Heat Kernel and Analysis on Manifolds
Excerpt with Exercises

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2000 *Mathematics Subject Classification.* Primary 58J35, Secondary 31B05, 35K05, 35K10, 35P15, 47D07, 53C20

Key words and phrases. Heat kernel, heat semigroup, heat equation, Laplace operator, eigenvalues of the Laplace operator, Gaussian estimates, Riemannian manifolds, weighted manifolds, regularity theory

Abstract. The book contains a detailed introduction to Analysis of the Laplace operator and the heat kernel on Riemannian manifolds, as well as some Gaussian upper bounds of the heat kernel.

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CHAPTER 1

Laplace operator and the heat equation in $\mathbb{R}^n$

1.1. Denote by $S_r(x)$ the sphere of radius $r > 0$ centered at the point $x \in \mathbb{R}^n$, that is

$$S_r(x) = \{y \in \mathbb{R}^n : |x - y| = r\}.$$

Let σ be the $(n-1)$-volume on $S_r(x)$, and note that $\sigma(S_r(x)) = \omega_n r^{n-1}$ where ω_n is the area of the unit $(n-1)$-sphere in $\mathbb{R}^n$. Prove that, for any $f \in C^2(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$,

$$\frac{1}{\omega_n r^{n-1}} \left(\int_{\mathbb{S}_r(x)} f d\sigma \right) - f(x) = \Delta f(x) \frac{r^2}{2n} + \bar{o}\left(r^2\right) \quad \text{as } r \to 0. \tag{1.1}$$

1.2. Denote a round ball in $\mathbb{R}^n$ by

$$B_R(x) = \{y \in \mathbb{R}^n : |x - y| < R\}$$

and note that its volume is equal to $c_n R^n$ where c_n is the volume of the unit ball in $\mathbb{R}^n$. Prove that, for any $f \in C^2(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$,

$$\frac{1}{c_n R^n} \left(\int_{B_R(x)} f(y)\, dy \right) - f(x) = \Delta f(x) \frac{R^2}{2(n+2)} + \bar{o}\left(R^2\right) \quad \text{as } R \to 0. \tag{1.2}$$

1.3. Prove the following properties of the heat kernel.

(a) For all $t > 0$ and $\xi \in \mathbb{R}^n$,

$$\widehat{p_t}(\xi) = e^{-t|\xi|^2}. \tag{1.3}$$

(b) $\int_{\mathbb{R}^n} p_t(x)\, dx = 1$.
(c) For all $t, s > 0$, $p_t * p_s = p_{t+s}$.
(c) $\frac{\partial p_t}{\partial t} = \Delta p_t$.

1.4. Fix a function $f \in L^2(\mathbb{R}^n)$ and set $u_t = p_t * f$ for any $t > 0$. Prove the following properties of the function u_t.

(a) $\widehat{u_t}(\xi) = e^{-t|\xi|^2} \widehat{f}(\xi)$.
(b) $u_t(x)$ is smooth and satisfies the heat equation in $\mathbb{R}_+ \times \mathbb{R}^n$.
(c) $\|u_t\|_{L^2} \le \|f\|_{L^2}$ for all $t > 0$.
(d) $u(t, x) \to f(x)$ as $t \to 0$ in the norm of $L^2(\mathbb{R}^n)$.
(e) If $\widehat{f} \in L^1(\mathbb{R}^n)$ then $u(t, x) \to f(x)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^n$.

1.5. Prove the following properties of the heat kernel.

(a) For any $\varepsilon > 0$, $p_t(x) \to 0$ as $t \to 0$ uniformly in $\{x : |x| > \varepsilon\}$.
(b) $p_t(x) \to 0$ as $x \to \infty$ uniformly in $t \in (0, +\infty)$.
(c) For any $\varepsilon > 0$, $p_t(x)$ is continuous in $\{x : |x| > \varepsilon\}$ uniformly in $t \in (0, +\infty)$.

1.6. *(Elliptic maximum principle)* Let Ω be a bounded open set in $\mathbb{R}^n$, and consider the following differential operator in Ω

$$L = \Delta + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j},$$

where b_j are smooth bounded functions in Ω.

(a) Show that there exists a function $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lv > 0$ in Ω.

(b) Prove that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $Lu \geq 0$ in Ω then

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

1.7. Evaluate the bounded solution $u(t,x)$ of the Cauchy problem with the initial function $f(x) = \exp(-|x|^2)$.

CHAPTER 2

Function spaces in $\mathbb{R}^n$

2.1. Prove that $L^q_{loc}(\Omega) \hookrightarrow L^p_{loc}(\Omega)$ for all $1 \le p < q \le +\infty$.

2.2. Let $\{f_k\}$ be a sequence of functions from $L^p(\Omega)$ that converges to a function f in L^p norm, $1 \le p \le \infty$. Prove that if $f_k \ge 0$ a.e. then also $f \ge 0$ a.e..

2.3. Prove that if $f \in L^\infty(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ then $f * g \in L^\infty(\mathbb{R}^n)$ and

$$\|f * g\|_{L^\infty} \le \|f\|_{L^\infty} \|g\|_{L^1}.$$

2.4. Prove that if $f, g \in L^1(\mathbb{R}^n)$ then $f * g \in L^1(\mathbb{R}^n)$ and

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}.$$

2.5. Prove that if $f, g, h \in L^1(\mathbb{R}^n)$ then $f * g = g * f$ and

$$(f * g) * h = f * (g * h).$$

2.6. Prove that if $C^k(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ then, for any multiindex α with $|\alpha| \le k$,

$$\partial^\alpha (f * \varphi) = (\partial^\alpha f) * \varphi.$$

2.7. Prove that if $f \in C^k(\mathbb{R}^n)$ and φ is a mollifier in $\mathbb{R}^n$ then $f * \varphi_\varepsilon \to f$ as $\varepsilon \to 0$ in the topology of $C^k(\mathbb{R}^n)$.

2.8. Let $f \in L^1_{loc}(\Omega)$. Prove that $f \ge 0$ a.e. if and only if

$$\int_\Omega f \psi d\mu \ge 0,$$

for all non-negative function $\psi \in C_0^\infty(\Omega)$.

2.9. For a function f on $\mathbb{R}$, denote by f'_{dist} its distributional derivative, reserving f' for the classical derivative.

- (a) Prove that if $f \in C^1(\mathbb{R})$ then $f'_{dist} = f'$.
- (b) Prove that the same is true if f is continuous and piecewise continuously differentiable.
- (c) Evaluate f'_{dist} for $f(x) = |x|$.
- (d) Let $f = 1_{[0,+\infty)}$. Prove that $f'_{dist} = \delta$, where δ is the Dirac delta-function at 0.

2.10. Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that two distributions $u, v \in \mathcal{D}'(\Omega)$ are equal on an open subset $U \subset \Omega$ if $(u, \varphi) = (v, \varphi)$ for all $\varphi \in \mathcal{D}(U)$.

- (a) Let $\{\Omega_\alpha\}$ be a family of open subsets of Ω. Prove that if u and v are equal on each of the sets Ω_α then they are equal on their union $\cup_\alpha \Omega_\alpha$.
- (b) Prove that for any $u \in \mathcal{D}'(\Omega)$ there exists the maximal open set $U \subset \Omega$ such that $u = 0$ in U.

REMARK. The closed set $\Omega \setminus U$ is called the support of the distribution u and is denoted by $\operatorname{supp} u$.

2.11. For any function $u(x)$, defined pointwise in Ω, set

$$S(u) = \overline{\{x \in \Omega : u(x) \neq 0\}},$$

where the bar means the closure in Ω.

(a) Prove that if $u \in C(\Omega)$ then its support $\operatorname{supp} u$ in the distributional sense coincides with $S(u)$.

(b) If $u \in L^1_{loc}(\Omega)$ then its support $\operatorname{supp} u$ in the distributional sense can be identified by

$$\operatorname{supp} u = \bigcap_{v=u \text{ a.e.}} S(v),$$

where the intersection is taken over all functions v in Ω, defined pointwise, which are equal to u almost everywhere.

2.12. Prove the product rule: if $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$ then

$$\partial^\alpha (fu) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \, \partial^\beta u, \tag{2.1}$$

where

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \dots \binom{\alpha_n}{\beta_n}$$

is the product of the binomial coefficients, and $\beta \leq \alpha$ means that $\beta_i \leq \alpha_i$ for all $i = 1, \dots, n$.

2.13. Let $\{u_k\}$ be a sequence of distributions in Ω such that $u_k \xrightarrow{\mathcal{D}'(\Omega)} u$.

(a) Prove that $\partial^\alpha u_k \xrightarrow{\mathcal{D}'} \partial^\alpha u$ for any multiindex α.

(b) Prove that $f u_k \xrightarrow{\mathcal{D}'} fu$ for any $f \in C^\infty(\Omega)$.

2.14. Let X be a topological space. Prove that a sequence $\{x_k\} \subset X$ converges to $x \in X$ (in the topology of X) if and only if any subsequence of $\{x_k\}$ contains a sub-subsequence that converges to x.

2.15. Prove that the convergence "almost everywhere" is not topological, that is, it is not determined by any topology.

2.16. Prove that the convergence in the space $\mathcal{D}(\Omega)$ is topological.

2.17. Prove that if $u, v \in L^2(\mathbb{R}^n)$ and $\partial_i u, \partial_i v \in L^2(\mathbb{R}^n)$ for some index i, then

$$(\partial_i u, v)_{L^2} = -(v, \partial_i v)_{L^2}. \tag{2.2}$$

2.18. Let $1 < p < \infty$, $u \in L^p(\mathbb{R}^n)$, and φ be a mollifier in $\mathbb{R}^n$.

(a) Prove that $u * \varphi \in L^p$ and

$$\|u * \varphi\|_{L^p} \leq \|u\|_{L^p}.$$

(b) Prove that

$$u * \varphi_\varepsilon \xrightarrow{L^p} u \text{ as } \varepsilon \to 0.$$

2.19. Prove that if $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and $g \in L^1(\mathbb{R}^n)$ then $f * g$ exists, belongs to $L^p(\mathbb{R}^n)$, and

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

2.20. *(Lemma of Schur)* Let (M, μ) be a measure space with a σ-finite measure μ. Let $q(x, y)$ be a non-negative measurable function $M \times M$ such that, for a constant K,

$$\int_M q(x, y)\, d\mu(y) \le K \text{ for almost all } x \tag{2.3}$$

and

$$\int_M q(x, y)\, d\mu(x) \le K \text{ for almost all } y. \tag{2.4}$$

Prove that, for any $f \in L^r(M, \mu)$, $1 \le r \le \infty$, the function

$$Qf(x) := \int_M q(x, y) f(y)\, d\mu(y)$$

belongs to $L^r(M, \mu)$ and

$$\|Qf\|_{L^r} \le K \|f\|_{L^r}. \tag{2.5}$$

2.21. Under the condition of Exercise 2.20, assume in addition that, for some constant C,

$$q(x, y) \le C,$$

for almost all $x, y \in M$. Prove that, for any $f \in L^r(M, \mu)$, $1 \le r \le +\infty$, the function Qf belongs to $L^s(M, \mu)$ for any $s \in (r, +\infty]$ and

$$\|Qf\|_{L^s} \le C^{1/r - 1/s} K^{1/r' + 1/s} \|f\|_{L^r}, \tag{2.6}$$

where r' is the Hölder conjugate to r.

2.22. A function f on a set $S \subset \mathbb{R}^n$ is called Lipschitz if, for some constant L, called the Lipschitz constant, the following holds:

$$|f(x) - f(y)| \le L\,|x - y| \quad \text{for all } x, y \in S.$$

Let U be an open subset of $\mathbb{R}^n$ and let f be a Lipschitz function in U with the Lipschitz constant L. For any $\varepsilon > 0$, set

$$U_\varepsilon = \left\{ x \in U : \overline{B_\varepsilon(x)} \subset U \right\}.$$

Let φ be a mollifier in $\mathbb{R}^n$.

(a) Show that U_ε is an open set and

$$U = \bigcup_{k=1}^{\infty} U_{1/k}. \tag{2.7}$$

Extend f to $\mathbb{R}^n$ by setting $f = 0$ outside U. Prove that $f * \varphi_\varepsilon$ is Lipschitz in U_ε with the same Lipschitz constant L.

(b) Prove that, for any $\delta > 0$, $f * \varphi_\varepsilon \rightrightarrows f$ in U_δ as $\varepsilon \to 0$.

2.23. Prove that if f is a Lipschitz function in an open set $U \subset \mathbb{R}^n$ then all the distributional partial derivatives $\partial_j f$ belong to $L^\infty(U)$ and $|\nabla f| \le L$ a.e. where

$$|\nabla f| := \left(\sum_{j=1}^{n} (\partial_j f)^2 \right)^{1/2}$$

and L is the Lipschitz constant of f.

2.24. Prove that if f and g are two bounded Lipschitz functions in an open set $U \subset \mathbb{R}^n$ then fg is also Lipschitz. Prove the product rule for the distributional derivatives:

$$\partial_j (fg) = (\partial_j f)\, g + f\, (\partial_j g)\, .$$

2.25. Let $f(x)$ be a Lipschitz function on an interval $[a,b] \subset \mathbb{R}$. Prove that if f' is its distributional derivative then

$$\int_a^b f'(x)\, dx = f(b) - f(a)\, .$$

Prove that if g is another Lipschitz function on $[a,b]$ then

$$\int_a^b f' g dx = [fg]_a^b - \int_a^b f g' dx. \tag{2.8}$$

2.26. Let $f \in C^k(\Omega)$, where k is a non-negative integer.

(a) Prove that if

$$\|f\|_{C^k(\Omega)} < \infty$$

then, for any $u \in W^k(\Omega)$, also $fu \in W^k(\Omega)$ and

$$\|fu\|_{W^k(\Omega)} \le C \|f\|_{C^k(\Omega)} \|u\|_{W^k(\Omega)}, \tag{2.9}$$

where the constant C depends only on k, n.

(b) Prove that if $u \in W^k_{loc}(\Omega)$ then $fu \in W^k_{loc}(\Omega)$.

2.27. Assume that $f_k \to f$ in W^k and $\partial^\alpha f \to g$ in W^k, for some multiindex α such that $|\alpha| \le k$. Prove that $g = \partial^\alpha f$.

2.28. Prove that, for any open set $\Omega \subset \mathbb{R}^n$, the space $W^k(\Omega)$ is complete.

2.29. Denote by $W^k_c(\Omega)$ the subset of $W^k(\Omega)$, which consists of functions with compact support in Ω. Prove that $\mathcal{D}(\Omega)$ is dense in $W^k_c(\Omega)$.

2.30. Prove that $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^k(\mathbb{R}^n)$, for any non-negative integer k. Warning: for an arbitrary open set $\Omega \subset \mathbb{R}^n$, $\mathcal{D}(\Omega)$ may not be dense in $W^k(\Omega)$.

2.31. Denote by $W^1_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^1(\Omega)$. Prove that, for any $u \in W^1(\Omega)$ and $v \in W^1_0(\Omega)$,

$$(\partial_i u, v)_{L^2} = -(u, \partial_i v)_{L^2}\, . \tag{2.10}$$

2.32. Let $u \in L^2(\mathbb{R}^n)$ and $\partial^\alpha u \in L^2(\mathbb{R}^n)$ for some multiindex α.

(a) Prove that

$$\widehat{\partial^\alpha u} = (i\xi)^\alpha\, \widehat{u}(\xi)\, , \tag{2.11}$$

where $\widehat{u}$ is the Fourier transform of u and $\xi^\alpha \equiv \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$, $i^\alpha \equiv i^{|\alpha|}$.

(b) Prove the following identity

$$\|\partial^\alpha u\|^2_{L^2} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2\, |\xi^\alpha|^2\, d\xi. \tag{2.12}$$

2.33. Let $u \in L^2(\mathbb{R}^n)$. Prove that if the right hand side of (2.12) is finite then $\partial^\alpha u$ belongs to $L^2(\mathbb{R}^n)$ and, hence, the identity (2.12) holds.

2.34. Prove that the space $W^k(\mathbb{R}^n)$ (where k is a positive integer) can be characterized in terms of the Fourier transform as follows: a function $u \in L^2(\mathbb{R}^n)$ belongs to $W^k(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^k d\xi < \infty.$$

Moreover, the following relation holds:

$$\|u\|_{W^k}^2 \simeq \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^k d\xi, \tag{2.13}$$

where the sign $\simeq$ means that the ratio of the both sides is bounded from above and below by positive constants.

2.35. Let k be a positive integer. Prove that if $u \in W^{-k}(\mathbb{R}^n)$ and φ is a mollifier in $\mathbb{R}^n$ then

$$\|u * \varphi\|_{W^{-k}} \leq \|u\|_{W^{-k}}. \tag{2.14}$$

2.36. Prove that, for any positive integer k, the space W^{-k} with the norm $\|\cdot\|_{W^{-k}}$ is a Hilbert space.

2.37. Evaluate function $\varphi(t)$ from Lemma 2 for $f(x) = \exp\left(-|x|^2\right)$.

2.38. Show that Lemma 2 remains true for $f \in C_b^\infty(\mathbb{R}^n)$.

2.39. Give an alternative proof of Lemma 2 using the Fourier transform and Exercises 1.4, 2.32.

CHAPTER 3

Laplace operator on a Riemannian manifold

3.1. Prove that, on any C-manifold M, there exists a countable sequence $\{\Omega_k\}$ of relatively compact open sets such that $\Omega_k \Subset \Omega_{k+1}$ and the union of all Ω_k is M. Prove also that if M is connected then the sets Ω_k can also be taken connected.

