Daily Assignments and Lectures
Based on
Shahriar Shahriari, *Approximately Calculus*, AMS, 2006.

Audience. The class plan presented on the following pages has been tried—I would say successfully—in a second semester honors calculus class at Pomona College. The students are predominately first year students who have had an AB Calculus Advanced Placement Class in high school. The mathematical maturity of the students covers quite a range. A few of the students are considering a mathematics major but the primary intellectual interests of most of the students lies elsewhere. For many of the students, this is the first time, in a mathematics class, that they have been asked to think about the underlying concepts and to try to see *why* some methods work and to actively take part in formulating questions.

Class Organization. It is quite straightforward to just choose chapters from the book and cover them one by one as in a traditional class. However, here we present quite a different paradigm based on two principles:

(1) The students take an active part in learning the material by first reading the text and attempting the problems. The class discussions and lecture come *later* and after the students have already grappled with the concepts.

(2) In order to allow the students to slowly develop each topic and at the same time cover a substantial amount of material, we cover several "threads" simultaneously. Hence for each class period the students work on problems from several Chapters.

Usually, in each class period, I do not try to cover the material that is necessary for the next period's assignment. Rather I talk about the problems that were already handed in. Often, I use the opportunity to go much beyond the questions asked in the problem and hence turn the discussion into a mini-lecture. Sometimes, from a sequence of problems, I assign only some of them, but then, when I go over these problems, I incorporate a discussion of the others as well.

Finally, I use the "plan" below more as a guideline—in the spirit of Captain Barbossa in the *Pirates of the Caribbean*—than a strict rule. If students' questions and comments lead us elsewhere, I gladly follow.

The duration and number of classes. The particular syllabus here assumes a class meeting three times weekly for fifty minutes at a time. There are 41 class periods with 4 of them set aside for reviews and midterms.

The Assignments and Lecture Ideas. For each lesson, an assignment is given that is due the day of the lesson. The assumption is that the students work on the assignment *before* the lesson and the actual class time is spent discussing the topics that arise from the problems for that day. For each lesson, some of the problems and the discussion/lecture topics that they lead to are listed as well.

Mini-Projects. In addition to daily homework assignments and exams, I allow the students to hand in up to two optional mini-projects. The problems in many sections of the text are suitable to be assigned for independent study and can be the basis of an expository write up. A list of mini-projects is available on the book's web page hosted by the AMS.

Patterns; Derivatives

Lesson 1. Introduction to the philosophy, topics, and the mechanics of the class. Discuss the importance of experimenting and looking at examples. Do an example like Example 1.1 of the text. Emphasize the value of clearly communicating mathematics (to others and to yourself).

Lesson 2. Read the Preface, Section 1.1, and Section 4.1. Do Problems 1.1.1, 4.1.2, 4.1.5, 19.1.1, 19.1.8.

Problem 1.1.1 $\longrightarrow$ Finding Patterns. How do we know that patterns continue?

Problems 4.1.2 and 4.1.5 $\longrightarrow$ What is the definition of the derivative of a function at a point?

Induction; Fundamental Theorem of Calculus; Linearization

Lesson 3. Read Section 5.1. Do Problems 1.1.2, 1.1.3, 4.1.3, 5.1.4, 19.1.4.

Problems 1.1.2 and 1.1.3 $\longrightarrow$ Symbolic mathematical software and basic Maple syntax.

Problem 5.1.4 $\longrightarrow$ What does the integral mean? Why can we find areas by finding antiderivatives. Discuss the "proof" of the Fundamental Theorem of Calculus in Section 5.1.

Lesson 4. Read Section 1.2. Do Problems 1.1.4, 1.2.1, 4.1.8, 5.1.2, 5.1.5.

Problem 1.2.1 $\longrightarrow$ Proof by induction and writing of complete mathematical sentences.

Problem 5.1.5 $\longrightarrow$ Inscribed and circumscribed rectangles and further discussion of the fundamental theorem of calculus.

Lesson 5. Read Section 4.3. Do Problems 1.2.3, 1.2.4, 4.1.14, 4.3.6, 19.1.3.

Problem 1.2.3 $\longrightarrow$ Finding the partial sums of the geometric series with and without induction.

Problem 1.2.4 $\longrightarrow$ More practice with proof by induction.

Problem 4.1.14 $\longrightarrow$ Historically, the definition of the derivative using limits was not accepted easily. In fact, Descartes, and others, had alternate (limit-free) methods for finding derivatives of certain functions.

Problem 4.3.6 $\longrightarrow$ This is a key problem. If we have a complicated function, we are interested in the behavior near a point, and the function is differentiable, then Calculus allows us to replace the function with a line (the tangent line).

Division Algorithm; Differentiability; Taylor Polynomials; Approximating Integrals; Functions Defined by Integrals; Writing Math in Paragraphs

Lesson 6. Read Sections 2.1 and 5.2. Do Problems 1.2.6, 2.1.1, 4.3.7, 5.1.6, 5.2.1.

