Prime Numbers

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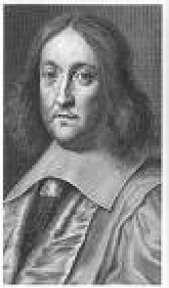
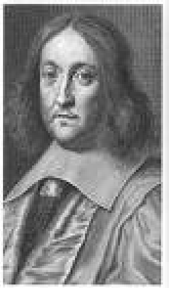
Question

Are there any nice formulas for finding primes?



Fermat (1601–1661)



Fermat (1601–1661)

Definition (Fermat Numbers)

$$F_n = 2^{(2^n)} + 1 \quad \text{for} \quad n = 0, 1, \ldots$$

Example

The first few Fermat numbers $(2^{(2^n)} + 1)$ are:

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537,$$

$$F_5 = 4294967297, \quad F_6 = 18446744073709551617.$$

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$$F_5 = 4294967297, \quad F_6 = 18446744073709551617.$$

Conjecture (Fermat)

Fermat: F_0, F_1, F_2, F_3, and F_4 are all prime numbers and so F_n should always be prime.

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- *Are there any other Fermat numbers that are prime?*
- *Is the number of Fermat primes finite?*

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Question

- *Are there any other Fermat numbers that are prime?*
- *Is the number of Fermat primes finite?*
- *Is F_{33} prime or composite?*

Definition (Mersenne Numbers)

Father Marin Mersenne (1588–1648) suggested that the numbers

$$M_p = 2^p - 1 \quad \text{where} \quad p \quad \text{is a prime,}$$

called *Mersenne numbers*, may generate many primes.

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called *Mersenne numbers*, may generate many primes.

Up to September 2006, 44 Mersenne primes were found. M_p is a prime for the following values of p:

2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657.

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See http://www.mersenne.org/ to join the hunt and to see the latest records for Mersenne primes.

Theorem (Lucas-Lehmer Test)

p a prime greater than two. Construct the sequence:

$$4, 14, 194, 37634, 1416317954, 2005956546822746114, \ldots$$

where the first term is $r_1 = 4$ and $r_n = r_{n-1}^2 - 2$.

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where the first term is $r_1 = 4$ and $r_n = r_{n-1}^2 - 2$.
Now $M_p = 2^p - 1$ is a prime if and only if M_p is a divisor of r_{p-1}, the $(p-1)$st term of the above sequence.

$$
\begin{aligned}
&F(a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z) = \\
&\quad [k+2][1-(wz+h+j-q)^2-(2n+p+q+z-e)^2 \\
&\quad -(y^2(a^2-1)+1-x^2)^2-((e^4+2e^3)(a+1)^2+1-o^2)^2 \\
&\quad -(16(k+1)^3(k+2)(n+1)^2+1-f^2)^2 \\
&\quad -\left(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2\right)^2 \\
&\quad -(ai+k+1-l-i)^2-(16r^2y^4(a^2-1)+1-u^2)^2 \\
&\quad -((g(k+2)+k+1)(h+j)+h-z)^2 \\
&\quad -(p-m+l(a-n-1)+b(2a(n+1)-n(n+2)-2))^2 \\
&\quad -(z-pm+pl(a-p)+t(2ap-p^2-1))^2 \\
&\quad -(q-x+y(a-p-1)+s(2a(p+1)-p(p+2)-2))^2 \\
&\quad -(l^2(a^2-1)+1-m^2)^2-(n+l+v-y)^2].
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\end{aligned}
$$

Theorem

p is a prime if and only if p is a positive value of the polynomial F.

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- $$\lim_{x \to \infty} \frac{\pi(2x)}{\pi(x)} = 2$$
- $$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{p_n} + \cdots$$

  diverges.

