a way to define $\zeta(z)$ for all complex numbers $z = r + it$ except $z = 1$.
He then proves the following *exact* formula:

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho})$$

where $\rho \in \{\text{zeroes of the } \zeta \text{ function}\}$

Conjecture (The Riemann Hypothesis)

The complex roots of the Riemann zeta function are of the form $\frac{1}{2} + it$.

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This conjecture—which has been the object of intense recent effort—is one of the most important outstanding mathematics problems.