Problem 2.1.1 $\longrightarrow$ Finding Remainders. If the remainder of n when divided by m is r then the remainder of n^k when divided by m is the same as the remainder of r^k when divided by m. Prove why this is true for $k = 2$ and $k = 3$. Argue why it must be true in general.

Problem 4.3.7 $\longrightarrow$ Taylor Polynomial of degree two at a point or approximating a twice differentiable function locally with a parabola.

Problem 5.2.1 $\longrightarrow$ Inscribed and Circumscribed rectangles. Often approximations are of limited use if not accompanied by some measure of how good the approximation is. One way of doing this is to find numbers a and b such that $a < L < b$ where L is the quantity being approximated. The smaller $b - a$, the better the approximation.

Lesson 7. Read Sections 4.2 and 5.3. Do Problems 1.2.13, 2.1.2, 4.2.4, 4.3.8, 5.3.1.

Problem 1.2.13 $\longrightarrow$ Be careful with induction.

Problem 2.1.2 $\longrightarrow$ When finding the remainder of $n^2 - n$ when divided by 7, the only relevant information is the remainder of n when divided by 7.

Problem 4.2.4 $\longrightarrow$ When is a function continuous? What is the definition of continuity?

Problem 4.3.8 $\longrightarrow$ Taylor polynomials of degree n at the point $x = 0$. If we want to approximate a function at $x = 0$ with a polynomial of degree n then we want that polynomial and our function to agree at $x = 0$, have the same slope at $x = 0$, the same concavity at $x = 0$, ...

Problem 5.3.1 $\longrightarrow$ Can we define functions via integrals? How do you find the derivative of such a function? Is the procedure easier or harder than other functions? We will be defining many functions by integrals and we shall see that, because of the fundamental theorem as well as our ability to approximate integrals, these functions are easy to work with.

Lesson 8. Read Section 2.2 and 4.4. Do Problems 2.1.4, 2.2.1, 4.2.12, 4.4.1, 5.1.19.

Problem 2.1.4 $\longrightarrow$ Rather than use formal results about congruences we can draw conclusions by checking all cases and by making tables like:

If n has remainder	0,	1,	2,	3,	4,	5	6	when divided by 7,
then n^2 has remainder	0,	1,	4,	2,	2,	4,	1	when divided by 7.

In fact, change 7 to another small integer, make a table just like the above, add a row for remainders of n^3 and n^4, and discover your own theorem.

Problem 2.2.1 $\longrightarrow$ Proof by contradiction and irrationality proofs.

Problem 4.2.12 $\longrightarrow$ A function with a first but not a second derivative.

Problem 4.4.1 $\longrightarrow$ The formula for Taylor Polynomial of degree n at $x = 0$.

Problem 5.1.19 $\longrightarrow$ If a function is differentiable at the point $x = \alpha$ then the Lanczos derivative of the function will be the same as the usual derivative. Given the definition of the Lanczos derivative, this seems incredible. However, if you assume that f is equal to its Taylor series (or if you are willing to approximate f with its, say, second degree Taylor polynomial), then you can see how the derivative pops out of this definition. There are however functions that do not have a derivative at a point $x = \alpha$ for which the Lanczos derivative is defined.

Prime Numbers; Integers mod n; Differentiability implies Continuity

Lesson 9. Read Section 5.4. Do Problems 1.3.2, 2.1.6, 4.2.3. 4.3.2, 5.4.1.

Problem 1.3.2 $\longrightarrow$ When you have a recurrence relation, you can use induction to prove conjectures.

Problem 4.2.3 $\longrightarrow$ Explore the definition of continuity at a point.

Problem 5.4.1 $\longrightarrow$ Two functions that have the same derivative differ by a constant. Do the proof (the proof is outlined on page 139 of Section 8.3 and in Problem 8.3.16) pointing out the importance of the mean value theorem.

(As a prelude to Problem 3.1.1 that is assigned for the next lesson, the following short activity can be amusing: Ask each student to write down their favorite three digit number, then write it next to itself to get a six digit number. Then divide the six digit number by 7. If no remainder, then divide the quotient by 11. If no remainder, then divide the quotient by 13. What did you get? Everyone gets their original three digit number back. Why does this work? That is Problem 3.1.1.)

Lesson 10. Read Section 3.1. Do Problems 2.1.9, 2.1.11, 3.1.1, 4.2.6, 5.3.5.

Problem 3.1.1 $\longrightarrow$ Can we do a similar trick for four digit numbers? How do we factor 10,001? Can there be more than one factorization? Discuss the Fundamental Theorem of Arithmetic. If there was interest discuss the rules for integers being divisible by 3, 9, and 11 and why they work.

Problem 4.2.6 $\longrightarrow$ Following the outline in the text, prove that differentiability implies continuity.

Problem 5.3.5 $\longrightarrow$ We can define functions by integrals. Not only this is legitimate but straightforward to work with. The students may think that they know what the function $y = \sin(x)$ is. But do they know how to approximate $\sin 1$? Do they know how to find the derivative of $\sin x$ without just recalling the formula? If a function is defined as a definite integral, then answering these questions would be easy.

Lesson 11. Read Section 2.3. Do Problems 2.1.14, 2.3.1, 3.1.4, 4.4.5, 19.1.13.