Theorem (The Prime Number Theorem—PNT)

Let $\pi(x)$ denote the number of primes up to x then

$$\lim_{x\to\infty}\frac{\pi(x)}{x/\ln(x)}=1.$$

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Whenever he had a few spare minutes, Gauss would find the next so many primes. He had a list of primes up to 3 million. His list has only about 72 mistakes!

Carl Friedrich Gauss (1777–1855)

Gauss improves PNT

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Heuristic: The "probability" that a large number x is prime is

$$\approx \frac{1}{\ln(x)}$$

Li(x)

n	2	3	4	...	46	47
Prob n prime	1	1	0	...	0	1

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So

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Definition

$$\mathrm{Li}(x) = \text{Logarithmic integral} = \int_2^x \frac{1}{\ln(t)} dt$$

How good is Li(x)?

Gauss had found **6762** primes
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Maple gives the number of primes as 6765 and approximates the integral as 6761.3243

x	$\pi(x)$	$x/\ln x - \pi(x)$
10^2	25	-3
10^3	168	-23
10^4	1,229	-143
10^5	9,592	-906
10^6	78,498	$-6,116$
10^7	664,579	$-44,158$
10^8	5,761,455	$-332,774$
10^9	50,847,534	$-2,592,592$
10^{10}	455,052,511	$-20,758,029$

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x	$\pi(x)$	$x/\ln x - \pi(x)$	$\mathrm{Li}(x) - \pi(x)$
10^2	25	-3	5
10^3	168	-23	10
10^4	$1,229$	-143	
10^5	$9,592$	-906	
10^6	$78,498$	$-6,116$	
10^7	$664,579$	$-44,158$	
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10^7	664, 579	$-44, 158$	
10^8	5, 761, 455	$-332, 774$	
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10^8	5,761,455	$-332,774$	754
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10^7	$664,579$	$-44,158$	339
10^8	$5,761,455$	$-332,774$	754
10^9	$50,847,534$	$-2,592,592$	$1,701$
10^{10}	$455,052,511$	$-20,758,029$	$3,104$

Conjecture (Twin prime conjecture)

There are an infinite number of twin primes, i.e., primes of the form n *and* $n+2$*.*

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11 and 13, 101 and 103, 10006427 and 10006429, or $107570463 \times 10^{2250} - 1$ and $107570463 \times 10^{2250} + 1$.

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Conjecture (Schinzel's Conjecture)

There exists an infinite number of positive integers n *such that each of the numbers* $n + 1$, $n + 3$, $n + 7$, $n + 9$, *and* $n + 13$ *is a prime.*

Pick two random integers between 1 and n. Chances of both being prime is

$$\frac{1}{(\ln n)^2}$$

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A better (and harder) argument gives a better approximation for the number of twin primes up to n:

$$1.32032 \int_2^n \frac{1}{(\ln x)^2} dx$$

How good an approximation?

Intervals of 150,000 starting at	Twin Primes	
	expected	found
10^8	584	
10^9	461	
10^{10}	374	
10^{11}	309	
10^{12}	259	
10^{13}	221	
10^{14}	191	
10^{15}	166	

How good an approximation?

Intervals of 150,000	Twin Primes	
starting at	expected	found
10^8	584	601
10^9	461	466
10^{10}	374	389
10^{11}	309	276
10^{12}	259	276
10^{13}	221	208
10^{14}	191	
10^{15}	166	

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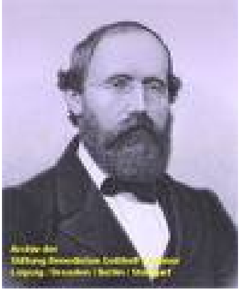

Georg Bernhard Riemann (1826–1866)

Riemann in 1860

Gauss counted primes as 1 and composites as 0 and got

$$\mathrm{Li}(x) \approx \pi(x)$$

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Riemann: squares of primes as 1/2, cubes of primes as 1/3, . . .