REMARK. An increasing sequence $\{\Omega_k\}$ of open subsets of M whose union is M, is called an *exhaustion sequence*. If in addition $\Omega_k \Subset \Omega_{k+1}$ (that is, Ω_k is relatively compact and $\overline{\Omega}_k \subset \Omega_{k+1}$) then the sequence $\{\Omega_k\}$ is called a *compact exhaustion sequence*.

3.2. Prove that, on any C-manifold M, there is a countable locally finite family of relatively compact charts covering all M. (A family of sets is called locally finite if any compact set intersects at most finitely many sets from this family).

3.3. Prove the product rule for d and ∇:

$$d\,(uv) = udv + vdu$$

and

$$\nabla\,(uv) = u\nabla v + v\nabla u, \tag{3.1}$$

where u and v are smooth function on M.

3.4. Prove the chain rule for d and ∇:

$$df\,(u) = f'\,(u)\,du$$

and

$$\nabla f\,(u) = f'\,(u)\,\nabla u$$

where u and f are smooth functions on M and $\mathbb{R}$, respectively.

3.5. Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metric tensors on a smooth manifold M and let g and $\widetilde{g}$ be the matrices of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ respectively in some coordinate system. Prove that the ratio

$$\frac{\det \widetilde{g}}{\det g}$$

does not depend on the choice of the coordinates (although separately $\det g$ and $\det \widetilde{g}$ do depend on the coordinate system).

3.6. Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metric tensors on a smooth manifold M such that

$$\frac{\widetilde{\mathbf{g}}}{\mathbf{g}} \le C, \tag{3.2}$$

that is, for all $x \in M$ and $\xi \in T_xM$,

$$\widetilde{\mathbf{g}}\,(\xi,\xi) \le C\mathbf{g}\,(\xi,\xi)\,.$$

(a) Prove that if ν and $\widetilde{\nu}$ are the Riemannian volumes of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively. then
$$\frac{d\widetilde{\nu}}{d\nu} \leq C^{n/2},$$
where $n = \dim M$.

(b) Prove that, for any smooth function f on M,
$$|\nabla f|^2_{\mathbf{g}} \leq C\, |\nabla f|^2_{\widetilde{\mathbf{g}}}.$$

3.7. *(Product rule for divergence)* Prove that, for any smooth function u and any smooth vector field ω,
$$\operatorname{div}_\mu (u\omega) = \langle \nabla u, \omega \rangle + u \operatorname{div}_\mu \omega \tag{3.3}$$

3.8. *(Product rule for the Laplacian)* Prove that, for any two smooth functions u and v,
$$\Delta_\mu (uv) = u\Delta_\mu v + 2\langle \nabla u, \nabla v \rangle_{\mathbf{g}} + (\Delta_\mu u)\, v. \tag{3.4}$$

3.9. *(Chain rule for the Laplacian)* Prove that
$$\Delta_\mu f(u) = f''(u)\, |\nabla u|^2_{\mathbf{g}} + f'(u)\, \Delta_\mu u,$$
where u and f are smooth functions on M and $\mathbb{R}$, respectively.

3.10. The *Hermite polynomials* $h_k(x)$ are defined by
$$h_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2},$$
where $k = 0, 1, 2, \ldots$. Show that the Hermite polynomials are the eigenfunctions of the operator (**??**).

3.11. Let $a(x)$, $b(x)$ be smooth positive functions on a weighted manifold $(M, \mathbf{g}, \mu)$, and define new metric $\widetilde{\mathbf{g}}$ and measure $\widetilde{\mu}$ by
$$\widetilde{\mathbf{g}} = a\mathbf{g} \quad \text{and} \quad d\widetilde{\mu} = b d\mu.$$
Prove that the Laplace operator $\widetilde{\Delta}_{\widetilde{\mu}}$ of the weighted manifold $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$ is given by
$$\widetilde{\Delta}_{\widetilde{\mu}} = \frac{1}{b} \operatorname{div}_\mu \left(\frac{b}{a} \nabla \right).$$
In particular, if $a = b$ then
$$\widetilde{\Delta}_{\widetilde{\mu}} = \frac{1}{a} \Delta_\mu.$$

3.12. Consider the following operator L on a weighted manifold $(M, \mathbf{g}, \mu)$:
$$Lu = \frac{1}{b} \operatorname{div}_\mu (A \nabla u),$$
where $b = b(x)$ is a smooth positive function on M and $A = A(x)$ is a smooth field of positive definite symmetric operators on $T_x M$. Prove that L coincides with the Laplace operator $\widetilde{\Delta}_{\widetilde{\mu}}$ of the weighted manifold $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$ where
$$\widetilde{\mathbf{g}} = b\mathbf{g}A^{-1} \text{ and } d\widetilde{\mu} = b d\mu.$$

3.13. Consider the following operator L on a weighted manifold $(M, \mathbf{g}, \mu)$:
$$Lu = \Delta_\mu u + \langle \nabla v, \nabla u \rangle_{\mathbf{g}},$$
where v is a smooth function on M. Prove that $L = \Delta_{\widetilde{\mu}}$ for some measure $\widetilde{\mu}$, and determine this measure.

3.14. Let M be a smooth manifold of dimension n and N be its submanifold of dimension $n-1$ given by the equation $F(x)=0$ where F is a smooth function on M such that $dF\neq 0$ on N. Prove that, for any $x\in N$, the tangent space T_xN is determined as a subspace of T_xM by the equation

$$T_xN=\{\xi\in T_xM:\langle dF,\xi\rangle=0\}. \tag{3.5}$$

In the case when $M=\mathbb{R}^n$, show that the tangent space T_xN can be naturally identified with the hyperplane in $\mathbb{R}^n$ that goes through x and has the normal

$$\nabla F=\left(\frac{\partial F}{\partial x^1},...,\frac{\partial F}{\partial x^n}\right).$$

In other words, the tangent space T_xN is identified with the tangent hyperplane to the hypersurface N at the point x.

3.15. Prove that the Riemannian measure ν of the metric (3) is given by

$$d\nu=\psi^m(x)\,d\nu_X d\nu_Y, \tag{3.6}$$

and the Laplace operator Δ of this metric is given by

$$\Delta f=\Delta_X f+m\langle\nabla_X\log\psi,\nabla_X f\rangle_{\mathbf{g}_X}+\frac{1}{\psi^2(x)}\Delta_Y f, \tag{3.7}$$

where ∇_X is gradient on X.

3.16. Let q be the south pole of $\mathbb{S}^n$. For any point $x\in\mathbb{S}^n\setminus\{q\}$, its stereographic projection is the point y at the subspace

$$\mathbb{R}^n=\left\{z\in\mathbb{R}^{n+1}:z^{n+1}=0\right\},$$

which belongs to the straight line through x and q. Show that the stereographic projection is a bijection $x\leftrightarrow y$ between $\mathbb{S}^n\setminus\{q\}$ and $\mathbb{R}^n$ given by

$$y=\frac{x'}{x^{n+1}+1},$$

where $x=\left(x^1,...,x^{n+1}\right)$ and $x'=\left(x^1,...,x^n\right)$. Prove that, in the Cartesian coordinates $y^1,...,y^n$, the canonical spherical metric has the form

$$\mathbf{g}_{\mathbb{S}^n}=\frac{4}{\left(1+|y|^2\right)^2}\mathbf{g}_{\mathbb{R}^n},$$

where $|y|^2=\sum\left(y^i\right)^2$ and $\mathbf{g}_{\mathbb{R}^n}=\left(dy^1\right)^2+...+\left(dy^n\right)^2$ is the canonical Euclidean metric.

3.17. Prove that the canonical hyperbolic metric $\mathbf{g}_{\mathbb{H}^n}$ is positive definite using directly the definition of $\mathbf{g}_{\mathbb{H}^n}$ as the restriction of the Minkowski metric to the hyperboloid.

3.18. Show that the equation

$$y=\frac{x'}{x^{n+1}+1} \tag{3.8}$$

determines a bijection of the hyperboloid $\mathbb{H}^n$ onto the unit ball $\mathbb{B}^n=\{|y|<1\}$ in $\mathbb{R}^n$. Prove that, in the Cartesian coordinates $y^1,...,y^n$ in $\mathbb{B}^n$, the canonical hyperbolic metric has the form

$$\mathbf{g}_{\mathbb{H}^n}=\frac{4}{\left(1-|y|^2\right)^2}\mathbf{g}_{\mathbb{R}^n}, \tag{3.9}$$

where $|y|^2 = \sum (y^i)^2$ and $\mathbf{g}_{\mathbb{R}^n} = (dy^1)^2 + ... + (dy^n)^2$ is the canonical Euclidean metric.

REMARK. The ball $\mathbb{B}^n$ with the metric (3.9) is called the *Poincaré model* of the hyperbolic space. Representation of the metric $\mathbf{g}_{\mathbb{H}^n}$ in this form gives yet another proof of its positive definiteness.

3.19. Prove that the relation between the polar coordinates (r, θ) in $\mathbb{H}^n$ and the coordinates $y^1, ..., y^n$ in the Poincaré model of Exercise 3.18 are given by

$$\cosh r = \frac{1 + |y|^2}{1 - |y|^2} \text{ and } \theta = \frac{y}{|y|}.$$

3.20. Let ω_n be defined by (**??**).

(a) Use (3) to obtain a recursive formula for ω_n.

(b) Evaluate ω_n for $n = 3, 4$ given $\omega_2 = 2\pi$. Evaluate the volume functions of $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$ for $n = 2, 3, 4$.

3.21. Prove that, for any $n \geq 1$,

$$\omega_n = 2\frac{\pi^{n/2}}{\Gamma(n/2)}, \tag{3.10}$$

where Γ is the gamma function (cf. Section **??**).

3.22. Using (**??**), obtain a full expansion of $\Delta_{\mathbb{S}^n}$ in the polar coordinates for $n = 2, 3$. Hence, obtain a full expansion of $\Delta_{\mathbb{R}^n}$ and $\Delta_{\mathbb{H}^n}$ in the polar coordinates for $n = 2, 3$.

3.23. Consider in $\mathbb{H}^3$ a function u given in the polar coordinates by $u = \frac{r}{\sinh r}$.

(a) Prove that, in the domain of the polar coordinates, this function satisfies the equation

$$\Delta_{\mathbb{H}^3} u + u = 0. \tag{3.11}$$

(b) Prove that function u extends to a smooth function in the whole space $\mathbb{H}^3$ and, hence, satisfies (3.11) in $\mathbb{H}^3$.

HINT. Write function u in the coordinates of the Poincaré model (cf. Exercises 3.18 and 3.19).

3.24. Let M be a weighted model of radius r_0 and $u = u(r)$ be a smooth function on $M \setminus \{o\}$ depending only on the polar radius. Let $S(r)$ be its area function. Prove that u is harmonic, that is, $\Delta_\mu u = 0$, if and only if

$$u(r) = C \int_{r_1}^{r} \frac{dr}{S(r)} + C_1,$$

where C, C_1 arbitrary reals and $r_1 \in (0, r_0)$. Hence or otherwise, find all radial harmonic functions in $\mathbb{R}^n$, $\mathbb{S}^2$, $\mathbb{S}^3$, $\mathbb{H}^2$, $\mathbb{H}^3$.

3.25. Let M be a weighted model of radius r_0. Fix some $0 < a < b < r_0$ and consider the annulus

$$A = \{x \in M : a < |x| < b\}.$$

Prove the following Green formulas for any two function u, v of the class $C^2(A) \cap C^1(\overline{A})$:

$$\int_A (\Delta_\mu u)\, v \, d\mu = -\int_A \langle \nabla u, \nabla v \rangle d\mu + \int_{S_b} u_r v d\mu_{S_b} - \int_{S_a} u_r v d\mu_{S_a} \tag{3.12}$$

and

$$\int_A (\Delta_\mu u)\, v\, d\mu - \int_A (\Delta_\mu v)\, u\, d\mu = \int_{S_b} (u_r v - v_r u)\, d\mu_{S_b} - \int_{S_a} (u_r v - v_r u)\, d\mu_{S_a}, \tag{3.13}$$

where $u_r = \frac{\partial u}{\partial r}$.

3.26. Let S be a surface of revolution in $\mathbb{R}^{n+1}$ given by the equation

$$|x'| = \Phi\left(x^{n+1}\right),$$

where Φ is a smooth positive function defined on an open interval.

(a) Prove that S is a submanifold of $\mathbb{R}^{n+1}$ of dimension n.

(b) Prove that the induced metric $\mathbf{g}_S$ of S is given in the coordinates $t = x^{n+1}$ and $\theta = \frac{x'}{|x'|} \in \mathbb{S}^{n-1}$ by

$$\mathbf{g}_S = \left(1 + \Phi'(t)^2\right) dt^2 + \Phi^2(t)\, \mathbf{g}_{\mathbb{S}^{n-1}}.$$

(c) Show that the change of the coordinate

$$\rho = \int \sqrt{1 + \Phi'(t)^2} dt$$

brings the metric $\mathbf{g}_S$ to the model form

$$\mathbf{g}_S = d\rho^2 + \Psi^2(\rho)\, \mathbf{g}_{\mathbb{S}^{n-1}}, \tag{3.14}$$

where Ψ is a smooth positive function.

3.27. Represent in the model form (3.14) the induced metric of the cylinder

$$Cyl = \left\{x \in \mathbb{R}^{n+1} : |x'| = 1\right\}$$

and that of the cone

$$Cone = \left\{x \in \mathbb{R}^{n+1} : x^{n+1} = |x'| > 0\right\}.$$

3.28. The pseudo-sphere PS is defined as follows

$$PS = \left\{ x \in \mathbb{R}^{n+1} : 0 < |x'| < 1, \quad x^{n+1} = -\sqrt{1 - |x'|^2} + \log \frac{1 + \sqrt{1 - |x'|^2}}{|x'|} \right\}.$$

Show that the model form (3.14) of the induced metric of PS is

$$\mathbf{g}_{PS} = d\rho^2 + e^{-2\rho} \mathbf{g}_{\mathbb{S}^{n-1}}.$$

Hint. Use a variable s defined by $|x'| = \frac{1}{\cosh s}$.

3.29. For any two-dimensional Riemannian manifold $(M, \mathbf{g})$, the Gauss curvature $K_{M,\mathbf{g}}(x)$ is defined in a certain way as a function on M. It is known that if the metric $\mathbf{g}$ has in coordinates x^1, x^2 the form

$$\mathbf{g} = \frac{\left(dx^1\right)^2 + \left(dx^2\right)^2}{f^2(x)}, \tag{3.15}$$

where f is a smooth positive function, then the Gauss curvature can be computed in this chart as follows

$$K_{M,\mathbf{g}} = f^2 \Delta \log f, \tag{3.16}$$

where $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2}$ is the Laplace operator of the metric $\left(dx^1\right)^2 + \left(dx^2\right)^2$.

(a) Using (3.16), evaluate the Gauss curvature of $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$.

(b) Consider in the half-plane $\mathbb{R}^2_+ := \left\{\left(x^1, x^2\right) \in \mathbb{R}^2 : x_2 > 0\right\}$ the metric

$$\mathbf{g} = \frac{\left(dx^1\right)^2 + \left(dx^2\right)^2}{\left(x^2\right)^2}.$$

Evaluate the Gauss curvature of this metric.

3.30. Let $\mathbf{g}$ be the metric (3.15) on a two-dimensional manifold M. Consider the metric $\widetilde{\mathbf{g}} = \frac{1}{h^2}\mathbf{g}$ where h is a smooth positive function on M. Prove that

$$K_{M,\widetilde{\mathbf{g}}} = \left(K_{M,\mathbf{g}} + \Delta_{\mathbf{g}} \log h\right) h^2,$$

where $\Delta_{\mathbf{g}}$ is the Laplace operator of the metric $\mathbf{g}$.

3.31. Let the metric $\mathbf{g}$ on a two-dimensional manifold M have in coordinates (r, θ) the form

$$\mathbf{g} = dr^2 + \psi^2(r)\, d\theta^2. \tag{3.17}$$

Prove that

$$K_{M,\mathbf{g}} = -\frac{\psi''(r)}{\psi(r)}. \tag{3.18}$$

3.32. Using (3.18), evaluate the Gauss curvature of the two-dimensional manifolds $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$, Cyl, $Cone$, PS.

3.33. Find all metrics $\mathbf{g}$ of the form (3.17) with constant Gauss curvature.

3.34. Prove that the length $\ell(\gamma)$ does not depend on the parametrization of the path γ as long as the change of the parameter is monotone.

3.35. Prove that the geodesic distance $d(x, y)$ is finite if and only if the points x, y belong to the same connected component of M.

3.36. Let $(M, \mathbf{g})$ be a Riemannian model, and let x', x'' be two points on M with the polar coordinates (r', θ') and (r'', θ''), respectively.

(a) Prove that, for any smooth path γ on M connecting the points x' and x'',

$$\ell(\gamma) \geq \left|r' - r''\right|.$$

Consequently, $d(x', x'') \geq |r' - r''|$.

(b) Show that if $\theta' = \theta''$ then there exists a path γ of length $|r' - r''|$ connecting the points x' and x''. Consequently, $d(x', x'') = |r' - r''|$.