Problem 2.1.14 $\longrightarrow$ At least one of k consecutive integers is divisible by k. Follows that the product of 4 consecutive integers is divisible by 24.

Problem 2.3.1 $\longrightarrow$ Modular arithmetic or arithmetic in the ring $\mathbb{Z}/n\mathbb{Z}$.

Problem 3.1.4 $\longrightarrow$ There does not seem to be any pattern to the differences between consecutive primes. Twin primes differ by only two and it is conjectured that there are an infinite number of these. On the other hand, do you think that the difference between two consecutive primes can ever be as large as one million?

Problem 4.4.5 $\longrightarrow$ Taylor polynomials at $x = a$.

Infinitude of Primes; Taylor Polynomial and Differential Equations; Integrals and Sums

Lesson 12. Read Section 6.1. Do Problems 2.3.5, 3.1.5, 4.4.12, 5.2.3, 6.1.1.

Problem 2.3.5 $\longrightarrow$ **Theorem.** Let x, y, and n be integers. Assume that x and n are relatively prime and the remainder of xy when divided by n is x. Then the remainder of y when divided by n is 1.

Problem 3.1.5 $\longrightarrow$ p_n and $\pi(x)$ and the strange way primes are distributed.

Problem 4.4.12 $\longrightarrow$ Our general principle at work: If a function is too complicated then approximate it locally using Taylor polynomials. Here we approximate the solution to a differential equation.

Problem 5.2.3 $\longrightarrow$ Exploring the close relation of integrals and sums. Not only we can use sums (of areas of rectangles) to approximate integrals, we can use integrals to approximate sums.

Problem 6.1.1 $\longrightarrow$ If we have a functional equation satisfied by a function, we may be able to find the derivative of the function. We can then express the function as an integral.

Lesson 13. Do Problems 3.1.8, 4.4.9, 5.2.4, 5.4.3, 6.1.2.

Problem 3.1.8 $\longrightarrow$ The existence of prime deserts (i.e., long stretches of composite numbers). You can build on this problem and do Euclid's proof of the Infinitude of primes (Problems 3.1.9 and 3.1.10).

Problem 4.4.9 $\longrightarrow$ Discussion of Taylor polynomial approximations. As the degree of the polynomial goes up, do we get a better and better approximation? If we find the Taylor polynomial at $x = \alpha$, how far away from α is the approximation good? Can we go further if we raise the degree of the polynomial? Or are there natural barriers? Does the answer depend on the function? Problem 4.4.10 assigned later continues this discussion. Hint at Taylor series, analytic functions, and the value of estimating the error.

Problem 5.2.4 $\longrightarrow$ Can use Integrals to approximate sums.

Problem 5.4.3 $\longrightarrow$ When derivatives are the same, the two functions differ by a constant.

Problem 6.1.2 $\longrightarrow$ When you have the derivative of a function and an initial value then you can write an expression for the function using a definite integral.

Lesson 14. Review. Problems in Section 13.1 provide a good review of the calculus material. The additional problems in Chapters 2 and 4 (Sections 2.4 and 4.5) also provide review material.

Lesson 15. MIDTERM I.

Fermat's Little Theorem; Distribution of Primes; The Log Function; Sequences.

Lesson 16. Read Section 14.1. Do Problems 3.1.11, 3.3.1, 4.4.10, 6.1.3, 14.1.8.

Problem 3.3.1 $\longrightarrow$ The first step toward conjecturing (and then proving) Fermat's little Theorem.

Problem 4.4.10 $\longrightarrow$ Sometimes (in general, not easy to predict when, although for rational functions a nice theorem in Complex Analysis provides the answer) no matter how high the degree of the Taylor polynomial, for x outside of some range, the Taylor polynomial approximation will not be any good. It is surprising that the Taylor polynomial for $\frac{1}{x-1}$ at $x = 0$ does not give a good approximation for the value of the function at $x = -1.5$ no matter how high the degree of the polynomial. Compare with Problem 4.4.9 when the same question for $\sin x$ was considered.

Problem 6.1.3 $\longrightarrow$ Explore the distribution of primes. We are looking for a smooth function that approximates $x/\pi(x)$. Instead of directly finding this function, we first specify a functional equation that our mystery function should satisfy. We then use this functional equation (and Problem 6.1.1) to find the derivative of the function. We get that the derivative should be $1/x$. Imposing an initial condition of $f(1) = 0$ (and mimicking Problem 6.1.2) we can write $f(x)$ as an integral. *Define* natural log of x as this integral.

Problem 14.1.8 $\longrightarrow$ Find the partial sum of a geometric series.

Lesson 17. Read Section 7.1. Do Problems 3.1.12, 3.3.2, 7.1.1, 7.1.2, 14.1.9.

Problem 3.1.12 $\longrightarrow$ A (useless) formula for generating all the primes (and many composite numbers). Note that $f(x) = x$ would have done the same thing.

Problem 3.3.2 $\longrightarrow$ Conjecture Fermat's Little Theorem.