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$$\Rightarrow \mathrm{Li}(x) \approx \pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \cdots$$

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Or equivalently

$$\pi(x) \approx \underbrace{\mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(\sqrt{x}) - \frac{1}{3}\mathrm{Li}(\sqrt[3]{x}) - \ldots}_{R(x)}$$

Riemann found that

$$R(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n\zeta(n+1)} \frac{(\ln x)^n}{n!}$$

where

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

is the Riemann zeta function.

How good is $R(x)$?

x	$\pi(x)$
$100,000,000$	$5,761,455$
$200,000,000$	$11,078,937$
$300,000,000$	$16,252,323$
$400,000,000$	$21,336,326$
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PNT and Li(x)

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How good is $R(x)$?

x	$\pi(x)$	$R(x)$
$100,000,000$	$5,761,455$	$5,761,552$
$200,000,000$	$11,078,937$	$11,079,090$
$300,000,000$	$16,252,323$	$16,252,355$
$400,000,000$	$21,336,326$	$21,336,185$
$500,000,000$	$26,355,867$	
$600,000,000$	$31,324,703$	
$700,000,000$	$36,252,931$	
$800,000,000$	$41,146,179$	
$900,000,000$	$46,009,215$	
$1,000,000,000$	$50,847,534$	

PNT and $\mathrm{Li}(x)$

How good is $R(x)$?

x	$\pi(x)$	$R(x)$
100,000,000	5,761,455	5,761,552
200,000,000	11,078,937	11,079,090
300,000,000	16,252,323	16,252,355
400,000,000	21,336,326	21,336,185
500,000,000	26,355,867	26,355,517
600,000,000	31,324,703	31,324,622
700,000,000	36,252,931	36,252,719
800,000,000	41,146,179	
900,000,000	46,009,215	
1,000,000,000	50,847,534	

PNT and Li(x)

How good is $R(x)$?

x	$\pi(x)$	$R(x)$
$100,000,000$	$5,761,455$	$5,761,552$
$200,000,000$	$11,078,937$	$11,079,090$
$300,000,000$	$16,252,323$	$16,252,355$
$400,000,000$	$21,336,326$	$21,336,185$
$500,000,000$	$26,355,867$	$26,355,517$
$600,000,000$	$31,324,703$	$31,324,622$
$700,000,000$	$36,252,931$	$36,252,719$
$800,000,000$	$41,146,179$	$41,146,248$
$900,000,000$	$46,009,215$	$46,009,949$
$1,000,000,000$	$50,847,534$	

PNT and Li(x)

How good is $R(x)$?

x	$\pi(x)$	$R(x)$
100, 000, 000	5, 761, 455	5, 761, 552
200, 000, 000	11, 078, 937	11, 079, 090
300, 000, 000	16, 252, 323	16, 252, 355
400, 000, 000	21, 336, 326	21, 336, 185
500, 000, 000	26, 355, 867	26, 355, 517
600, 000, 000	31, 324, 703	31, 324, 622
700, 000, 000	36, 252, 931	36, 252, 719
800, 000, 000	41, 146, 179	41, 146, 248
900, 000, 000	46, 009, 215	46, 009, 949
1, 000, 000, 000	50, 847, 534	50, 847, 455

PNT and Li(x)

Riemann does even more!

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$$

is only defined for $z > 1$.

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is only defined for $z > 1$.

Riemann finds a way to define $\zeta(z)$ for all complex numbers $z = r + it$ except $z = 1$.
He then proves the following *exact* formula:

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho})$$

where $\rho \in \{\text{zeroes of the } \zeta \text{ function}\}$

Conjecture (The Riemann Hypothesis)

The complex roots of the Riemann zeta function are of the form $\frac{1}{2} + it$.

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- By 1986, some 1,500,000,000 of the zeros were found to satisfy the conjecture.
- In 1915, Godfrey Hardy had proved that an infinite number of the zeros do occur on the critical line.
- In 1989, Brian Conrey showed that over 40% of the zeros are on the line predicted by the hypothesis.