3.37. Let $(M, \mathbf{g})$ be a Riemannian model. Prove that, for any point $x = (r, \theta)$, we have $d(0, x) = r$.

Hence or otherwise prove that in $\mathbb{R}^n$ the geodesic distance $d(x, y)$ coincides with $|x - y|$.

3.38. Let γ be a shortest geodesics between points x, y and let z be a point on the image of γ. Prove that the part of γ connecting x and z is a shortest geodesics between x and z.

3.39. Fix a point p on a Riemannian manifold M and consider the function $f(x) = d(x, p)$. Prove that if $f(x)$ is finite and smooth in a neighborhood of a point x then $|\nabla f(x)| \leq 1$.

3.40. Let $(M, \mathbf{g})$ be a Riemannian model with infinite radius. Prove that, for any smooth even function a on $\mathbb{R}$, the function $a \circ r$ is smooth on M, where r is the polar radius on $(M, \mathbf{g})$.

3.41. Denote by $\mathcal{S}$ the class of all smooth, positive, even functions a on $\mathbb{R}$, such that

$$\int_0^\infty a(t)\,dt = \infty. \tag{3.19}$$

For any function $a \in \mathcal{S}$, let C_a be the conformal transformation of the metric of a Riemannian model $(M, \mathbf{g})$ with infinite radius given by

$$C_a \mathbf{g} = a^2(r)\,\mathbf{g}.$$

(*a*) Prove that $(M, C_a\mathbf{g})$ is also a Riemannian model with infinite radius and that the polar radius $\widetilde{r}$ on $(M, C_a\mathbf{g})$ is related to the polar radius r on $(M, \mathbf{g})$ by the identity

$$\widetilde{r} = \int_0^r a(s)\,ds.$$

(*b*) For any two functions $a, b \in \mathcal{S}$, consider the operation $a \star b$ defined by

$$(a \star b)(t) = a\left(\int_0^t b(s)\,ds\right) b(t). \tag{3.20}$$

Prove that $(\mathcal{S}, \star)$ is a group.

(*c*) Fix $m \in \mathbb{N}$ and set for any $v \in \mathbb{R}^m$

$$\log^{[v]} r = (\log r)^{v_1} (\log\log r)^{v_2} \ldots (\underbrace{\log\ldots\log}_{m \text{ times}} r)^{v_m}$$

assuming that r is a large enough positive number. Let a and b be functions from $\mathcal{S}$ such that, for large enough r,

$$a(r) \simeq r^{\alpha-1}\log^{[u]} r \quad \text{and} \quad b(r) \simeq r^{\beta-1}\log^{[v]} r,$$

for some $\alpha, \beta \in \mathbb{R}_+$ and $u, v \in \mathbb{R}^m$. Prove that

$$a \star b \simeq r^{\gamma-1}\log^{[w]} r,$$

where

$$\gamma = \alpha\beta \text{ and } w = u + \alpha v. \tag{3.21}$$

Remark. The identity (3.21) leads to the operation

$$(u, \alpha) \star (v, \beta) = (u + \alpha v, \alpha\beta),$$

that coincides with the group operation in the semi-direct product $\mathbb{R}^m \rtimes \mathbb{R}_+$, where the multiplicative group $\mathbb{R}_+$ acts on the additive group $\mathbb{R}^m$ by the scalar multiplication.

3.42. Let $J : M \to M$ be a Riemannian isometry and let S be a submanifold of M such that $J(S) = S$. Prove that $J|_S$ is a Riemannian isometry of S with respect to the induced metric of S.

3.43. Let $(M, \mathbf{g}_M)$ and $(N, \mathbf{g}_N)$ be Riemannian manifolds and $J : M \to N$ be a Riemannian isometry. Prove the following identities:

(*a*) For any smooth path γ on M,

$$\ell_{\mathbf{g}_M}(\gamma) = \ell_{\mathbf{g}_N}(J \circ \gamma).$$

(b) For any two points $x, y \in M$,

$$d_M(x, y) = d_N(Jx, Jy),$$

where d_M, d_N are the geodesic distances on M and N, respectively.

3.44. Let $(M, \mathbf{g}_M, \mu_M)$ and $(N, \mathbf{g}_N, \mu_N)$ be two weighted manifolds and $J : M \to N$ be a quasi-isometry. Prove the following relations.

(a) For all smooth paths γ on M,

$$\ell_{\mathbf{g}_M}(\gamma) \simeq \ell_{\mathbf{g}_N}(J \circ \gamma).$$

(b) For all couples of points $x, y \in M$,

$$d_M(x, y) \simeq d_N(Jx, Jy).$$

(c) For all non-negative measurable functions f on N,

$$\int_M (J_* f)\, d\mu_M \simeq \int_N f\, d\mu_N. \tag{3.22}$$

(d) For all smooth functions f on N,

$$\int_M |\nabla (J_* f)|^2_{\mathbf{g}_M}\, d\mu_M \simeq \int_N |\nabla f|^2_{\mathbf{g}_N}\, d\mu_N. \tag{3.23}$$

3.45. For any real α, consider the mapping $y = Jx$ of $\mathbb{R}^{n+1}$ onto itself given by

$$\begin{array}{l} y^1 = x^1 \\ \dots \\ y^{n-1} = x^{n-1} \\ y^n = x^n \cosh\alpha + x^{n+1}\sinh\alpha \\ y^{n+1} = x^n \sinh\alpha + x^{n+1}\cosh\alpha, \end{array} \tag{3.24}$$

which is called a hyperbolic rotation.

(a) Prove that J is an isometry of $\mathbb{R}^{n+1}$ with respect to the Minkowski metric

$$\mathbf{g}_{Mink} = \left(dx^1\right)^2 + \dots + (dx^n)^2 - \left(dx^{n+1}\right)^2.$$

(b) Prove that $J|_{\mathbb{H}^n}$ is a Riemannian isometry of the hyperbolic space $\mathbb{H}^n$ (cf. Section 3).

3.46. Prove that, for any four points $p, q, p', q' \in \mathbb{H}^n$ such that

$$d\left(p', q'\right) = d(p, q), \tag{3.25}$$

there exists a Riemannian isometry J of $\mathbb{H}^n$ such that $Jp' = p$ and $Jq' = q$.

CHAPTER 4

Laplace operator and heat equation in $L^2(M)$

4.1. Prove that if $\varphi_k \overset{\mathcal{D}}{\to} \varphi$ then

(a) $\varphi_k \rightrightarrows \varphi$ on M;
(b) $\Delta_\mu \varphi_k \overset{\mathcal{D}}{\longrightarrow} \Delta_\mu \varphi$;
(c) $f\varphi_k \overset{\mathcal{D}}{\longrightarrow} f\varphi_k$ for any $f \in C^\infty(M)$.

4.2. For any function $f \in C^\infty(M)$ and a distribution $u \in \mathcal{D}'(M)$, their product fu is defined as a distribution by

$$(fu, \varphi) = (u, f\varphi) \text{ for any } \varphi \in \mathcal{D}(M). \tag{4.1}$$

Prove the following assertions.

(a) If $u_k \overset{\mathcal{D}'}{\longrightarrow} u$ then $fu_k \overset{\mathcal{D}'}{\longrightarrow} fu$.
(b) $\operatorname{supp}(fu) \subset \operatorname{supp} f \cap \operatorname{supp} u$.
(c) Product rule:

$$\nabla(fu) = f\nabla u + (\nabla f)\, u,$$

where the product $f\nabla u$ of a smooth function by a distributional vector field and the product $(\nabla f)\, u$ of a smooth vector field by a distribution are defined similarly to (4.1).

4.3. Prove that if $f \in C^\infty(M)$ is such that $|f|$ and $|\nabla f|$ are bounded, and $u \in W^1(M)$ then $fu \in W^1(M)$ and

$$\|fu\|_{W^1} \le C\|u\|_{W^1},$$

where $C = 2\max(\sup|f|, \sup|\nabla f|)$.

4.4. Prove the extension of Theorem 2 to manifold: for any $1 \le p < \infty$ and for any weighted manifold $(M, \mathbf{g}, \mu)$, $\mathcal{D}(M)$ is dense in $L^p(M)$, and the space $L^p(M)$ is separable.

4.5. Prove that $\mathcal{D}(M)$ is dense in $C_0(M)$, where $C_0(M)$ is the space of continuous functions with compact support, endowed with the sup-norm.

4.6. Let $u \in \mathcal{D}'(M)$ and $(u, \varphi) = 0$ for all non-negative functions $\varphi \in \mathcal{D}(M)$. Prove that $u = 0$.

4.7. Let $u \in L^1_{loc}(M)$.

(a) Prove that if $(u, \varphi) \ge 0$ for all non-negative functions $\varphi \in \mathcal{D}(M)$, then $u \ge 0$ a.e.
(b) Prove that if $(u, \varphi) = 0$ for all non-negative functions $\varphi \in \mathcal{D}(M)$, then $u = 0$ a.e.

4.8. Let $\{u_k\}$ be a sequence from $L^2(M)$ such that $u_k \overset{\mathcal{D}'}{\longrightarrow} u$, where $u \in \mathcal{D}'(M)$.

(a) Prove that if the sequence of norms $\|u_k\|_{L^2}$ is bounded then $u \in L^2(M)$ and

$$\|u\|_{L^2} \le \liminf_{k\to\infty} \|u_k\|_{L^2}.$$

(b) Assume in addition that $\nabla u_k \in \vec{L}^2$ and that the sequence of norms $\|\nabla u_k\|_{L^2}$ is bounded. Prove that $u \in W^1(M)$ and

$$\|\nabla u\|_{L^2} \le \liminf_{k\to\infty} \|\nabla u_k\|_{L^2}.$$

4.9. Prove that the space $\vec{L}^p(M,\mu)$ is complete.

4.10. Define the divergence of a distributional vector field $v \in \vec{\mathcal{D}}'$ by

$$(\operatorname{div}_\mu v, \varphi) = -(v, \nabla\varphi) \quad \text{for all } \varphi \in \mathcal{D}.$$

Prove that, for any distribution $u \in \mathcal{D}'$,

$$\Delta_\mu u = \operatorname{div}_\mu(\nabla u),$$

where all operators Δ_μ, ∇, and div_μ are understood in the distributional sense.

4.11. Let $(M,\mathbf{g},\mu)$ be a weighted manifold and U be a chart on M with coordinates $x^1,...,x^n$. Let $f \in L^2_{loc}(U)$.

(a) Assume that all distributional partial derivatives $\frac{\partial f}{\partial x^j}$ are in $L^2_{loc}(U)$, considering U as a part of $\mathbb{R}^n$. Prove that the distributional gradient $\nabla_{\mathbf{g}} f$ in U is given by

$$(\nabla_{\mathbf{g}} f)^i = g^{ij}\frac{\partial f}{\partial x^j},$$

and

$$|\nabla_{\mathbf{g}} f|^2_{\mathbf{g}} = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}. \tag{4.2}$$

Conclude that $\nabla_{\mathbf{g}} f \in \vec{L}^2_{loc}(U)$.

(b) Assuming that $\nabla_{\mathbf{g}} f \in \vec{L}^2_{loc}(U)$, prove that distributional partial derivatives $\frac{\partial f}{\partial x^j}$ are given by

$$\frac{\partial f}{\partial x^j} = g_{ij}(\nabla_{\mathbf{g}} f)^i$$

and that the identity (4.2) holds. Conclude that $\frac{\partial f}{\partial x^j} \in L^2_{loc}(U)$.

4.12. For an open set $\Omega \subset \mathbb{R}^n$, let $W^1(\Omega)$ be the Sobolev space defined in Section **??**, and $W^1(\Omega,\mathbf{g},\lambda)$ be the Sobolev space defined in Section 4, where $\mathbf{g}$ is the canonical Euclidean metric and λ is the Lebesgue measure. Prove that these two Sobolev spaces are identical.

4.13. Denote by ∇_{dist} the distributional gradient in $\mathbb{R}^n$ $(n \ge 2)$ reserving ∇ for the gradient in the classical sense, and the same applies to the Laplace operators Δ_{dist} and Δ.

(a) Let $f \in C^1(\mathbb{R}^n \setminus \{o\})$ and assume that

$$f \in L^2_{loc}(\mathbb{R}^n) \quad \text{and} \quad \nabla f \in \vec{L}^2_{loc}(\mathbb{R}^n).$$

Prove that $\nabla_{dist} f = \nabla f$.

(b) Let $f \in C^2(\mathbb{R}^n \setminus \{o\})$ and assume that

$$f \in L^2_{loc}(\mathbb{R}^n), \quad \nabla f \in \vec{L}^2_{loc}(\mathbb{R}^n), \quad \text{and} \quad \Delta f \in L^2_{loc}(\mathbb{R}^n).$$

Prove that $\Delta_{dist} f = \Delta f$.

(c) Consider in $\mathbb{R}^3$ the function $f(x) = |x|^{-1}$. Show that $f \in L^2_{loc}(\mathbb{R}^3)$ and $\Delta f = 0$ in $\mathbb{R}^3 \setminus \{o\}$. Prove that $\Delta_{dist} f = -4\pi\delta$ where δ is the Dirac delta-function at the origin o.

4.14. Consider in $\mathbb{R}^n$ $(n \geq 2)$ the function $f(x) = |x|^\alpha$, where α is a real parameter.

(a) Prove that $f \in L^2_{loc}$ provided $\alpha > -n/2$.

(b) Prove that $f \in L^2_{loc}$ and $\nabla f \in \vec{L}^2_{loc}$ provided $\alpha > 1 - n/2$. Show that in this case $\nabla_{dist} f = \nabla f$.

(c) Prove that $f \in L^2_{loc}$, $\nabla f \in \vec{L}^2_{loc}$, and $\Delta u \in L^2_{loc}$ provided $\alpha > 2 - n/2$. Show that in this case $\Delta_{dist} f = \Delta f$..

4.15. Prove that if $\{u_k\}$ is a sequence of functions from W^1 that is bounded in the norm of W^1 then there exists a subsequence $\{u_{k_i}\}$ that converges to a function $u \in W^1$ weakly in W^1 and weakly in L^2.

4.16. Prove that if $\{u_k\}$ is a sequence of functions from W^1 that converges weakly in W^1 to a function $u \in W^1$ then there is a subsequence $\{u_{k_i}\}$ such that

$$u_{k_i} \overset{L^2}{\rightharpoonup} u \text{ and } \nabla u_{k_i} \overset{L^2}{\rightharpoonup} \nabla u,$$

where $\rightharpoonup$ stands for the weak convergence.

4.17. Let $\{u_k\}$ be a sequence of functions from W^1 that converges weakly in L^2 to a function $u \in L^2$.

(a) Prove that if the sequence $\{u_k\}$ is bounded in the norm W^1 then $u \in W^1$ and $u_k \overset{W^1}{\rightharpoonup} u$.

(b) Prove that if in addition $\|u_k\|_{W^1} \to \|u\|_{W^1}$ then $u_k \overset{W^1}{\to} u$.

4.18. Let $\{u_k\}$ be an increasing sequence of non-negative functions from W^1 that converges almost everywhere to a function $u \in L^2$. Prove that if

$$\|\nabla u_k\|_{L^2} \leq c$$

for some constant c and all k, then $u \in W^1$, $u_k \overset{W^1}{\rightharpoonup} u$, and $\|\nabla u\|_{L^2} \leq c$.

4.19. Let M be the unit ball B in $\mathbb{R}^n$. Prove that the Laplace operator Δ with domain

$$\left\{f \in C^2(B) : \Delta f \in L^2(B)\right\}$$

is not symmetric in $L^2(B)$.

4.20. Let A be an operator in $L^2(M)$ defined by $Af = -\Delta_\mu f$ with $\operatorname{dom} A = C_0^\infty(M)$. Prove that operator A is unbounded.

4.21. Prove that if $f \in C_0^\infty(M)$ and $u \in W_0^1$ then $fu \in W_0^1(M)$.

4.22. Prove that the spaces $W_0^2(M)$ and $W^2(M)$, endowed with the inner product

$$(u, v)_{W^2} = (u, v)_{W^1} + (\Delta_\mu u, \Delta_\mu v)_{L^2}, \tag{4.3}$$

are Hilbert spaces.

4.23. Prove that, for any $u \in W_0^2(M)$,

$$\|u\|^2_{W^1} \leq c\left(\|u\|^2_{L^2} + \|\Delta_\mu u\|^2_{L^2}\right), \tag{4.4}$$

where $c = \frac{1+\sqrt{2}}{2}$.

4.24. Let $\{E_\lambda\}$ be the spectral resolution of the Dirichlet Laplace operator $\mathcal{L}$ in $L^2(M)$. Prove that, for any $f \in W_0^2(M)$,

$$\|\nabla f\|_{L^2}^2 = \int_0^\infty \lambda \|dE_\lambda f\|_{L^2}^2. \tag{4.5}$$

4.25. Prove that $\operatorname{dom} \mathcal{L}^{1/2} = W_0^1(M)$ and that (4.5) holds for any $f \in W_0^1(M)$.

Hint. Use Exercise 17.13.