Problems 7.1.1 and 7.1.2 $\longrightarrow$ Using the definition of $\ln x$, find its properties. Stress the importance of definitions in mathematics.

Problem 14.1.9 $\longrightarrow$ The sum of a geometric series. Discuss the meaning of the convergence of a series, and the connection to the sequence of partial sums. Beware that students often confuse sequences and series and that convergence is a subtle concept. I have found that going over the Toy and Coupon Problem of Example 14.4 (continued in Example 14.6) is often helpful. Make sure that it is clear when geometric series converge and to what (i.e., Problem 14.2.1).

Properties of $\ln x$; Mean Value Theorem; L'Hospital's Rule; Series.

Lesson 18. Read Sections 7.2 and 14.2. Do Problems 3.3.3, 7.1.3, 7.1.8, 7.2.1, 14.1.14.

Problem 3.3.3 $\longrightarrow$ Use Fermat's Little Theorem. This problem is not too big for Maple. It still can provide the answer.

Problem 7.1.3 $\longrightarrow$ Bringing together work done in several of previous problems, we now conjecture the Prime Number Theorem (PNT):

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$

Problem 7.1.8 $\longrightarrow$ An easy fact using circumscribed rectangles that is later used to show (Problem 14.1.10) that the Harmonic Series diverges.

Problem 7.2.1 $\longrightarrow$ Using calculus to prove algebraic identities. Guess a relationship between two functions, the prove it by showing that both sides have identical derivatives and satisfy the same initial conditions.

Problem 14.1.14 $\longrightarrow$ Theon's ladder gives a way of approximating $\sqrt{2}$. Use induction to prove the recurrence relation, and then use the relation to find the limit.

Lesson 19. Read Sections 3.3 and 8.3. Do Problems 3.3.8, 7.1.9, 7.2.2, 8.3.7, 14.2.2.

Problem 3.3.8 $\longrightarrow$ A slight variation to the Fermat's little theorem. This will be the version that we will prove later.

Problem 7.1.9 $\longrightarrow$ An increasing bounded sequence. A version of this will be used to show the existence of Euler's gamma constant (Problem 14.5.1). Discuss the convergence of monotone bounded sequences (Section 14.3).

Problem 7.2.2 $\longrightarrow$ More algebraic properties of $\ln x$ using calculus.

Problem 8.3.7 $\longrightarrow$ Discuss the statement of the mean value theorem and give an idea of the proof.

Problem 14.2.2 $\longrightarrow$ By calculating the partial sums, show that $1-1+1-1+\cdots$ diverges. If so inclined mention Cesàro summability.

Lesson 20. Read Sections 3.4 and 9.1. Do Problems 3.4.1, 7.2.4, 7.2.10, 9.1.1, 14.1.10.

Problem 3.4.1 $\longrightarrow$ First in a series of problems that explore the language and basic concepts of Dynamical Systems in anticipation of the proof of Fermat's little theorem in Problem 3.4.22. For now, discuss the notation $f : [0,1] \to [0,1]$ and forward orbits of points.

Problem 7.2.4 $\longrightarrow$ Discuss the meaning and definition of $\lim_{x\to\infty} f(x) = \infty$ (i.e., No matter how big a number M we choose, we can guarantee that $f(x)$ will eventually become bigger than M and stay bigger than M). Using this definition, and the relation between $\ln 4^M$ and M find $\lim_{x\to\infty} \ln(x)$. $\ln x$ goes to infinity but very slowly.

Problem 7.2.10 $\longrightarrow$ Show that $\int \frac{1}{x} dx = \ln|x| + c$.

Problem 9.1.1 $\longrightarrow$ L'Hospital's rule is very handy, is an example of linearization (replacing a function by a line), and does not always work.

Problem 14.1.10 $\longrightarrow$ Prove using Problem 7.1.8 (using rectangles we have found that $\ln n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$) that the harmonic series diverges. Convergence of a series is about whether the terms of the series are getting smaller fast enough.

Dynamical Systems; Logarithmic Differentiation; The Exponential Function; Monotone Bounded Sequences.

Lesson 21. Read Section 7.3. Do Problems 3.4.2, 7.2.12, 7.2.13, 7.3.1, 8.3.2.

Problem 3.4.2 ⟶ Continuing with Dynamical Systems, discuss the iterates g, g^2, g^3, ... of a function.

Problem 7.2.12 ⟶ Taylor polynomial of $\ln(x+1)$ at $x = 0$. Conjecture the range of x for which the Taylor series for $\ln(x+1)$ converges. We anticipate that for some functions the Taylor series will converge to the function for all values of x while for others this happens only for a specific range of values for x.

Problem 7.2.13 ⟶ Logarithmic Differentiation is useful when we have products and exponents. Historically, logarithms were developed to aid in calculations involving large numbers (for example, in astronomy). Logarithms were a useful tool because they turned multiplication into addition and exponentiation into multiplication.

Problem 7.3.1 ⟶ Discuss the meaning of e. It can be defined in a variety of ways but, in this book, we have defined it as the number whose natural log is one. This allows us to approximate e as closely as we want.

Problem 8.3.2 ⟶ More practice with the Mean Value Theorem.

Lesson 22. Read Section 14.3. Do Problems 3.4.3, 5.2.8, 7.3.2, 8.3.12, 14.3.6.

Problem 3.4.3 ⟶ Finding the composition of two functions.

Problem 5.2.8 ⟶ Using integrals to find sums.

Problem 7.3.2 ⟶ Definitions of $\ln(x)$, $\exp(x)$, and e^x. Go over the discussion in Section 7.3 about the way we have defined $\exp(x)$, and, the somewhat subtle fact, that we have to show that for rational values of x we have $\exp(x) = e^x$ before we can say $\exp(x) = e^x$. Also it should be clear that a^x for irrational values of x does need a definition.

Problem 8.3.12 ⟶ Discuss the Mean Value Theorem for Integrals and its role in (our) proof of the Fundamental Theorem of Calculus.

Problem 14.3.6 ⟶ What is an upper bound, and a least upper bound? A bounded increasing sequence has a *least upper bound* and the sequence converges to it.

The meaning of $2^{\sqrt{3}}$; Periodic points; Error in Taylor Polynomial Approximation

Lesson 23. Read Section 8.4. Do Problems 3.4.6, 7.2.15, 7.3.3, 8.4.1, 12.2.1.

Problem 3.4.6 ⟶ The relation between the fixed points of f^n and the periodic points of f.

Problems 7.2.15 and 7.3.3 ⟶ More practice with logs and exponential functions.

Problem 8.4.1 ⟶ Section 8.4 is an important section where we discuss the error term in Taylor polynomial approximations. Without a sense of how big the error can be, the Taylor polynomial approximation cannot be that useful. Finding an expression for the error term serves several purposes. We can estimate the error for any given approximation and we can see the effect of increasing the degree of the Taylor polynomial. In fact, using some expression for the error term is one of the best ways of deciding if the Taylor series of a function converges to the function.

Problem 12.2.1 ⟶ This is a simple limit problem with a message. Just because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, it doesn't mean that we can replace $f(x)$ by $g(x)$ when finding limits of expressions involving $f(x)$. This warning will be useful in doing Problem 12.2.3 correctly.

Lesson 24. Read Section 14.4. Do Problems 3.4.8, 7.2.16, 7.3.19, 8.4.2, 14.4.3.

Problem 3.4.8 ⟶ We want the students to be used to going back and forth between the number of fixed points of the iterates of f and the number of period-n points of f.

Problem 8.4.2 ⟶ Students find working with the error term difficult. Hence a bit more practice can be useful in understanding the concept.

Problem 14.4.3 ⟶ The nth term test discussed and proved.

Lesson 25. Do Problems 3.4.15, 7.3.11, 7.3.22, 8.4.3, 14.4.9.

Problem 3.4.15 ⟶ The function $f_a(x) = \mathrm{frac}(ax)$ appears in the proof of Fermat's little Theorem that will be done in Problem 3.4.22. Here we want to understand the function f_5^2. The students are asked to just try some inputs and graph the function. In the discussion you can actually do Problem 3.4.16 and prove a formula (i.e., $f_a^2 = f_{a^2}$) for f_5^2. (See the Hint for Problem 3.4.16.)

Problem 7.3.22 ⟶ What does $2^{\sqrt{3}}$ mean? Give a rigorous definition based on the function $1/x$. Many students assume that they know what the function $y = 2^x$ means and are surprised that you can give a precise definition using the function $y = 1/x$.

Problem 8.4.3 ⟶ More practice with the error of Taylor polynomial approximation.

Problem 14.4.9 ⟶ Practice using the Comparison test. Discuss this test and the intuition behind it. (A proof is outlined in problem 14.4.13 which can be assigned as a writing mini-project.) In this course we study various test for convergence of series not so much because we value the students' ability to apply the tests but rather a discussion of tests for convergence provides an opportunity to think about the concept of convergence. The fact that we can add an infinite number of positive numbers and sometimes get infinitely large and sometimes get a finite number takes some getting used to. It is particularly disturbing to the students that the converse of the n-th term test is not true. If the terms of a series go to zero then a series may or may not converge. It is all about whether the terms go to zero fast enough. In the process of thinking about tests of convergence, the students grapple with these ideas.

Improper Integrals

Lesson 26. Read Section 11.1. Do Problems 3.4.17, 8.4.5, 9.1.2, 11.1.1, 14.4.10.

Problem 3.4.17 ⟶ The function T_a is basically the same as $\mathrm{frac}(ax)$. It has been modified slightly to assure that it has exactly a fixed points. The students should be clear about the function T_a and the number of fixed points that it has.

Problem 8.4.5 ⟶ Taylor Error Formula for $\ln x$ at $x = 1$. By thinking through whether you are getting an overestimate or an underestimate you can find slightly better bounds.

Problem 9.1.2 ⟶ Discuss l'Hospital's rule as an example of linearization. If we have two functions and we want to find the limit of their ratios, then we may be able to replace the functions with a linear (or possibly higher order) approximations.