4.26. Prove that $\operatorname{dom} \mathcal{L}^{1/2} = \operatorname{dom} (\mathcal{L} + \mathrm{id})^{1/2}$ and, for any $f \in W_0^1(M)$,

$$\|f\|_{W^1} = \|(\mathcal{L} + \mathrm{id})^{1/2} f\|_{L^2}. \tag{4.6}$$

4.27. Prove that, for all $f \in W_0^1(M)$,

$$\|\nabla f\|_{L^2}^2 \geq \lambda_{\min} \|f\|_{L^2}^2, \tag{4.7}$$

where

$$\lambda_{\min} := \inf \operatorname{spec} \mathcal{L}. \tag{4.8}$$

4.28. Assuming that $\lambda_{\min} > 0$, prove that the weak Dirichlet problem on M

$$\begin{cases} -\Delta_\mu u = f, \\ u \in W_0^1(M), \end{cases} \tag{4.9}$$

has a unique solution u for any $f \in L^2(M)$, and that for this solution

$$\|u\|_{L^2} \leq \lambda_{\min}^{-1} \|f\|_{L^2} \tag{4.10}$$

and

$$\|\nabla u\|_{L^2} \leq \lambda_{\min}^{-1/2} \|f\|_{L^2}. \tag{4.11}$$

4.29. Consider the following version of the weak Dirichlet problem: given a real constant α and functions $f \in L^2(M)$, $w \in W^1(M)$, find a function $u \in L^2(M)$ that satisfies the conditions

$$\begin{cases} \Delta_\mu u + \alpha u = f, \\ u = w \mod W_0^1(M), \end{cases} \tag{4.12}$$

where the second condition means $u - w \in W_0^1(M)$. Prove that if $\alpha < \lambda_{\min}$ then the problem (4.12) has exactly one solution.

4.30. Let A be a bounded self-adjoint operator in L^2 such that, for a constant $\alpha > 0$ and for any function $f \in L^2(M)$,

$$\alpha^{-1} \|f\|_2^2 \leq (Af, f)_{L^2} \leq \alpha \|f\|_2^2.$$

(a) Prove that the bilinear form

$$\{f, g\} := (\nabla f, \nabla g) + (Af, g)$$

defines an inner product in W_0^1, and that W_0^1 with this inner product is a Hilbert space.

(b) Prove that, for any $h \in L^2$, the equation

$$-\Delta_\mu u + Au = h$$

has exactly one solution $u \in W_0^2$.

4.31. Prove that, for any $\alpha > 0$ and $f \in L^2(M)$, the function $u = R_\alpha f$ is the only minimizer of the functional

$$E(v) := \|\nabla v\|_2^2 + \alpha \|v - f\|_2^2,$$

in the domain $v \in W_0^1(M)$.

4.32. Prove that for any $\alpha > 0$ the operators $\nabla \circ R_\alpha : L^2(M) \to \vec{L}^2(M)$ and $\mathcal{L} \circ R_\alpha : L^2(M) \to L^2(M)$ are bounded and

$$\|\nabla \circ R_\alpha\| \le \alpha^{-1/2}, \tag{4.13}$$

$$\|\mathcal{L} \circ R_\alpha\| \le 1. \tag{4.14}$$

4.33. Prove that, for any $f \in L^2(M)$,

$$\alpha R_a f \xrightarrow{L^2} f \text{ as } \alpha \to +\infty.$$

Prove that if $f \in \operatorname{dom} \mathcal{L}$ then

$$\|\alpha R_\alpha f - f\|_{L^2} \le \frac{1}{\alpha} \|\mathcal{L} f\|_{L^2}.$$

4.34. Prove that, for all $\alpha, \beta > 0$,

$$R_\alpha - R_\beta = (\beta - \alpha) R_\alpha R_\beta. \tag{4.15}$$

4.35. Fix a function $f \in L^2$.

(a) Prove that the function $\varphi(t) := (P_t f, f)$ on $t \in [0, +\infty)$ is non-negative, decreasing, continuous, and log-convex.

(b) Prove that the function $\psi(t) := \|\nabla P_t f\|_2^2$ is decreasing on $(0, +\infty)$ and

$$\int_0^\infty \psi(t)\, dt \le \frac{1}{2} \|f\|_{L^2}^2.$$

4.36. Prove that, for any $f \in W_0^1$, such that $\|f\|_{L^2} = 1$,

$$\|P_t f\|_{L^2} \ge \exp\left(-t \int_M |\nabla f|^2 d\mu\right), \tag{4.16}$$

for any $t > 0$.

HINT. Use Exercise 4.25 and 4.35.

4.37. Prove that, for all $f \in L^2$ and all $t > 0$,

$$\|\Delta_\mu (P_t f)\|_{L^2} \le \frac{e}{t} \|f\|_{L^2}. \tag{4.17}$$

and

$$\|\nabla (P_t f)\|_{L^2} \le \sqrt{\frac{e}{t}} \|f\|_{L^2}. \tag{4.18}$$

4.38. For any $t > 0$, define a quadratic form $\mathcal{E}_t(f)$ by

$$\mathcal{E}_t(f) = \left(\frac{f - P_t f}{t}, f\right)_{L^2}, \tag{4.19}$$

for all $f \in L^2$.

(a) Prove that $\mathcal{E}_t(f)$ is increasing as t is decreasing.

(*b*) Prove that $\lim_{t\to 0}\mathcal{E}_t(f)$ is finite if and only if $f\in W_0^1$, and

$$\lim_{t\to 0}\mathcal{E}_t(f)=\int_M |\nabla f|^2\,d\mu.$$

(*c*) Define a bilinear form $\mathcal{E}_t(f,g)$ in L^2 by

$$\mathcal{E}_t(f,g)=\left(\frac{f-P_tf}{t},g\right)_{L^2}.$$

Prove that if $f,g\in W_0^1$ then

$$\mathcal{E}_t(f,g)\to\int_M\langle\nabla f,\nabla g\rangle\,d\mu\ \text{ as } t\to 0. \tag{4.20}$$

4.39. Prove that if $f\in W_0^2$ then, for all $t>0$,

$$\|P_tf-f\|_{L^2}\le t\|\Delta_\mu f\|_{L^2}, \tag{4.21}$$

REMARK. *Recall that, by Theorem 4, if $f\in L^2$ then $P_tf\overset{L^2}{\to}f$ as $t\to 0$. The estimate (4.21) implies a linear decay of $\|P_tf-f\|_{L^2}$ as $t\to 0$ provided $f\in W_0^2$.*

4.40. Prove that if $f\in W_0^1$ then

$$\|P_tf-f\|_{L^2}\le t^{1/2}\|\nabla f\|_{L^2}. \tag{4.22}$$

HINT. Use Exercise 4.25 or argue as in Lemma 2.

4.41. Prove that if $f\in W_0^2$ then

$$\frac{P_tf-f}{t}\overset{L^2}{\to}\Delta_\mu f \text{ as } t\to 0. \tag{4.23}$$

4.42. Prove that, for any $f\in L^2$,

$$\frac{P_tf-f}{t}\overset{\mathcal{D}'}{\longrightarrow}\Delta_\mu f\ \text{ as } t\to 0,$$

where $\Delta_\mu f$ is understood in the distributional sense.

4.43. Prove that if $f\in L^2$ and, for some $g\in L^2$,

$$\frac{P_tf-f}{t}\overset{L^2}{\to}g \text{ as } t\to 0$$

then $f\in W_0^2$ and $g=\Delta_\mu f$.

4.44. Let $f\in W_0^2$ be such that $\Delta_\mu f=0$ in an open set $\Omega\subset M$. Consider a path

$$u(t)=\begin{cases}P_tf, & t>0,\\ f, & t\le 0.\end{cases}$$

Prove that $u(t)$ satisfies in $\mathbb{R}\times\Omega$ the heat equation $\frac{du}{dt}=\Delta_\mu u$ in the following sense: the path $t\mapsto u(t)$ is strongly differentiable in $L^2(\Omega)$ for all $t\in\mathbb{R}$ and the derivative $\frac{du}{dt}$ is equal to $\Delta_\mu u$ where Δ_μ is understood in the distributional sense.

4.45. Prove that if $f\in W_0^1$ then

$$P_tf\overset{W^1}{\longrightarrow}f\ \text{ as } t\to 0.$$

and if $f\in W_0^2$ then

$$P_tf\overset{W^2}{\longrightarrow}f\ \text{ as } t\to 0.$$

4.46. (*Product rule for strong derivatives*)

(*a*) Let $\mathcal{H}$ be a Hilbert space, I be an interval in $\mathbb{R}$, and $u(t), v(t) : I \to \mathcal{H}$ be strongly differentiable paths. Prove that

$$\frac{d}{dt}(u,v) = (u, \frac{dv}{dt}) + (\frac{du}{dt}, v).$$

(*b*) Consider the mappings $u : I \to L^p(M)$ and $v : I \to L^q(M)$ where I is an interval in $\mathbb{R}$ and $p, q \in [1, +\infty]$. Prove that if u and v are continuous then the function $w(t) = u(t)v(t)$ is continuous as a mapping from I to $L^r(M)$ where r is defined by the equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

(*c*) Prove that if u and v as above are strongly differentiable then w is also strongly differentiable and

$$\frac{dw}{dt} = u\frac{dv}{dt} + \frac{du}{dt}v.$$

4.47. For any open set $\Omega \subset M$, denote by $C_b(\Omega)$ the linear space of all bounded continuous functions on Ω with the sup-norm. Let $u(t,x)$ be a continuous function on $I \times M$ where I is an open interval in $\mathbb{R}$, and let the partial derivative $\frac{\partial u}{\partial t}$ be also continuous in $I \times M$. Prove that, for any relatively compact open set $\Omega \subset M$, the path $u(t, \cdot) : I \to C_b(\Omega)$ is strongly differentiable, and its strong derivative $\frac{du}{dt}$ coincides with the partial derivative $\frac{\partial u}{\partial t}$.

4.48. Let $\mathcal{H}$ be a Hilbert space.

(*a*) Let $u(t) : [a,b] \to \mathcal{H}$ be a continuous path. Prove that, for any $x \in \mathcal{H}$, the functions $t \mapsto (u(t), x)$ and $t \mapsto \|u(t)\|$ are continuous in $t \in [a,b]$, and

$$\left| \int_a^b (u(t), x)\, dt \right| \le \left(\int_a^b \|u(t)\| dt \right) \|x\|.$$

Conclude that there exists a unique vector $U \in \mathcal{H}$ such that

$$\int_a^b (u(t), x)\, dt = (U, x) \quad \text{for all } x \in \mathcal{H},$$

which allows to define $\int_a^b u(t)\, dt$ by

$$\int_a^b u(t)\, dt := U.$$

Prove that

$$\left\| \int_a^b u(t)\, dt \right\| \le \int_a^b \|u(t)\|\, dt.$$

(*b*) (*Fundamental theorem of calculus*) Let $u(t) : [a,b] \to \mathcal{H}$ be a strongly differentiable path. Prove that if the strong derivative $u'(t)$ is continuous in $[a,b]$ then

$$\int_a^b u'(t)\, dt = u(b) - u(a).$$

4.49. Let $u : [a,b] \to L^1(M, \mu)$ be a continuous paths in L^1. Prove that there exists an function $w \in L^1(N, d\nu)$ where $N = [a,b] \times M$ and $d\nu = dt d\mu$, such that $w(t, \cdot) = u(t)$ for any $t \in [a,b]$.

4.50. (*Chain rule for strong derivatives*) Let $u(t) : (a,b) \to L^2(M)$ be a strongly differentiable path. Consider a function $\psi \in C^1(\mathbb{R})$ such that

$$\psi(0) = 0 \text{ and } \sup |\psi'| < \infty. \tag{4.24}$$

Prove that the path $\psi(u(t))$ is also strongly differentiable in $t \in (a,b)$ and

$$\frac{d\psi(u)}{dt} = \psi'(u)\frac{du}{dt}.$$

4.51. Let $\Phi(\lambda)$ be a continuous function on $[0,+\infty)$ of a subexponential growth; that is, for any $\varepsilon > 0$,

$$\Phi(\lambda) = o\left(e^{\varepsilon\lambda}\right) \quad \text{as } \lambda \to +\infty. \tag{4.25}$$

Let $\mathcal{L}$ be a non-negative definite self-adjoint operator in a Hilbert space $\mathcal{H}$. Fix $f \in \mathcal{H}$ and consider the path $v : \mathbb{R}_+ \to \mathcal{H}$ defined by

$$v(t) := \int_0^\infty \Phi(\lambda) e^{-t\lambda} dE_\lambda f, \tag{4.26}$$

where $\{E_\lambda\}$ is the spectral resolution of $\mathcal{L}$. Prove that, for any $t > 0$, $v(t) \in \operatorname{dom}\mathcal{L}$, the strong derivative $\frac{dv}{dt}$ exists, and

$$\frac{dv}{dt} = -\int_0^\infty \lambda\Phi(\lambda) e^{-t\lambda} dE_\lambda f = -\mathcal{L}v(t). \tag{4.27}$$

Conclude that the strong derivative $\frac{d^k v}{dt^k}$ of any order $k \in \mathbb{N}$ exists and

$$\frac{d^k v}{dt^k} = (-\mathcal{L})^k v(t). \tag{4.28}$$

4.52. Let $\mathcal{L}$ be a non-negative definite self-adjoint operator in a Hilbert space $\mathcal{H}$. For any $t \in \mathbb{R}$, consider *the wave operators*

$$C_t = \cos\left(t\mathcal{L}^{1/2}\right) \quad \text{and} \quad S_t = \sin\left(t\mathcal{L}^{1/2}\right).$$

(*a*) Prove that C_t and S_t are bounded self-adjoint operators.

(*b*) Prove that, for all $f, g \in \operatorname{dom}\mathcal{L}^{1/2}$, the function

$$u(t) = C_t f + S_t g$$

is strongly differentiable in t and satisfies the initial data

$$u|_{t=0} = f \quad \text{and} \quad \left.\frac{du}{dt}\right|_{t=0} = \mathcal{L}^{1/2} g.$$

(*c*) Prove that, for any $f \in \operatorname{dom}\mathcal{L}$, both functions $C_t f$ and $S_t f$ are twice strongly differentiable in t and satisfy the wave equation

$$\frac{d^2 u}{dt^2} = -\mathcal{L}u,$$

where $\frac{d^2}{dt^2}$ is the second strong derivative.

(*d*) (*A transmutation formula*) Prove the following relation between the heat and wave operators:

$$e^{-t\mathcal{L}} = \int_0^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) C_s ds, \tag{4.29}$$

where the integral is understood in the sense of the weak operator topology (cf. Lemma 5).

4.53. Let $\varphi(t)$ be a continuous real-valued function on an interval (a,b), $a<b$, and assume that $\varphi(t)$ is right differentiable at any point $t\in(a,b)$. Prove that if $\varphi'(t)\leq 0$ for all $t\in(a,b)$ (where φ' stands for the right derivative) then function φ is monotone decreasing on (a,b).

4.54. Consider the right Cauchy problem in a Hilbert space $\mathcal{H}$: to find a path $u:(0,+\infty)\to\mathcal{H}$ so that the following conditions are satisfied:

(i) $u(t)$ is continuous and strongly right differentiable for all $t>0$;

(ii) For any $t>0$, $u(t)\in\operatorname{dom}\mathcal{L}$ and

$$\frac{du}{dt}=-\mathcal{L}u,$$

where $\frac{du}{dt}$ is the strong right derivative of u.

(iii) $u(t)\to f$ as $t\to 0$, where f is a given element of $\mathcal{H}$.

Prove the uniqueness of the path $u(t)$ for any given f.

CHAPTER 5

Weak maximum principle and related topics

5.1. Let $\psi(t)$ and $\varphi(t)$ be functions satisfying the conditions (5) and (**??**) of Theorem 5. Prove that $\psi'_{dist} = \varphi$.

5.2. Let $\psi \in C^1(\mathbb{R})$ be such that

$$\psi(0) = 0 \quad \text{and} \quad \sup\left|\psi'\right| < \infty.$$

Prove that the functions ψ and $\varphi := \psi'$ satisfy the conditions (5) and (**??**) of Theorem 5.

5.3. Prove that if $u, v \in W_0^1(M)$ then also $\max(u, v)$ and $\min(u, v)$ belong to $W_0^1(M)$.

5.4. Prove that if M is a compact manifold then $W^1(M) = W_0^1(M)$.

5.5. Prove that if $u \in W^1(M)$ then, for any real constant c, $\nabla u = 0$ a.e. on the set $\{x \in M : u(x) = c\}$.

5.6. Prove that, for any $u \in W^1(M)$,

$$(u - c)_+ \stackrel{W^1}{\longrightarrow} u_+ \text{ as } c \to 0+.$$

5.7. Let $f \in W^1(M)$ and assume that $f(x) \to 0$ as $x \to \infty$ (the latter means that, for any $\varepsilon > 0$, the set $\{|f| \geq \varepsilon\}$ is relatively compact). Prove that $f \in W_0^1(M)$.

5.8. Prove that if $u \in W_{loc}^1(M)$ and φ, ψ are functions on $\mathbb{R}$ satisfying the conditions of Theorem 5 then $\psi(u) \in W_{loc}^1(M)$ and $\nabla\psi(u) = \varphi(u)\nabla u$.

5.9. Define the space $W_{loc}^2(M)$ by

$$W_{loc}^2 = \left\{f \in W_{loc}^1 : \Delta_\mu f \in L_{loc}^2\right\}.$$

Prove the Green formula (**??**) for any two functions $u \in W_c^1$ and $v \in W_{loc}^2$.