Problem 11.1.1 ⟶ The definition of an improper integral and the similarity (and the actual relationship) between convergence of series and improper integrals.

Problem 14.4.10 ⟶ More practice with the Comparison test.

Lesson 27. Review.

Lesson 28. MIDTERM II.

Proof of Fermat's Little Theorem; Proof of Taylor Remainder; Differential Equations

Lesson 29. Read Section 18.1. Do Problems 3.4.20, 7.3.26, 8.4.9, 11.1.3, 18.1.1.

Problem 3.4.20 $\longrightarrow$ Using the understanding that has been gained through this series of problems, we prove that $T_a(T_b(x)) = T_{ab}(x)$. We then use this and induction to prove that $(T_a)^p = T_{a^p}$. The latter is a key ingredient of the proof of Fermat's little Theorem.

Problem 7.3.26 $\longrightarrow$ Discuss the derivative and the anti-derivative of 2^x.

Problem 8.4.9 $\longrightarrow$ The error term for Taylor polynomial of degree 0 is exactly the Mean Value Theorem. Hence the error term for the Taylor polynomials is a generalization of the Mean Value Theorem. Hence very useful.

Problem 11.1.3 $\longrightarrow$ This improper integral will be used to determine the convergence (or divergence) of the p-series.

Problem 18.1.1 $\longrightarrow$ Differential Equations and the modeling process. Our plan for Chapter 18 is to start with a simple exponential population model, compare its results with data, and modify it to get a more reasonable logistic model. We also want to do a qualitative analysis of the long term behavior of solutions to certain kinds of differential equations. Certainly, if we are interested in the local behavior of a solution, we can use Taylor approximations. However, this may be just the wrong thing to do if we are interested in long term behaviors. We begin—in this problem—by setting up of a simple exponential growth model.

Lesson 30. Do Problems 3.4.21, 9.1.11, 14.2.8, 14.4.6, 18.1.3.

Problem 3.4.21 $\longrightarrow$ We use several of problems that we have done (Problems 3.4.20, 3.4.18 or 3.4.17, and 3.4.7) to find the number of period-p points of T_a. If you are comfortable with the notions (i.e., period-n points, fixed points, iteration) then this is really not that hard and could have all been done in one sitting. However, we assumed that the students were not familiar with any of these and wanted to develop the concepts slowly in order to have an opportunity to build intuition for what is happening.

Problem 9.1.11 $\longrightarrow$ This limit is used later in finding $\lim_{x\to\infty} \frac{\pi(2x)}{\pi(x)}$.

Problem 14.2.8 $\longrightarrow$ Sometimes we can see a pattern for partial sums, prove it using induction, and use this to find the sum of a series.

Problem 14.4.6 $\longrightarrow$ It is possible for the even partial sums and the odd partial sums converge to different values. Does the series converge then? This cautionary tale is helpful in understanding the proof of the convergence of the alternating harmonic series later in Problem 14.5.3.

Problem 18.1.3 $\longrightarrow$ A straightforward exponential growth model.

Lesson 31. Do Problems 3.4.22, 8.4.10, 11.1.5, 14.4.7, 18.1.5.

Problem 3.4.22 $\longrightarrow$ Proof of Fermat's Little Theorem using the language of dynamical systems. We follow the outline presented in the text, notice that each orbit consisting of period-p points has p elements, and just count the number of orbits.

Problem 8.4.10 $\longrightarrow$ We prove the error term for Taylor polynomial approximation. This problem asks for a proof for Taylor polynomials of degree 1 but the general proof for polynomials of degree n is very similar.

Problem 11.1.5 $\longrightarrow$ Straightforward problem that will be used in the proof of the integral test for the convergence of series.

Problem 14.4.7 $\longrightarrow$ Continuing Problem 14.4.6, we explore the relationship of odd and even partial sums and the nth term of a series.

Problem 18.1.5 $\longrightarrow$ Radioactive Decay can be modeled in the same way as population growth.

Methods of Integration; Euler's Constant; Alternating Harmonic Series; Integral Test; The Logistic Equation

Lesson 32. Read Sections 11.2 and 14.5. Do Problems 9.1.13, 11.2.3, 14.1.13, 14.5.1, 18.1.7.

Problem 9.1.13 ⟶ Practice l'Hospital's rule by finding $\lim_{n\to\infty}(1+\frac{3}{n})^n$.

Problem 11.2.3 ⟶ While, in this course, we have not spent much time on techniques of integration, some experience with methods of integration is healthy. Here we do partial fractions.

Problem 14.1.13 ⟶ We can use our results—from Section 8.4—on the error in the Taylor approximation prove convergence of certain series constructed from Taylor polynomials. You can hint at Taylor Series and questions regarding the convergence of Taylor series.

Problem 14.5.1 ⟶ A proof that Euler's γ constant exists. This is used in Problem 14.5.2 to prove the convergence of the alternating harmonic series.

Problem 18.1.7 ⟶ Is exponential growth a reasonable model for predicting the population of U.S. over a long period? Does the model overestimate or underestimate the data? To make it more reasonable should we add a positive or negative term to the original differential equation? We will later choose an appropriate term for modifying the equation.

Lesson 33. Read Section 14.6. Do Problems 3.3.13, 11.2.4, 14.5.3, 14.6.1, 18.1.9.

Problem 3.3.13 ⟶ In a final sequence of problems from Chapter 3, we try to give some idea of the use of prime numbers in public key cryptography and the role of number theory in all of this. This will be an opportunity to talk about application of mathematics in internet security as well as discuss issues regarding complexity of algorithms. All of this will come later. For now, this simple problem introduces the Euler ϕ function.

Problem 11.2.4 ⟶ Integration by parts.

Problem 14.5.3 ⟶ the Alternating Harmonic Series converges to $\ln 2$. For the students to be able to do the proof, it is useful to have done Problems 14.4.6 and/or 14.4.7 regarding even and odd partial sums. The Euler gamma constant makes an interesting appearance (and disappearance) in this proof.

Problem 14.6.1 ⟶ A warm up to understand and use the integral test for convergence of series.

Problem 18.1.9 ⟶ An attempt for justifying the interaction term in the logistic equation.

Lesson 34. Read Sections 12.1 and 12.2. Do Problems 12.1.3, 12.2.2, 13.1.6, 14.6.2, 18.1.10.

Problem 12.1.3 ⟶ The PNT gives an amazingly good approximation for the number of primes in the interval proposed by this problem. Will it always be this good? Think about prime deserts (Problem 3.1.8).

Problem 12.2.2 ⟶ Our aim is to find $\lim_{x\to\infty}\frac{\pi(2x)}{\pi(x)}$ in Problem 12.2.3. Because of the warning in Problem 12.2.1, we have to be careful on how to proceed. Here we do the preliminary needed calculations.

Problem 13.1.6 ⟶ If we know the derivative of f and we know the value of f at 0, how do we approximate the value of f at 2? This is an open-ended question and a number of answers can be given. In this method the close relationship of two methods (Euler's method of successive tangent line approximations and the method of writing the unknown as an integral and approximating the integral using inscribed and circumscribed rectangles) is explored. A useful problem to bring together several themes that have been studied throughout the semester. The assumption is that the students had already worked out some of the more concrete problems in this section as a preparation for the first midterm.

Problem 14.6.2 $\longrightarrow$ Discuss the integral test for the convergence of series. This provides an opportunity to tie in the convergence of series with the intuition that students have developed using inscribed and circumscribed rectangles to approximate integrals. The problem asks the students to prove one case of the integral test.

Problem 18.1.10 $\longrightarrow$ Set up the logistic equation as a possible improvement to our population model. Discuss the kinds of questions that we may be interested in (e.g., what is the long term behavior of the population), and the fact that, as our models become more sophisticated, our ability to find closed form solutions diminishes. This is why we are interested in analyzing differential equations qualitatively. This will be explored later.

Euler's Theorem; Primes between n and $2n$; Taylor Series; Qualitative Analysis of DEs

Lesson 35. Read Sections 15.1, 15.2, and 18.2. Do Problems 3.3.14, 12.2.3, 14.6.3, 15.2.1, 18.2.1.

Problem 3.3.14 $\longrightarrow$ Generalize Fermat's little Theorem and conjecture Euler's Theorem.

Problem 12.2.3 $\longrightarrow$ Assuming the Prime Number Theorem, find $\lim_{x\to\infty}\frac{\pi(2x)}{\pi(x)}$ and discuss its consequences. This doesn't prove Bertrand's postulate (there is always at least one prime between x and $2x$) but it shows that, for large x, approximately there is as many primes between x and $2x$ as there are between 1 and x. This is somewhat surprising—it challenges the intuition that the primes thin out as we go out—but follows directly from the PNT.

Problem 14.6.3 $\longrightarrow$ The convergence of p-series is discussed. The relevant results are proven using the integral test. This series is important for the use with the comparison test. You can mention the Riemann zeta function, its values for even integers, and open questions regarding its values for odd integers.

Problem 15.2.1 $\longrightarrow$ Discuss Taylor Series, Power Series in general, and the ratio test. Point out that for Taylor series there are two sets of questions. For which x does the Taylor series converge? For which x does the Taylor series converge to the original function? The former can be answered (except for possibly a couple of values of x) using the ratio test. The latter cannot be. We need the error term in Taylor polynomial approximations to answer the second question.

Problem 18.2.1 $\longrightarrow$ Problems 18.2.1 and 18.2.2 get the students to think through the project of analyzing a DE qualitatively. It is imperative that they understand Problem 18.2.1 before attempting 18.2.2. In particular, it is confusing that, even though we eventually want the graph of $p(t)$ as a function of t, we begin by drawing a graph of dp/dt as a function of p. Interpreting this graph is a good exercise.

Lesson 36. Read Section 12.4. Do Problems 3.3.15, 12.4.1, 14.6.4, 15.2.2, 18.2.2.

Problem 3.3.15 $\longrightarrow$ The students are asked to conjecture a formula for $\phi(pq)$ where p and q are primes. In the discussion you can provide a proof as well (see the hint for Problem 3.3.16).