5.10. Let R_α be the resolvent defined by (**??**).

(*a*) Prove that, for any $f \in L^2$ and $\alpha > 0$,

$$P_t f = \lim_{\alpha \to +\infty} e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\alpha^{2k} t^k}{k!} R_\alpha^k f. \tag{5.1}$$

(*b*) Using (5.1), give an alternative proof of the fact that $f \leq 1$ implies $P_t f \leq 1$.

5.11. For all $\alpha, k > 0$, define R_α^k as $\varphi(R_\alpha)$ where $\varphi(\lambda) = \lambda^k$.

(*a*) Prove that, for all $\alpha, k > 0$,

$$R_\alpha^k = \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-\alpha t} P_t dt, \tag{5.2}$$

where the integral is understood in the weak sense, as in Lemma 5, and Γ is the gamma function (cf. Section **??**).

(*b*) Write for simplicity $R_1 = R$. Prove that

$$R^k R^l = R^{k+l} \text{ for all } k, l > 0.$$

Prove that if $f \in L^2(M)$ then $f \geq 0$ implies $R^k f \geq 0$ and $f \leq 1$ implies $R^k f \leq 1$, for all $k \geq 0$.

(*c*) Prove that $R^k = e^{-kL}$ where $L = \log(\mathrm{id} + \mathcal{L})$ and $\mathcal{L}$ is the Dirichlet Laplace operator.

Remark. The semigroup $\left\{R^k\right\}_{k \geq 0}$ is called the Bessel semigroup, and the operator $\log(\mathrm{id} + \mathcal{L})$ is its generator.

5.12. Prove that, for any non-negative function $f \in L^2(M)$ and all $t, \alpha > 0$,

$$P_t R_\alpha f \leq e^{\alpha t} R_\alpha f.$$

5.13. Let $\mathcal{L}$ be the Dirichlet Laplace operator on $\mathbb{R}^1$.

(*a*) Prove that the resolvent $R_\lambda = (\mathcal{L} + \lambda\,\mathrm{id})^{-1}$ is given for any $\lambda > 0$ by the following formula:

$$R_\lambda f = \frac{1}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} e^{-\sqrt{\lambda}|x-y|} f(y)\, dy, \tag{5.3}$$

for any and $f \in L^2\left(\mathbb{R}^1\right)$.

(*b*) Comparing (5.3) with

$$R_\lambda = \int_0^\infty e^{-\lambda t} P_t dt$$

and using the explicit formula for the heat kernel in $\mathbb{R}^1$, establish the following identity:

$$e^{-t\sqrt{\lambda}} = \int_0^\infty \frac{t}{\sqrt{4\pi s^3}} \exp\left(-\frac{t^2}{4s}\right) e^{-s\lambda} ds, \tag{5.4}$$

for all $t > 0$ and $\lambda \geq 0$.

Remark. The function $s \mapsto \frac{t}{\sqrt{4\pi s^3}} \exp\left(-\frac{t^2}{4s}\right)$ is the density of a probability distribution on $\mathbb{R}_+$, which is called the *Levy distribution*.

5.14. Let $\mathcal{L}$ be the Dirichlet Laplace operator on an arbitrary weighted manifold, and consider the family of operators $Q_t = \exp\left(-t\mathcal{L}^{1/2}\right)$, where $t \geq 0$.

(*a*) Prove the identity

$$Q_t = \int_0^\infty \frac{t}{\sqrt{4\pi s^3}} \exp\left(-\frac{t^2}{4s}\right) P_s ds. \tag{5.5}$$

(*b*) Let $f \in L^2(M)$. Prove that $f \geq 0$ implies $Q_t f \geq 0$ and $f \leq 1$ implies $Q_t f \leq 1$.

(*c*) Prove that in the case $M = \mathbb{R}^n$, Q_t is given explicitly by

$$Q_t f = \int_{\mathbb{R}^n} q_t(x-y) f(y)\, dy$$

where

$$q_t(x) = \frac{2}{\omega_{n+1}} \frac{t}{\left(t^2 + |x|^2\right)^{\frac{n+1}{2}}}. \tag{5.6}$$

REMARK. The semigroup $\{Q_t\}_{k\geq 0}$ is called the Cauchy semigroup, and the operator $\mathcal{L}^{1/2}$ is its generator.

5.15. Let Ψ be a C^∞-function on $\mathbb{R}$ such that $\Psi(0) = \Psi'(0) = 0$ and $0 \leq \Psi''(s) \leq 1$ for all s.

(a) Prove that, for any $f \in L^2(M)$, the following function

$$F(t) := \int_M \Psi(P_t f)\, d\mu \tag{5.7}$$

is continuous and decreasing in $t \in [0, +\infty)$.

(b) Using part (a), give yet another proof of the fact that $f \leq 1$ implies $P_t f \leq 1$, without using the resolvent.

5.16. Give an example of a manifold M and a non-negative function $u \in W^1_{loc}(M)$ such that

$$u \leq 0 \mod W^1_0(M)$$

but $u \notin W^1(M)$.

5.17. Let the paths $w : (0,T) \to W^1(M)$ and $v : (0,T) \to W^1_0(M)$ satisfy the same heat equation

$$\frac{du}{dt} = \Delta_\mu u \ \text{ for all } t \in (0,T),$$

where $\frac{du}{dt}$ is the strong derivative in $L^2(M)$ and $\Delta_\mu u$ is understood in the distributional sense. Prove that if

$$w(t,\cdot) - v(t,\cdot) \overset{L^2(M)}{\longrightarrow} 0 \ \text{ as } t \to 0,$$

and $w \geq 0$ then $w(t,\cdot) \geq v(t,\cdot)$ for all $t \in (0,T)$.

5.18. Let $v_\alpha(x)$ be a real valued function on a non-compact smooth manifold M depending on a parameter $\alpha \in A$, and let $c \in \mathbb{R}$. Prove that the following conditions are equivalent (all convergences are inform in $\alpha \in A$):

(i) $v_\alpha(x) \rightrightarrows c$ as $x \to \infty$.

(ii) For any sequence $\{x_k\}_{k=1}^\infty$ that eventually leaves any compact set $K \subset M$, $v_\alpha(x_k) \rightrightarrows c$ as $k \to \infty$.

(iii) For any sequence $\{x_k\}$ on M that eventually leaves any compact set $K \subset M$, there is a subsequence $\{x_{k_i}\}$ such that $v_\alpha(x_{k_i}) \rightrightarrows c$ as $i \to \infty$.

(iv) For any $\varepsilon > 0$, the set

$$V_\varepsilon = \left\{ x \in M : \sup_{\alpha \in A} |v_\alpha(x) - c| \geq \varepsilon \right\} \tag{5.8}$$

is relatively compact.

Show that these conditions are also equivalent for $c = \pm\infty$ provided (5.8) is appropriately adjusted.

5.19. Referring to Exercise 5.18, let $M = \Omega$ where Ω is an unbounded open subset of $\mathbb{R}^n$. Prove that the condition (i) is equivalent to

(v) $v_\alpha(x_k) \rightrightarrows c$ for any sequence $\{x_k\} \subset \Omega$ such that either $x_k \to x \in \partial\Omega$ or $|x_k| \to \infty$.

5.20. Let a function $v \in C^2(M)$ satisfy the conditions:

(i) $-\Delta_\mu v + \alpha v \le 0$ on M, for some $\alpha > 0$;
(ii) $v_+(x) \to 0$ as $x \to \infty$ in M.

Prove that $v \le 0$ in M.

5.21. Prove that the statement of Corollary 5 remains true if the condition $u_+(t,\cdot) \overset{L^2_{loc}(M)}{\longrightarrow} 0$ as $t \to 0$ is replaced by

$$u_+(t,\cdot) \overset{L^1_{loc}(M)}{\longrightarrow} 0 \text{ as } t \to 0.$$

5.22. Let u be a function from $C(M) \cap W_0^1(M)$. For any $a > 0$, set

$$U_a = \{x \in M : u(x) > a\}.$$

Prove that $(u-a)_+ \in W_0^1(U_a)$.

5.23. Let Ω be an open subset of a weighted manifold M and K be a compact subset of Ω. Let f be a non-negative function from $L^2(M)$. Prove that, for all $\alpha > 0$,

$$R_\alpha f - R_\alpha^\Omega f \le \operatorname*{esup}_{M \setminus K} R_\alpha f. \tag{5.9}$$

5.24. Under the hypotheses of Exercise 5.23, prove that, for all $t > 0$,

$$P_t f - P_t^\Omega f \le \sup_{s \in [0,t]} \operatorname*{esup}_{M \setminus K} P_s^\Omega f. \tag{5.10}$$

5.25. Let $\{\Omega_i\}_{i=1}^\infty$ be an increasing sequence of open subsets of M, $\Omega = \bigcup_{i=1}^\infty \Omega_i$, and $f \in L^2(\Omega_1)$. Prove that the family of functions $\left\{P_t^{\Omega_i} f\right\}_{i=1}^\infty$ considered as the paths in $L^2(\Omega)$, is equicontinuous in $t \in [0,+\infty)$ with resect to the norm in $L^2(\Omega)$.

5.26. Let A be the multiplication operator by a bounded,non-negative measurable function a on M.

(a) Prove that A is a bounded, non-negative definite, self-adjoint operator in L^2 and, for any non-negative $f \in L^2$ and $t \ge 0$,

$$0 \le e^{-tA} f \le f. \tag{5.11}$$

(b) Prove that, for any non-negative $f \in L^2$ and $t \ge 0$,

$$0 \le e^{-t(\mathcal{L}+A)} f \le e^{-t\mathcal{L}} f. \tag{5.12}$$

(c) Using part (b), give an alternative proof of the fact that $P_t^\Omega f \le P_t f$.

Hint. In part (b) use the Trotter product formula:

$$e^{-t(A+B)} f = \lim_{n \to \infty} \left(e^{-\frac{t}{n}A} e^{-\frac{t}{n}B}\right)^n f, \tag{5.13}$$

that is true for any two non-negative definite self-adjoint operators A, B in L^2.

CHAPTER 6

Regularity theory in $\mathbb{R}^n$

6.1. Show that the delta function δ in $\mathbb{R}^n$ belongs to W^{-k} for any $k > n/2$.

6.2. The purpose of this problem is to give an alternative proof of Theorem 6 by means of the Fourier transform. Let Ω be a bounded open set in $\mathbb{R}^n$. Recall that $W_0^1(\Omega)$ can be considered as a subspace of $W^1(\mathbb{R}^n)$ by extending functions by 0 outside Ω.

(a) Prove that, for all $f \in W_0^1(\Omega)$ and $g \in C^\infty(\mathbb{R}^n)$,

$$\int_\Omega (\partial_j f)\, g\, dx = -\int_\Omega f\, \partial_j g\, dx. \tag{6.1}$$

(b) Prove that, for any $f \in W_0^1(\Omega)$ and for any $\xi \in \mathbb{R}^n$,

$$\left(f, e^{i\xi x}\right)_{W^1(\Omega)} = \left(1 + |\xi|^2\right) \widehat{f}(\xi), \tag{6.2}$$

where $\widehat{f}(\xi)$ is the Fourier transform of f.

(c) Let $\{f_k\}$ be a sequence from $W_0^1(\Omega)$ such that f_k converges weakly in $W^1(\mathbb{R}^n)$ to a function $f \in W^1(\mathbb{R}^n)$. Prove that $\widehat{f_k}(\xi) \to \widehat{f}(\xi)$, for any $\xi \in \mathbb{R}^n$. Prove that also $\widehat{f_k} \to \widehat{f}$ in $L^2_{loc}(\mathbb{R}^n)$.

(d) Finally, prove that if $\{f_k\}$ is a bounded sequence in $W_0^1(\Omega)$ then $\{f_k\}$ contains a subsequence that converges in $L^2(\Omega)$.

Hint. Use Exercises 2.28 and 2.34.

6.3. Prove that, for any open set $\Omega' \Subset \Omega$, for any $m \geq -1$, and for any $u \in \mathcal{D}(\Omega')$,

$$\|u\|_{W^{m+2}} \leq C\|Lu\|_{W^m}, \tag{6.3}$$

where a constant C depends on Ω', L, m.

Hint. Use Lemma 6 for the inductive basis and prove the inductive step as in Lemma 6.

6.4. Consider a more general operator

$$L = \partial_i\left(a^{ij}(x)\,\partial_j\right) + b^j(x)\,\partial_j + c(x), \tag{6.4}$$

where a^{ij} is as before, and b^j and c are smooth functions in Ω. Prove that if $u \in \mathcal{D}'(\Omega)$ and $Lu \in W^m_{loc}(\Omega)$ for some $m \in \mathbb{Z}$ then $u \in W^{m+2}_{loc}(\Omega)$. Conclude that $Lu \in C^\infty$ implies $u \in C^\infty$.

6.5. Let $\Omega' \Subset \Omega$ be open sets and $m \geq -1$ be an integer.

(a) Prove that, for any $u \in \mathcal{D}(\Omega')$,

$$\|u\|_{V^{m+2}(\Omega)} \leq C\|\mathcal{P}u\|_{V^m(\Omega)}, \tag{6.5}$$

where a constant C depends on $\Omega', \mathcal{P}, m$.

(b) Using part (a), prove that, for any $u \in C^\infty(\Omega)$,

$$\|u\|_{V^{m+2}(\Omega')} \leq C\left(\|u\|_{L^2(\Omega)} + \|\mathcal{P}u\|_{V^m(\Omega)}\right). \tag{6.6}$$

Remark. The estimate (6.6) was proved in Theorem 6. In the case $u \in C^\infty$, it is easier to deduce it from (6.5).

6.6. Consider a more general parabolic operator

$$\mathcal{P} = \rho\partial_t - \partial_i\left(a^{ij}(x)\,\partial_j\right) - b^j(x)\,\partial_j - c(x),$$

where a^{ij} and ρ are as before, and b^j and c are smooth functions in Ω. Prove that if $u \in \mathcal{D}'(\Omega)$ and $\mathcal{P}u \in V^m_{loc}(\Omega)$ for some $m \in \mathbb{Z}$ then $u \in V^{m+2}_{loc}(\Omega)$. Conclude that $\mathcal{P}u \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.

CHAPTER 7

The heat kernel on a manifold

7.1. Prove that $\mathcal{W}_0^s$ is a Hilbert space.

7.2. Prove that $\mathcal{W}_0^1 = W_0^1$ and $\mathcal{W}_0^2 = W_0^2$ including the equivalence (but not necessarily the identity) of the norms.

7.3. Prove that if k is a positive integer then $f \in \mathcal{W}_0^{2k}$ if and only if

$$f,\ \mathcal{L}f,\ ...,\ \mathcal{L}^{k-1}f \in W_0^1(M) \ \text{ and } \ \mathcal{L}^k f \in L^2(M). \tag{7.1}$$

7.4. Prove that $\mathcal{W}_0^{2k} \subset \mathcal{W}^{2k}$ and that the norms in $\mathcal{W}_0^{2k}$ and $\mathcal{W}^{2k}$ are equivalent.

7.5. Prove that if $f \in \mathcal{W}_0^{2k}$ then, for all integer $0 \le l \le k$,

$$\|\mathcal{L}^l f\|_{L^2} \le \|f\|_{L^2}^{(k-l)/k} \|\mathcal{L}^k f\|_{L^2}^{l/k}. \tag{7.2}$$

7.6. Let M be a connected weighted manifold. Prove that if $f \in L^2_{loc}(M)$ and $\nabla f = 0$ on M then $f = \text{const}$ on M.

7.7. Let M be a connected manifold and Ω be an open subset of M such that and $M \setminus \overline{\Omega}$ is non-empty. Prove that $1_\Omega \notin W^1(M)$ and $1_\Omega \notin W_0^1(\Omega)$.

Remark. If in addition $\mu(\Omega) < \infty$ then clearly $1_\Omega \in L^2(\Omega)$ and $\nabla 1_\Omega = 0$ in Ω whence $1_\Omega \in W^1(\Omega)$. In this case we obtain an example of a function that is in $W^1(\Omega)$ but not in $W_0^1(\Omega)$.

7.8. *(The exterior maximum principle)*Let M be a connected weighted manifold and Ω be a non-empty open subset of M such that $M \setminus \overline{\Omega}$ is non-empty. Let u be a function from $C(M) \cap W_0^1(M)$ such that $\Delta_\mu u = 0$ in Ω. Prove that

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Prove that if in addition Ω is the exterior of a compact set, then the hypothesis $u \in C(M) \cap W_0^1(M)$ can be relaxed to $u \in C(\overline{\Omega}) \cap W_0^1(M)$.

7.9. Assume that $u \in L^2_{loc}(M)$ and $\Delta_\mu u \in L^2_{loc}(M)$. Prove that $u \in W^1_{loc}(M)$ and, moreover, for any couple of open sets $\Omega' \Subset \Omega'' \Subset M$,

$$\|u\|_{W^1(\Omega')} \le C\left(\|u\|_{L^2(\Omega'')} + \|\Delta_\mu u\|_{L^2(\Omega'')}\right), \tag{7.3}$$

where the constant C depends on $\Omega', \Omega'', \mathbf{g}, \mu, n$. The space $W^1_{loc}(M)$ is defined in Exercise 5.8 by (**??**).

7.10. Prove that if $u \in \mathcal{D}'(M)$ and $\Delta_\mu u \in C^\infty(M)$ then $u \in C^\infty(M)$.

7.11. A function u on a weighted manifold M is called harmonic if $u \in C^\infty(M)$ and $\Delta_\mu u = 0$. Prove that if $\{u_k\}_{k=1}^\infty$ is a sequence of harmonic functions such that

$$u_k \xrightarrow{L^2_{loc}} u \in L^2_{loc}(M)$$

then (a version of) u is also harmonic. Moreover, prove that, in fact, $u_k \xrightarrow{C^\infty} u$.

7.12. Let $\{u_k\}$ be a sequence of functions from $L^2_{loc}(M)$ such that

$$-\Delta_\mu u_k + \alpha_k u_k = f_k, \tag{7.4}$$

for some $\alpha_k \in \mathbb{R}$ and $f_k \in \mathcal{W}^{2m}_{loc}(M)$, with a fixed non-negative integer m. Assume further that, as $k \to \infty$,

$$\alpha_k \to \alpha, \quad f_k \overset{\mathcal{W}^{2m}_{loc}}{\longrightarrow} f \text{ and } u_k \overset{L^2_{loc}}{\longrightarrow} u.$$

Prove that function u satisfies the equation

$$-\Delta_\mu u + \alpha u = f, \tag{7.5}$$

and that

$$u_k \overset{\mathcal{W}^{2m+2}_{loc}}{\longrightarrow} u. \tag{7.6}$$

Prove that if in addition $f_k \in C^\infty(M)$ and $f_k \overset{C^\infty}{\longrightarrow} f$ then (versions of) u_k and u belong to $C^\infty(M)$ and $u_k \overset{C^\infty}{\longrightarrow} u$.

7.13. Let $\{u_k\}$ be a sequence of non-negative functions from $C^\infty(M)$, which satisfy (7.4) with $\alpha_k \in \mathbb{R}$ and $f_k \in C^\infty(M)$. Assume further that, as $k \to \infty$,

$$\alpha_k \to \alpha, \quad f_k \overset{C^\infty}{\longrightarrow} f \text{ and } u_k(x) \uparrow u(x) \text{ for any } x \in M,$$

where $u(x)$ is a function from L^2_{loc} that is defined pointwise. Prove that $u \in C^\infty(M)$ and $u_k \overset{C^\infty}{\longrightarrow} u$.

7.14. Prove that, for any relatively compact open set $\Omega \subset M$, for any set $K \Subset \Omega$, and for any $\alpha \in \mathbb{R}$, there exists a constant $C = C(K, \Omega, \alpha)$ such that, for any smooth solution to the equation $-\Delta_\mu u + \alpha u = 0$ on M,

$$\sup_K |u| \le C \|u\|_{L^2(\Omega)}.$$

7.15. Let R_α be the resolvent operator defined in Section 4, that is, $R_\alpha = (\mathcal{L} + \alpha\,\mathrm{id})^{-1}$, where $\alpha > 0$. Prove that if $f \in L^2 \cap C^\infty(M)$ then also $R_\alpha f \in L^2 \cap C^\infty(M)$.

7.16. Let $\{\Omega_i\}$ be an exhaustion sequence in M. Prove that, for any non-negative function $f \in L^2 \cap C^\infty(M)$ and any $\alpha > 0$,

$$R^{\Omega_i}_\alpha f \overset{C^\infty}{\longrightarrow} R_\alpha f \text{ as } i \to \infty.$$

Hint. Use that $R^{\Omega_i}_\alpha f \overset{L^2}{\longrightarrow} R_\alpha f$ (cf. Theorem 5).

7.17. Prove that, for any compact set $K \subset M$, for any $f \in L^2(M, \mu)$, and for any positive integer m,

$$\sup_K \left|\Delta^m_\mu (P_t f)\right| \le C t^{-m} \left(1 + t^{-\sigma}\right) \|f\|_2, \tag{7.7}$$

where σ is the smallest integer larger than $n/4$.

7.18. Let f be a non-negative function from $L^2(M)$ and $\{\Omega_i\}$ be an exhaustion sequence in M. Prove that

$$P^{\Omega_i}_t f \overset{C^\infty(\mathbb{R}_+ \times M)}{\longrightarrow} P_t f \text{ as } i \to \infty.$$

Hint. Use the fact that, for any $t > 0$,

$$P^{\Omega_i}_t f \overset{\text{a.e.}}{\longrightarrow} P_t f \text{ as } i \to \infty$$

(cf. Theorem 5).

7.19. Prove that if $f \in C_0^\infty(M)$ then

$$P_t f \overset{C^\infty(M)}{\longrightarrow} f \text{ as } t \to 0.$$

7.20. Consider the cos-wave operator

$$C_t = \cos\left(t\mathcal{L}^{1/2}\right)$$

(cf. Exercise 4.52). Prove that, for any $f \in C_0^\infty(M)$, the function

$$u(t,x) = C_t f(x)$$

belongs to $C^\infty(\mathbb{R} \times M)$ and solves in $\mathbb{R} \times M$ the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta_\mu u$$

with the initial conditions

$$u(0,x) = f(x) \text{ and } \frac{\partial u}{\partial t}(0,x) = 0.$$

7.21. Prove that, for all $x, y \in M$ and $t > 0$,

$$p_t(x,y) \le \sqrt{p_t(x,x)\, p_t(y,y)}. \tag{7.8}$$

7.22. Prove that, for all $x \in M$, the functions $p_t(x,x)$ and $\|p_{t,x}\|_2$ are non-increasing in t.

7.23. Let $K \subset M$ be a compact set.

(a) Prove that the function

$$S(t) := \sup_{x,y \in K} p_t(x,y)$$

is non-increasing in $t > 0$.

(b) Prove that, for all $t > 0$,

$$S(t) \le C\left(1 + t^{-\alpha}\right),$$

for some constants $\alpha, C > 0$, where C depends on K.

7.24. Let J be an isometry of a weighted manifold M (see Section 3). Prove that

$$p_t(Jx, Jy) \equiv p_t(x,y).$$

7.25. Prove that, for any two non-negative measurable functions f and g on M,

$$(P_t(fg))^2 \le P_t\left(f^2\right) P_t\left(g^2\right).$$

Prove that

$$(P_t f)^2 \le P_t\left(f^2\right).$$

7.26. Prove that the following dichotomy takes place: either $\sup P_t 1 = 1$ for all $t > 0$ or there is $c > 0$ such that

$$\sup P_t 1 \le \exp(-ct)$$

for all large enough t.

7.27. Prove that, for any fixed $t > 0$ and $x \in M$, the heat kernel $p_t(x,y)$ is a bounded function of $y \in M$.

7.28. Let $\mathcal{F}$ be a set of functions on M such that $f \in \mathcal{F}$ implies $|f| \in \mathcal{F}$ and $P_t f \in \mathcal{F}$.

(a) Prove that the semigroup identity
$$P_t P_s = P_{t+s}$$
holds in $\mathcal{F}$.

(b) Assume in addition that $\mathcal{F}$ is a normed linear space such that, for any $f \in \mathcal{F}$,
$$\|P_t f\|_{\mathcal{F}} \le \|f\|_{\mathcal{F}}$$
and
$$\|P_t f - f\|_{\mathcal{F}} \to 0 \text{ as } t \to 0.$$
Prove that, for any $s > 0$,
$$\|P_t f - P_s f\|_{\mathcal{F}} \to 0 \ \text{ as } t \to s.$$

7.29. Let $f \in W^1_{loc}(M)$ be a non-negative function such that $\Delta_\mu f \le 0$ in the distributional sense. Prove that $P_t f \le f$ for all $t > 0$.

7.30. Let $f \in L^1_{loc}(M)$ be a non-negative function such that $P_t f \le f$ for all $t > 0$.

(a) Prove that $P_t f(x)$ is decreasing in t for any $x \in M$.
(b) Prove that $P_t f$ is a smooth solution to the heat equation in $\mathbb{R}_+ \times M$.
(c) Prove that $P_t f \xrightarrow{L^1_{loc}} f$ as $t \to 0$.
(d) Prove that $\Delta_\mu f \le 0$ in the distributional sense.

7.31. Under the conditions of Exercise 7.30, assume in addition that $\Delta_\mu f = 0$ in an open set $U \subset M$. Prove that the function
$$u(t,x) = \begin{cases} P_t f(x), & t > 0, \\ f(x), & t \le 0, \end{cases}$$
is C^∞ smooth in $\mathbb{R} \times U$ and solves the heat equation in $\mathbb{R} \times U$.

REMARK. The assumption $P_t f \le f$ simplifies the proof but is not essential – cf. Exercise 9.8(c).

7.32. Let $f \in L^1_{loc}(M)$ be a non-negative function such that $P_t f \in L^1_{loc}(M)$ for all $t \in (0,T)$ (where $T > 0$) and $P_t f \ge f$ for all $t \in (0,T)$.

(a) Prove that $P_t f(x)$ is increasing in t for any $x \in M$.
(b) Prove that $P_t f$ is a smooth solution to the heat equation in $(0,T) \times M$.
(c) Prove that $P_t f \xrightarrow{L^1_{loc}} f$ as $t \to 0$.
(d) Prove that $\Delta_\mu f \ge 0$ in the distributional sense.
(e) Show that the function $f(x) = \exp\left(\frac{|x|^2}{4T}\right)$ in $\mathbb{R}^n$ satisfies the above conditions.

7.33. Let $f \in L^\infty(M)$. Prove that $P_t f \in L^\infty(M)$ for any $t > 0$,
$$\|P_t f\|_{L^\infty} \le \|f\|_{L^\infty},$$
and the function $u(t,x) = P_t f(x)$ is C^∞ smooth in $\mathbb{R}_+ \times M$ and satisfies the heat equation.

7.34. Let $\Omega \subset M$ be an open set, and consider the function
$$f(x) = 1_\Omega(x) := \begin{cases} 1, & x \in \Omega \\ 0, & x \in M \setminus \Omega. \end{cases}$$
Prove that
$$\lim_{t \to 0} P_t f(x) = f(x) \text{ for all } x \in M \setminus \partial\Omega, \tag{7.9}$$

and the convergence is locally uniform in x.

7.35. Prove that if a function $f \in L^{\infty}(M)$ is continuous at a point $x \in M$ then

$$P_t f(x) \to f(x) \text{ as } t \to 0. \tag{7.10}$$

7.36. Let $1 \le r \le \infty$ and $f \in L^r(M)$.

(a) Prove that $P_t f \in L^r(M)$ for any $t > 0$, and

$$\|P_t f\|_{L^r} \le \|f\|_{L^r}. \tag{7.11}$$

(b) Prove that $P_t f(x)$ is a smooth function of $(t,x) \in \mathbb{R}_+ \times M$ and satisfies the heat equation.

7.37. Prove that if $1 < r < \infty$ and $f \in L^r(M)$ then $P_t f \xrightarrow{L^r} f$ as $t \to 0$.

7.38. Assume that

$$F(t) := \sup_{x \in M} p_t(x,x) < \infty.$$

Prove that, for all $1 \le r < s \le +\infty$, $f \in L^r(M)$ implies $P_t f \in L^s(M)$ and

$$\|P_t f\|_{L^s} \le F(t)^{1/r - 1/s} \|f\|_{L^r}. \tag{7.12}$$

7.39. Let $f : M \to [-\infty, +\infty]$ be a measurable function on M.

(a) Prove that, if $f \ge 0$ then the function

$$P_t f(x) := \int_M p_t(x,y) f(y) \, d\mu(y) \tag{7.13}$$

is measurable on M for any $t > 0$.

(b) Prove that if f is signed and the integral (7.13) converges for almost all x then $P_t f(x)$ is measurable on M.

(c) Prove the identity

$$P_{t+s} f = P_t(P_s f)$$

for any non-negative measurable function f.

7.40. For any open set $\Omega \subset M$, denote by $p_t^{\Omega}(x,y)$ the heat kernel of the manifold $(\Omega, \mathbf{g}, \mu)$.

(a) Prove that $p_t^{\Omega}(x,y) \le p_t(x,y)$ for all $x, y \in \Omega$ and $t > 0$.

(b) Let $\{\Omega_i\}$ be an exhaustion sequence in M. Prove that

$$p_t^{\Omega_i}(x,y) \xrightarrow{C^{\infty}(\mathbb{R}_+ \times M \times M)} p_t(x,y) \quad \text{as } i \to \infty.$$

(c) Prove that, for any non-negative measurable function $f(x)$,

$$P_t^{\Omega_i} f(x) \to P_t f(x) \quad \text{as } i \to \infty,$$

for any fixed $t > 0$ and $x \in M$.

(d) Prove that if $f \in C_b(M)$ then

$$P_t^{\Omega_i} f(x) \xrightarrow{C^{\infty}(\mathbb{R}_+ \times M)} P_t f(x) \quad \text{as } i \to \infty.$$

7.41. Let $(X, \mathbf{g}_X, \mu_X)$ and $(Y, \mathbf{g}_Y, \mu_Y)$ be two weighted manifold and $(M, \mathbf{g}, \mu)$ be their direct product (see Section 3). Denote by p_t^X and p_t^Y the heat kernels on X and Y, respectively. Prove that the heat kernel p_t on M satisfies the identity

$$p_t\left((x,y),(x',y')\right) = p_t^X(x,x')\, p_t^Y(y,y'), \tag{7.14}$$

for all $t > 0$, $x, x' \in X$, $y, y' \in Y$ (note that (x,y) and (x',y') are points on M).

7.42. For any $t > 0$, consider the quadratic form in $L^2(M)$, defined by

$$\mathcal{E}_t(f) = \left(\frac{f - P_t f}{t}, f\right)_{L^2}$$

(cf. Exercise 4.38). Prove that if the heat kernel is stochastically complete, that is, for all $x \in M$ and $t > 0$,

$$\int_M p_t(x, y)\, d\mu(y) = 1, \tag{7.15}$$

then the following identity holds:

$$\mathcal{E}_t(f) = \frac{1}{2t}\int_M \int_M (f(x) - f(y))^2 p_t(x, y) d\mu(y) d\mu(x), \tag{7.16}$$

for all $t > 0$ and $f \in L^2(M)$.

7.43. Prove that, for any real $k > 0$ and for any $f \in L^2(M)$,

$$(\mathcal{L} + \mathrm{id})^{-k} f(x) = \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-t} P_t f(x)\, dt, \tag{7.17}$$

for almost all $x \in M$, where Γ is the gamma function.

HINT. Use Exercise 5.11.

7.44. Assume that the heat kernel satisfies the following condition

$$p_t(x, x) \leq c t^{-\gamma} \text{ for all } x \in M \text{ and } 0 < t < 1. \tag{7.18}$$

where $\gamma, c > 0$. Fix a real number $k > \gamma/2$.

(a) Prove that, for any $f \in L^2(M)$, the function $(\mathcal{L} + \mathrm{id})^{-k} f$ is continuous and

$$\sup_M \left|(\mathcal{L} + \mathrm{id})^{-k} f\right| \leq C\|f\|_{L^2}, \tag{7.19}$$

where $C = C(c, \gamma, k)$.

(b) Prove that, for any $u \in \operatorname{dom} \mathcal{L}^k$, we have $u \in C(M)$ and

$$\sup_M |u| \leq C\left(\|u\|_{L^2} + \|\mathcal{L}^k u\|_{L^2}\right). \tag{7.20}$$

7.45. Prove that if (7.20) holds for all $u \in \operatorname{dom} \mathcal{L}^k$ with some $k > 0$ then the heat kernel satisfies the estimate (7.18) with $\gamma = 2k$.

7.46. The purpose of this question is to give an alternative proof of Theorem 6 (Sobolev embedding theorem).

(a) Prove that if $u \in W^k(\mathbb{R}^n)$ where k is a positive integer then $u \in \operatorname{dom} \mathcal{L}^{k/2}$, where $\mathcal{L}$ is the Dirichlet Laplace operator in $\mathbb{R}^n$. Prove also that, for any $u \in W^k(\mathbb{R}^n)$,

$$\|(\mathcal{L} + \mathrm{id})^{k/2} u\|_{L^2} \leq C\|u\|_{W^k},$$

where C is a constant depending only on n and k.

(b) Prove that if $u \in W^k(\mathbb{R}^n)$ where k is an integer such that $k > n/2$ then $u \in C(\mathbb{R}^n)$ and

$$\sup_{\mathbb{R}^n} |u| \leq C\|u\|_{W^k}.$$

(c) Prove that if $k > m + n/2$ where m is a positive integer then $u \in W^k(\mathbb{R}^n)$ implies $u \in C^m(\mathbb{R}^n)$ and

$$\|u\|_{C^m(\mathbb{R}^n)} \leq C\|u\|_{W^k(\mathbb{R}^n)}.$$

(d) Prove that if Ω is an open subset of $\mathbb{R}^n$ and k and m are non-negative integers such that $k > m + n/2$ then $u \in W^k_{loc}(\Omega)$ implies $u \in C^m(\Omega)$. Moreover, for any open sets $\Omega' \Subset \Omega'' \Subset \Omega$,

$$\|u\|_{C^m(\Omega')} \leq C\|u\|_{W^k(\Omega'')},$$

with a constant C depending on $\Omega', \Omega'', k, m, n$.