Problem 12.4.1 $\longrightarrow$ Finding a limit using l'Hospitals rule. This limit together with the Prime Number Theorem will be used in Problem 12.4.2 to find an approximate value for the n-th prime.

Problem 14.6.4 $\longrightarrow$ A cautionary tale about using computers in order to do a comparison test.

Problem 15.2.2 $\longrightarrow$ Practice using the ratio test and building intuition about which series converge and which don't.

Problem 18.2.2 $\longrightarrow$ We continue the qualitative solution to the Logistic Equation. When dp/dt is negative, then p decreases. In the graph of dp/dt as a function of p, this means that, in such times, p moves to the left. This causes confusion for students who are used to the x-axis being time and hence always increasing.

Public Key Cryptography; Inverse Trigonometric Functions; Where is the nth prime?; Manipulating Power Series; Limit Comparison Test

Lesson 37. Read Sections 3.5 and 7.4. Do Problems 3.5.2, 7.4.1, 12.4.2, 15.2.6, 18.2.3.

Problem 3.5.2 $\longrightarrow$ Discuss the aims of Public Key Cryptography. In this problem we conjecture the mathematical theorem that allows it to work.

Problem 7.4.1 $\longrightarrow$ The definition of the inverse tangent function and some very basic properites.

Problem 12.4.2 $\longrightarrow$ Find an approximate value for the nth prime.

Problem 15.2.6 $\longrightarrow$ Routine finding of an interval of convergence for a power series.

Problem 18.2.3 $\longrightarrow$ In the case of the Logistic Equation, we can actually get a closed form using the method of partial fractions. In this problem we do so.

Lesson 38. Read Section 15.5. Do Problems 3.5.4, 7.4.2, 15.2.12, 15.5.1, 18.2.8.

Problem 3.5.4 $\longrightarrow$ Gain actual practice encoding a message.

Problem 7.4.2 $\longrightarrow$ Find the derivative of $\arctan x$.

Problem 15.2.12 $\longrightarrow$ Further practice for finding the interval of convergence.

Problem 15.5.1 $\longrightarrow$ Manipulation of power series. We often find new power series from old ones.

Problem 18.2.8 $\longrightarrow$ Further practice in finding qualitative solutions to DEs.

Lesson 39. Read Section 16.1. Do Problems 3.5.5, 7.4.6 15.2.16, 15.5.3, 16.1.1.

Problem 3.5.5 $\longrightarrow$ This is an opportunity to further discuss Public Key Cryptography. The RSA cryptosystem is based on the (recently proved) fact that we can efficiently decide whether an integer is a prime but we cannot factor large integers efficiently. You can discuss the $P \neq NP$ problem and issues regarding the complexity of algorithms.

Problem 7.4.6 $\longrightarrow$ Using derivatives to find algebraic identities.

Problem 15.2.16 $\longrightarrow$ The ratio test for convergence can be used to get some interesting results.

Problem 15.5.3 $\longrightarrow$ Manipulation of power series.

Problem 16.1.1 $\longrightarrow$ Discuss the limit comparison test for convergence of series.

Irrationality of e; The sum of reciprocals of Primes; Alternating Series Test

Lesson 40. Do Problems 7.4.7, 15.5.7, 16.1.4, 16.1.6, 18.2.10.

Problem 15.5.7 $\longrightarrow$ By manipulating power series we conjecture the Nilakantha-Gregory-Leibniz series for $\pi/4$. You can discuss the the theorem that allows for integrating and differentiating of power series. Does it give any guarantees at the end points of the interval of convergence? This theorem is not proved in the text since it depends on uniform convergence which is not discussed here. A proof of the series for $\pi/4$ is given in Problem 15.5.8.

Problem 16.1.4 $\longrightarrow$ Construct a convergent series whose terms are, in the limit, much bigger than $1/n^{3/2}$. Use the limit comparison test.

Problem 16.1.6 $\longrightarrow$ The harmonic series diverges. But what if we drop some of its terms? In this problem we drop any term whose denominator has a zero digit. Does the series now converge? What if we had dropped any term whose denominator had the digit one? What if we had dropped any term whose denominator was a composite number? Does your intuition about the distribution of primes lead you to believe that this series converges or diverges? (This is Problem 16.1.7.)

Problem 18.2.10 $\longrightarrow$ A final problem in the qualitative solutions to DEs.

Lesson 41. Read Section 16.2. Do Problems 7.4.8, 8.4.8, 16.1.7, 16.2.1, 16.3.1.

Problem 8.4.8 $\longrightarrow$ A proof that e is irrational using Taylor polynomials with remainder.

Problem 16.1.7 $\longrightarrow$ Using the prime number theorem and the limit comparison test, the convergence of the reciprocals of primes is investigated.

I usually end with a lecture on Prime Numbers and their Distribution using material from Sections 3.2 (Formulas for Primes), 3.6 (Open Conjectures about Primes), 12.3 (Logarithmic Integral), and particularly 12.5 (Primes and the Riemann Hypothesis). A Beamer presentation for this talk is available on the text's web page hosted by the AMS.