HINT. Use Exercise 4.25 for part (a) and Exercise 7.44 for part (b)

7.47. (*Compact embedding theorems*)

(a) Assume that $\mu(M) < \infty$ and

$$\sup_{x \in M} p_t(x, x) < \infty \text{ for all } t > 0. \tag{7.21}$$

Prove that the identical embedding $W_0^1(M) \hookrightarrow L^2(M)$ is a compact operator.

(b) Prove that, on any weighted manifold M and for any non-empty relatively open compact set $\Omega \subset M$, the identical embedding $W_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is a compact operator (cf. Theorem 6 and Corollary 10).

HINT. Use for part (a) the weak compactness of bounded sets in L^2 and Exercises 7.36, 4.40.

7.48. Let I be an open interval in $\mathbb{R}$ and $\mathcal{H}$ be a Hilbert space. Prove that if a mapping $h : I \to \mathcal{H}$ is weakly differentiable then h is strongly continuous.

CHAPTER 8

Positive solutions

8.1. Prove that if h is a non-negative function satisfying on M the equation

$$-\Delta_\mu h + \alpha h = 0,$$

where α is a real constant, then $P_t h \leq e^{\alpha t} h$ for all $t > 0$.

8.2. If $u \in L^2_{loc}(M)$ is a non-negative solution to the equation

$$-\Delta_\mu u + \alpha u = f$$

where $\alpha > 0$ and $f \in L^2_{loc}(M)$, $f \geq 0$. Prove that if

$$u(x) \to 0 \text{ as } x \to \infty,$$

then $u = R_\alpha f$.

8.3. Let $u \in L^2(M)$ satisfy in M the equation

$$\Delta_\mu u + \lambda u = 0,$$

where $\lambda \in \mathbb{R}$, and

$$u(x) \to 0 \text{ as } x \to \infty.$$

Prove that $u \in W^1_0(M)$.

Remark. Since by the equation $\Delta_\mu u \in L^2(M)$, it follows that $u \in \operatorname{dom}(\mathcal{L})$ and, hence, u satisfies the equation $\mathcal{L}u = -\lambda u$. Assuming that $u \neq 0$ we obtain that u is an eigenfunction of the Dirichlet Laplace operator.

8.4. Let M be a connected weighted manifold and E, F be two compact subsets of M. Prove that, for any real α there is a constant $C = C(\alpha, E, F)$ such that, for any non-negative α-superharmonic function u on M,

$$\inf_E u \leq C \inf_F u.$$

8.5. *(A version of the elliptic minimum principle)* Let M be a non-compact connected weighted manifold and let $u(t,x) \in C^2(M)$ be a superharmonic function. Prove that if

$$\limsup_{k\to\infty} u(x_k) \geq 0 \tag{8.1}$$

for any sequence $\{x_k\}$ such that $x_k \to \infty$ in M, then $u(x) \geq 0$ for all $x \in M$.

8.6. *(A version of the parabolic minimum principle)* Fix $T \in (0, +\infty]$ and consider the manifold $N = (0,T) \times M$. We say that a sequence $\{(t_k, x_k)\}_{k=1}^\infty$ of points in N escapes from N if one of the following two alternatives takes place as $k \to \infty$:

1. $x_k \to \infty$ in M and $t_k \to t \in [0,T]$;
2. $x_k \to x \in M$ and $t_k \to 0$.

Let $u(t,x) \in C^2(N)$ be a supersolution to the heat equation in N. Prove that if

$$\limsup_{k\to\infty} u(t_k, x_k) \geq 0 \tag{8.2}$$

for any sequence $\{(t_k, x_k)\}$ that escapes from N, then $u(t,x) \geq 0$ for all $(t,x) \in N$.

8.7. Prove that any compact weighted manifold is stochastically complete.

8.8. Prove that $\mathbb{R}^n$ is stochastically complete (cf. Exercise 8.11).

8.9. Prove that if $P_t 1(x) = 1$ for some $t > 0$, $x \in M$ then $P_t 1(x) = 1$ for all $t > 0$, $x \in M$.

8.10. Fix $\alpha > 0$. Prove that M is stochastically complete if and only if $R_\alpha 1 \equiv \alpha^{-1}$.

8.11. Prove the following claims.

(a) $\mathbb{R}^n$ is stochastically complete for all $n \geq 1$. (cf. Exercise 8.8).

(b) $\mathbb{R}^n \setminus \{0\}$ is stochastically complete if $n \geq 2$, whereas $\mathbb{R}^1 \setminus \{0\}$ is stochastically incomplete.

(c) Any open set $\Omega \subset \mathbb{R}^n$ such that $\overline{\Omega} \neq \mathbb{R}^n$, is stochastically incomplete.

8.12. Let Ω be an open subset of $\mathbb{R}^n$ and h be a positive smooth function in Ω such that

$$\begin{cases} \Delta h = 0 \text{ in } \Omega, \\ h(x) \to 0 \text{ as } x \to \partial\Omega, \\ h(x) = e^{O(|x|)} \text{ as } |x| \to \infty \end{cases}$$

Prove that $P_t^\Omega h = h$ for all $t > 0$.

8.13. Let f be a non-negative superharmonic function on M.

(a) Prove that the function

$$v(x) := \lim_{t\to\infty} P_t f(x) \tag{8.3}$$

satisfies the identity $P_t v = v$ for all $t > 0$ and, hence, is harmonic on M (the limit in (8.3) exists and is finite because by Exercise 7.29 the function $P_t f(x)$ is finite and decreases in t).

(b) Assume in addition that manifold M is stochastically complete and f is bounded. Prove that, for any non-negative harmonic function h on M, the condition $h \leq f$ implies $h \leq v$.

Remark. The maximal non-negative harmonic function that is bounded by f is called the largest harmonic minorant of f. Hence, the function v is the largest harmonic minorant of f.

8.14. Set $v(x) = \lim_{t\to\infty} P_t 1(x)$. Prove that either $v \equiv 0$ or $\sup v = 1$. Prove also that either $v \equiv 1$ or $\inf v = 0$.

8.15. Let Ω be the exterior of the unit ball in $\mathbb{R}^n$, $n \geq 2$. Evaluate $\lim_{t\to\infty} P_t^\Omega 1(x)$.

8.16. *(A model with two ends)* Set $M = \mathbb{R} \times \mathbb{S}^{n-1}$ (where $n \geq 1$) so that every point $x \in M$ can be represented as a couple (r, θ) where $r \in \mathbb{R}$ and $\theta \in \mathbb{S}^{n-1}$. Fix smooth positive functions $\psi(r)$ and $\Upsilon(r)$ on $\mathbb{R}$, and consider the Riemannian metric on M

$$\mathbf{g} = dr^2 + \psi^2(r)\, \mathbf{g}_{\mathbb{S}^{n-1}},$$

and measure μ on $(M, \mathbf{g})$ with the density function Υ. Define the area function $S(r)$ by

$$S(r) = \omega_n \Upsilon(r) \psi^{n-1}(r)$$

and volume function $V(R)$ by

$$V(R) = \int_{[0,R]} S(r)\, dr,$$

so that $V(R) \geq 0$.

(a) Show that the expression (**??**) for Δ_μ remains true in this setting.
(b) Prove that if function $V(r)$ is even then the following are equivalent:
(i) $(M, \mathbf{g}, \mu)$ is stochastically complete.
(ii) There is a non-constant non-negative harmonic function $u \in L^1(M, \mu)$.
(iii) $\int^\infty \frac{V(r)}{S(r)} dr = \infty$.
(c) Let $S(r)$ satisfy the following relations for some $\alpha > 2$:

$$S(r) = \begin{cases} \exp(r^\alpha), & r > 1, \\ \exp(-|r|^\alpha), & r < -1. \end{cases}$$

Prove that $(M, \mathbf{g}, \mu)$ is stochastically incomplete. Prove that any non-negative harmonic function $u \in L^1(M, \mu)$ is identical zero.

CHAPTER 9

Heat kernel as a fundamental solution

9.1. Let μ be a measure in $\mathbb{R}^n$ defined by

$$d\mu = \exp(2c \cdot x)\, dx$$

where dx is the Lebesgue measure and c is a constant vector from $\mathbb{R}^n$. Prove that the heat kernel of $(\mathbb{R}^n, \mathbf{g}_{\mathbb{R}^n}, \mu)$ is given by

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-c \cdot (x + y) - |c|^2 t - \frac{|x - y|^2}{4t}\right). \tag{9.1}$$

9.2. *(Heat kernel in half-space)* Let

$$M = \left\{ \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n : x^n > 0 \right\}.$$

Prove that the heat kernel of M with the canonical Euclidean metric and the Lebesgue measure is given by

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \left(\exp\left(-\frac{|x - y|^2}{4t}\right) - \exp\left(-\frac{|x - \overline{y}|^2}{4t}\right) \right) \tag{9.2}$$

where $\overline{y}$ is the reflection of y at the hyperplane $x^n = 0$, that is,

$$\overline{y} = \left(y^1, \ldots, y^{n-1}, -y^n\right).$$

9.3. *(Heat kernel in Weyl's chamber)* Let

$$M = \left\{ \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n : x^1 < x^2 < \cdots < x^n \right\}.$$

Prove that the heat kernel of M with the canonical Euclidean metric and the Lebesgue measure is given by

$$p_t(x, y) = \det\left(p_t^{\mathbb{R}^1}\left(x^i, y^j\right)\right)_{i,j=1}^{n}, \tag{9.3}$$

where $p_t^{\mathbb{R}^1}$ is the heat kernel in $\mathbb{R}^1$.

9.4. Let $(M, \mathbf{g}, \mu)$ be a weighted manifold, and let h be a smooth positive function on M satisfying the equation

$$-\Delta_\mu h + \Phi h = 0, \tag{9.4}$$

where Φ is a smooth function on M. Define measure $\widetilde{\mu}$ on M by $d\widetilde{\mu} = h^2 d\mu$.

(*a*) Prove that, for any $f \in C^\infty(M)$,

$$\Delta_\mu f - \Phi f = h \Delta_{\widetilde{\mu}}\left(h^{-1} f\right). \tag{9.5}$$

(*b*) Prove that, for any $f \in \mathcal{D}(M)$,

$$\int_M \left(|\nabla f|^2 + \Phi f^2\right) d\mu \geq 0. \tag{9.6}$$

9.5. Applying (9.6) in $\mathbb{R}^n\setminus\{0\}$ with suitable functions h and Φ, prove the *Hardy inequality*: for any $f\in\mathcal{D}\left(\mathbb{R}^n\setminus\{0\}\right)$,

$$\int_{\mathbb{R}^n}|\nabla f|^2\,dx\geq\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{f^2}{|x|^2}dx. \tag{9.7}$$

9.6. Prove that if u and v are two regular fundamental solutions at point $y\in M$ then the difference $u-v$ is a C^∞-smooth function on $\mathbb{R}\times M$ satisfying in $\mathbb{R}\times M$ the heat equation.

9.7. Let $\Omega\subset M$ be an open set. Prove that the function $u_t(x,y):=p_t(x,y)-p_t^\Omega(x,y)$ is C^∞ smooth jointly in $t\in\mathbb{R}$ and $x,y\in\Omega$.

9.8. Let a smooth function $u(t,x)$ on $\mathbb{R}_+\times M$ satisfy the following conditions

$$\begin{cases}\frac{\partial u}{\partial t}=\Delta_\mu u & \text{in } \mathbb{R}_+\times M,\\ u(t,\cdot)\overset{L^1_{loc}}{\longrightarrow}f & \text{as } t\to 0,\end{cases} \tag{9.8}$$

where $f\in L^1_{loc}(M)$. Extend $u(t,x)$ to $t\leq 0$ by setting $u(t,x)\equiv 0$.

(*a*) Prove that the function $u(t,x)$ satisfies in $\mathbb{R}\times M$ the equation

$$\frac{\partial u}{\partial t}-\Delta_\mu u=F, \tag{9.9}$$

where F is a distribution on $\mathbb{R}\times M$ defined by

$$(F,\varphi)=\int_M\varphi(0,x)\,f(x)\,d\mu(x),$$

for any $\varphi\in\mathcal{D}(\mathbb{R}\times M)$.

(*b*) Prove that if in (9.8) $f\equiv 0$ in M then $u\in C^\infty(\mathbb{R}\times M)$.

(*c*) Prove that if $f\in C^\infty(M)$ then

$$u(t,\cdot)\overset{C^\infty(M)}{\longrightarrow}f \text{ as } t\to 0+.$$

Consequently, the function

$$\widetilde{u}(t,x)=\begin{cases}u(t,x), & t>0,\\ f(x), & t\leq 0,\end{cases}$$

belongs to $C^\infty(\mathbb{R}\times M)$.

Hint. Use Exercise 7.19.

9.9. Prove that, on any weighted manifold M, for any open set Ω, any compact set $K\subset\Omega$, and any $N>0$,

$$\sup_{x\in K}\int_{\Omega^c}p_t(x,y)\,d\mu(y)=o\left(t^N\right) \text{ as } t\to 0. \tag{9.10}$$

9.10. Define the *resolvent kernel* $r_\alpha(x,y)$ by

$$r_\alpha(x,y)=\int_0^\infty e^{-\alpha t}p_t(x,y)\,dt. \tag{9.11}$$

Prove that, for any $\alpha>0$, $r_\alpha(x,y)$ is a non-negative smooth function on $M\times M\setminus\operatorname{diag}$. Furthermore, for any $y\in M$, $r_\alpha(\cdot,y)$ satisfies the equation

$$-\Delta_\mu r_\alpha+\alpha r_\alpha=\delta_y. \tag{9.12}$$

CHAPTER 10

Spectral properties

10.1. Let (X, d) be a separable metric space and $S \subset X$ be a subset of X. Prove that if all points of S are isolated then S is at most countable.

10.2. Prove that, for any Borel set U,

$$m(U) = \operatorname{trace} E_U.$$

10.3. Prove that if A is a non-negative definite self-adjoint operator with a finite trace then A is a compact operator.

10.4. For any non-negative definite operator A with $\operatorname{dom} A = \mathcal{H}$, define its trace by

$$\operatorname{trace} A = \sum_k (Av_k, v_k),$$

where $\{v_k\}$ is any orthonormal basis of $\mathcal{H}$. Prove that the trace does not depend on the choice of the basis $\{v_k\}$.

10.5. Prove that, for any $f \in L^2(M)$,

$$(P_t f, f) \le \exp(-\lambda_{\min}(M)\, t) \|f\|_{L^2}^2.$$

10.6. Prove the following properties of $\lambda_{\min}$ for subsets of a weighted manifold M.

(a) If $\Omega_1 \subset \Omega_2$ are two open sets then

$$\lambda_{\min}(\Omega_1) \ge \lambda_{\min}(\Omega_2).$$

(b) If $\{\Omega_k\}$ is a finite or countable sequence of disjoint open sets and $\Omega = \bigcup_k \Omega_k$ then

$$\lambda_{\min}(\Omega) = \inf_k \lambda_{\min}(\Omega_k).$$

(c) If $\{\Omega_k\}_{k=1}^{\infty}$ is an increasing sequence of open sets and $\Omega = \bigcup_k \Omega_k$ then

$$\lambda_{\min}(\Omega) = \lim_{k\to\infty} \lambda_{\min}(\Omega_k).$$

10.7. Let $(M, \mathbf{g}, \mu)$ and $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$ be two weighted manifolds based on the same smooth manifold M of dimension n. Assume that they are quasi-isometric, that is, for some positive constant A and B,

$$A^{-1} \le \frac{\widetilde{\mathbf{g}}}{\mathbf{g}} \le A \quad \text{and} \quad B^{-1} \le \frac{\widetilde{\Upsilon}}{\Upsilon} \le B, \tag{10.1}$$

where Υ and $\widetilde{\Upsilon}$ are the density functions of measures μ and $\widetilde{\mu}$ respectively. Prove that

$$C^{-1}\lambda_{\min}(M) \le \widetilde{\lambda}_{\min}(M) \le C\lambda_{\min}(M) \tag{10.2}$$

where $C = C(A, B, n)$ is a positive constant, $\lambda_{\min}(M)$ is the bottom of the spectrum of the Dirichlet Laplacian on $(M, \mathbf{g}, \mu)$, and $\widetilde{\lambda}_{\min}(M)$ is the bottom of the spectrum of the Dirichlet Laplacian on $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$.

10.8. (*Cheeger's inequality*) The Cheeger constant of a manifold is defined by

$$h(M) := \inf_{f \in C_0^\infty(M) \setminus \{0\}} \frac{\int_M |\nabla f| \, d\mu}{\int_M |f| \, d\mu}. \tag{10.3}$$

Prove that

$$\lambda_{\min}(M) \geq \frac{1}{4} h^2(M). \tag{10.4}$$

10.9. In the setting of Lemma 10, prove that the integral operator Q is compact without using the trace.

10.10. Let M be a compact weighted manifold, which has a finite number m of connected components.

(a) Prove that $\lambda_1(M) = ... = \lambda_m(M) = 0$ and $\lambda_{m+1}(M) > 0$.
(b) Show that the estimate (10) holds for all $k \geq m+1$ and does not hold for $k \leq m$.

10.11. Let M be a compact connected weighted manifold. Prove that

$$p_t(x,y) \rightrightarrows \frac{1}{\mu(M)} \text{ as } t \to \infty,$$

where the convergence is uniform for all $x, y \in M$.

10.12. Let Ω be a non-empty relatively compact connected open subset of a weighted manifold M. Using the notation of Theorem 10, prove that, for all $x, y \in \Omega$,

$$p_t^\Omega(x,y) \sim e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y) \text{ as } t \to \infty.$$

10.13. Prove that, under the conditions of Theorem 10,

$$\sup_{x \in \Omega} |\varphi_k(x)| \leq C\left(1 + \lambda_k^\sigma\right), \text{ for all } k \geq 1, \tag{10.5}$$

where σ is the exponent from (**??**) and C is a constant that does not depend on k.

10.14. Let $(M, \mathbf{g}, \mu)$ be a weighted manifold with the discrete spectrum. Let $\{\varphi_k\}$ be an orthonormal basis in $L^2(M)$ that consists of the eigenfunctions of M, and let λ_k be the eigenvalue of φ_k.

(a) Prove that, for any $f \in L^2(M)$, if $f = \sum_k a_k \varphi_k$ is the expansion of f in the basis $\{\varphi_k\}$ in $L^2(M)$ then

$$P_t f = \sum_k e^{-\lambda_k t} a_k \varphi_k, \tag{10.6}$$

where the series converges in $L^2(M)$ for any $t > 0$. Show also that the series converges in $C^\infty(\mathbb{R}_+ \times M)$.

(b) Assume in addition that

$$\operatorname{trace} P_t = \sum_k e^{-\lambda_k t} < \infty$$

for all $t > 0$. Prove that

$$p_t(x,y) = \sum_k e^{-\lambda_k t} \varphi_k(x) \varphi_k(y), \tag{10.7}$$

where the series converges in $C^\infty(\mathbb{R}_+ \times M \times M)$.

10.15. On an arbitrary weighted manifold, consider the resolvent $R = (\mathrm{id} + \mathcal{L})^{-1}$ and its powers $R^s = (\mathrm{id} + \mathcal{L})^{-s}$, where $\mathcal{L}$ is the Dirichlet Laplace operator and $s > 0$.

(a) Prove that

$$\operatorname{trace} R^s = \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} e^{-t} \operatorname{trace} P_t dt. \tag{10.8}$$

(b) Assuming in addition that $\mu(M) < \infty$ and

$$p_t(x,x) \leq C t^{-\nu} \text{ for all } 0 < t < 1,\ x \in M,$$

where C and ν are positive constants, prove that $\operatorname{trace} R^s$ is finite for all $s > \nu$.

10.16. Let Ω be a relatively compact open subset of a weighted manifold M of dimension n. Let $\{\varphi_k\}$ be an orthonormal basis in $L^2(\Omega)$ that consists of the eigenfunctions of M, and let $\{\lambda_k\}$ be the sequence of the corresponding eigenvalues.

(a) Prove that if $s > s_0 = s_0(n)$ then

$$\sum_{k:\lambda_k > 0}^{\infty} \lambda_k^{-s} < \infty. \tag{10.9}$$

(b) Prove that if $f \in C_0^\infty(\Omega)$ then the Fourier series

$$f = \sum_k c_k \varphi_k$$

of function f converges to f absolutely and uniformly in Ω.

10.17. Let $(M, \mathbf{g}, \mu)$ be a compact weighted manifold and $\{\varphi_k\}$ be an orthonormal basis in $L^2(M)$ that consists of the eigenfunctions of M. Prove that the set of all finite linear combinations of functions φ_k is dense in $C(M)$.

Remark. This can be considered as a generalization of the classical Stone-Weierstrass theorem that any continuous 2π-periodic function on $\mathbb{R}$ can be uniformly approximated by trigonometric polynomials.

10.18. In this problem, the circle $\mathbb{S}^1$ is identified with $\mathbb{R}/2\pi\mathbb{Z}$.

(i) Prove that the heat kernel $p_t(x,y)$ of $\mathbb{S}^1$ is given by

$$p_t(x,y) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t} \cos k(x-y). \tag{10.10}$$

(ii) Show that the heat kernel $p_t(x,y)$ of $\mathbb{S}^1$ can be obtained from the heat kernel $\widetilde{p}_t(x,y)$ of $\mathbb{R}^1$ by

$$p_t(x,y) = \sum_{n\in\mathbb{Z}} \widetilde{p}_t(x + 2\pi n, y). \tag{10.11}$$

(iii) Prove the Poisson summation formula

$$\sum_{k\in\mathbb{Z}} e^{-k^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\pi^2 n^2}{t}\right). \tag{10.12}$$

10.19. Let $P(x)$ be a homogeneous of degree k harmonic polynomial on $\mathbb{R}^{n+1}$. Prove that the function $f = P|_{\mathbb{S}^n}$ is an eigenfunction of the Laplacian of $\mathbb{S}^n$ with the eigenvalue $\alpha = k(k+n-1)$.

Remark. It is possible to prove that such eigenfunctions exhaust all eigenfunctions on $\mathbb{S}^n$.

10.20. Consider the weighted manifold $(\mathbb{R}, \mathbf{g}_{\mathbb{R}}, \mu)$ where $d\mu = e^{-x^2}dx$. Prove that the spectrum of this manifold is discrete, its eigenvalues are $\lambda_k = 2k$, $k = 0, 1, ...,$ and the eigenfunctions are $h_k(x)$ – the Hermite polynomials (see Exercise 3.10). Hence, show that the heat kernel of this manifold satisfies the identity

$$p_t(x,y) = \sum_{k=0}^{\infty} e^{-2kt} \frac{h_k(x)\, h_k(y)}{\sqrt{\pi} 2^k k!}. \tag{10.13}$$

Remark. The same heat kernel is given by the formula

$$p_t(x,y) = \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(\frac{2xye^{-2t} - (x^2+y^2)\, e^{-4t}}{1-e^{-4t}} + t\right),$$

cf. Example 9.

10.21. Let $(M, \mathbf{g}, \mu)$ be a weighted manifold with discrete spectrum, and let $\{\varphi_k\}$ be an orthonormal basis in $L^2(M)$ of the eigenfunctions of M with eigenvalues $\{\lambda_k\}$.

(a) Prove that $\{\varphi_k\}$ is an orthogonal basis also in $W_0^1(M)$.

(b) Let $f \in L^2(M)$ and assume that $f = \sum_k a_k \varphi_k$ is its expansion in the basis $\{\varphi_k\}$ in $L^2(M)$. Prove that if, in addition, $f \in W_0^1(M)$ then

$$\nabla f = \sum_k a_k \nabla \varphi_k \quad \text{in } \vec{L}^2(M) \tag{10.14}$$

and

$$\int_M |\nabla f|^2 \, d\mu = \sum_k \lambda_k a_k^2. \tag{10.15}$$

(c) Prove that if $f \in W_0^2(M)$ then

$$-\Delta_\mu f = \sum_k \lambda_k a_k \varphi_k \quad \text{in } L^2(M) \tag{10.16}$$

and

$$\int_M (\Delta_\mu f)^2 \, d\mu = \sum_k \lambda_k^2 a_k^2. \tag{10.17}$$

10.22. Let manifold M admit k non-zero functions $f_1, ..., f_k \in W_0^1(M)$ with disjoint supports such that $\mathcal{R}(f_i) \leq a$ for all $i = 1, ..., k$ and some number a. Assuming that the spectrum of $\mathcal{L}$ is discrete, prove that $\lambda_k(M) \leq a$.

10.23. Prove that if the spectrum of a weighted manifold $(M, \mathbf{g}, \mu)$ is discrete then also the spectrum of any non-empty open subset $\Omega \subset M$ is discrete.

10.24. Let $(M', \mathbf{g}', \mu')$ and $(M'', \mathbf{g}'', \mu'')$ be two weighted manifold with discrete spectra, whose eigenvalues are given by the sequences $\{\alpha_i\}$ and $\{\beta_j\}$, respectively (each eigenvalue is counted with multiplicity). Prove that the spectrum of the direct product $(M, \mathbf{g}, \mu)$ is also discrete, and the eigenvalues are given by the double sequence $\{\alpha_i + \beta_j\}$.

10.25. *(Compactness of the embedding $W^1_{loc} \hookrightarrow L^2_{loc}$)* Let $\{u_k\}$ be a sequence of functions from $W^1_{loc}(M)$ such that $\{u_k\}$ is bounded in $W^1(\Omega)$ for any relatively compact open set $\Omega \subset M$. Prove that there exists a subsequence $\{u_{k_i}\}$ that converges in $L^2_{loc}(M)$.

10.26. Let $f \in C^2(M)$ be a non-negative function on a connected weighted manifold M that satisfies the inequality

$$\Delta_\mu f + \alpha f \le 0$$

with a real constant α. Prove that either $f \equiv 0$ or $\alpha \le \lambda_{\min}(M)$.

REMARK. The converse is also true, that is, for any $\alpha \ge \lambda_{\min}(M)$ there exists a positive solution to the equation $\Delta_\mu f + \alpha f = 0$. This will be proved later in Chapter **??** (cf. Theorem 13). Exercise 10.27 contains a partial result in this direction.

10.27. Let α be a real number.

(a) Prove that if $\alpha < \lambda_{\min}(M)$ then the operator $\mathcal{L} - \alpha\,\mathrm{id}$ has the inverse in $L^2(M)$ and

$$(\mathcal{L} - \alpha\,\mathrm{id})^{-1} = \int_0^\infty e^{\alpha t} P_t dt. \tag{10.18}$$

(b) Prove that if $\mu(M) < \infty$ and $\alpha < \lambda_{\min}(M)$ then the weak Dirichlet problem

$$\begin{cases} \Delta_\mu u + \alpha u = 0 \\ u \in 1 \mod W^1_0(M) \end{cases}$$

has a unique solution that is given by the formula

$$u = 1 + \alpha \int_0^\infty e^{\alpha t} (P_t 1)\, dt \tag{10.19}$$

Deduce that $u > 0$.

10.28. (*Maximum principle*) Let Ω be a non-empty relatively compact open set in a connected weighted manifold M such that $M \setminus \overline{\Omega}$ is non-empty. Prove that if $u \in C\left(\overline{\Omega}\right) \cap C^2(\Omega)$ is a subharmonic function in Ω then

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u. \tag{10.20}$$

REMARK. Of course, this statement follows from Corollary 8. Find another proof using Theorem 10 and Exercise 4.28.

10.29. Prove that, for all $x, y \in M$ and $t \ge s > 0$,

$$p_t(x,y) \le \sqrt{p_s(x,x)\, p_s(y,y)} \exp\left(-\lambda_{\min}(M)(t-s)\right).$$

CHAPTER 11

Distance function and completeness

11.1. Let $\mathbf{g}$ be a metric in $\mathbb{R}^n$, which is given in the polar coordinates (r,θ) by

$$\mathbf{g} = dr^2 + \psi^2(r)\,\mathbf{g}_{\mathbb{S}^{n-1}}, \tag{11.1}$$

where $\psi(r)$ is a smooth positive function (cf. Sections 3 and **??**). Prove that the Riemannian model $(\mathbb{R}^n, \mathbf{g})$ is complete.

11.2. Prove the implication $(c) \Rightarrow (a)$ of Hopf-Rinow Theorem, that is, if all geodesic balls are relatively compact then (M,d) is a complete metric space.

11.3. Prove that a function $f \in C^1(M)$ is Lipschitz if and only if $|\nabla f|$ is bounded, and

$$\|f\|_{Lip} = \sup_M |\nabla f|\,.$$

11.4. Prove the following properties of Lipschitz functions.

(*a*) Let $f_1,...,f_m \in Lip(M)$ and let $I_k = f_k(M)$ be the range of f_k. Let φ be a Lipschitz function on the set $I_1 \times ... \times I_m \subset \mathbb{R}^m$. Then the composite function

$$\Phi(x) := \varphi(f_1(x),...,f_m(x))$$

is Lipschitz on M and

$$\|\Phi\|_{Lip} \le \|\varphi\|_{Lip}\left(\sum_{k=1}^m \|f_k\|_{Lip}^2\right)^{1/2}. \tag{11.2}$$

(*b*) If $f \in Lip_0(M)$ and $\varphi \in Lip(\mathbb{R})$ is such that $\varphi(0) = 0$ then $\varphi \circ f \in Lip_0(M)$.

11.5. Prove that $f,g \in Lip(M)$ then also the functions $f+g$, $\max(f,g)$, $\min(f,g)$ are Lipschitz; moreover, fg is also Lipschitz provided one of the functions f,g is bounded on the support of the other.

Hence show, that if $f,g \in Lip_0(M)$ then also the functions $f+g$, fg, $\max(f,g)$, $\min(f,g)$ belong to $Lip_0(M)$.

11.6. Prove that for any open set $\Omega \subset M$ and any compact set $K \subset \Omega$ there is a function $f \in Lip_0(\Omega)$ such that $0 \le f \le 1$ in Ω, $f|_K \equiv 1$, and $\|f\|_{Lip} \le \frac{2}{d(K,\Omega^c)}$.

Remark. A function f with the above properties is called a *Lipschitz cutoff function* of K in Ω.

11.7. Let f be a real valued function on a Riemannian manifold M.

(*a*) Prove that if $\{U_\alpha\}$ is a countable family of open sets covering the manifold M such that

$$C := \sup_\alpha \|f\|_{Lip(U_\alpha)} < \infty,$$

then $f \in Lip(M)$ and $\|f\|_{Lip(M)} \le C$.

(b) Prove that if E_1, E_2 are two closed sets in M such that $E_1 \cup E_2 = M$ and f is Lipschitz in each set E_1, E_2 with the Lipschitz constant C, then f is also Lipschitz in M with the Lipschitz constant C.

11.8. Prove that

$$C^1(M) \subset Lip_{loc}(M) \subset W^1_{loc}(M).$$

11.9. Prove that the set of functions from $Lip_{loc}(M)$ with compact support is identical to $Lip_0(M)$.

11.10. Prove that if $f_1, ..., f_m \in Lip_{loc}(M)$ and $\varphi \in Lip_{loc}(\mathbb{R}^m)$ then the composite function $\Phi(x) := \varphi(f_1(x), ..., f_m(x))$ is locally Lipschitz on M.

11.11. Prove that if $f, g \in Lip_{loc}(M)$ then the functions $f + g$, fg, $\max(f, g)$, $\min(f, g)$ are also in $Lip_{loc}(M)$.

11.12. Prove that if $f \in Lip_{loc}(M)$ then the distributional gradient ∇f belongs to $\vec{L}^\infty_{loc}(M)$.

11.13. *(Product rule for Lipschitz functions)*

(a) Prove that, for all $f, g \in Lip_{loc}(M)$,

$$\nabla(fg) = f\nabla g + g\nabla f. \tag{11.3}$$

(b) Prove that if $f \in Lip(M) \cap L^\infty(M)$ and $g \in W^1_0(M)$ then $fg \in W^1_0(M)$ and (11.3) holds.

(c) Prove that if $f \in Lip_0(M)$ and $g \in W^1_{loc}(M)$ then $fg \in W^1_0(M)$ and (11.3) holds.

11.14. *(Chain rule for Lipschitz functions)* Prove that if $f \in Lip_{loc}(M)$ and $\psi \in C^1(\mathbb{R})$, then $\psi(f) \in Lip_{loc}(M)$ and

$$\nabla\psi(f) = \psi'(f)\nabla f.$$

11.15. Prove that if $(M, \mathbf{g}, \mu)$ is a complete weighted manifold then $W^1_0(M) = W^1(M)$.

11.16. Let $(M, \mathbf{g}, \mu)$ be a complete weighted manifold.

(a) Let $\{u_k\}_{k=1}^\infty$ be a sequence from $W^1(M)$ such that, for all $\varphi \in C_0^\infty(M)$,

$$(u_k, \varphi)_{W^1} \to (u, \varphi)_{W^1} \tag{11.4}$$

for some $u \in W^1$, and

$$(u_k, \varphi)_{L^2} \to (v, \varphi)_{L^2}, \tag{11.5}$$

for some $v \in L^2(M)$. Prove that $u = v$.

(b) Show that without the hypothesis of completeness, the claim of (a) is not true in general.

11.17. Let $(M, \mathbf{g}, \mu)$ be a complete weighted manifold, and let h be a smooth positive function on M satisfying (9.4). Set $d\widetilde{\mu} = h^2 d\mu$.

(a) Let $\widetilde{\mathcal{L}} = -\Delta_{\widetilde{\mu}}|_{W_0^2}$ be the Dirichlet Laplace operator of $(M, \mathbf{g}, \widetilde{\mu})$. Prove that the operator $-\Delta_\mu + \Phi|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M, \mu)$, and its unique self-adjoint extension, denoted by $\mathcal{L}^\Phi$, is given by

$$\mathcal{L}^\Phi = J\widetilde{\mathcal{L}}J^{-1}, \tag{11.6}$$

where J is a bijection $L^2(M, \widetilde{\mu}) \to L^2(M, \mu)$ defined by $Jf = hf$.

(b) Prove that the heat semigroup $e^{-t\mathcal{L}^\Phi}$ of the operator $\mathcal{L}^\Phi$ in $L^2(M,\mu)$ has the integral kernel $p_t^\Phi(x,y)$, given by

$$p_t^\Phi(x,y) = h(x)\,h(y)\,\widetilde{p}_t(x,y)\,. \tag{11.7}$$

11.18. Consider in $\mathbb{R}$ the function $\Phi(x) = x^2 - 1$. Verify that the function $h(x) = e^{-\frac{1}{2}x^2}$ satisfies (9.4) with this function. Hence, prove that

$$p_t^\Phi(x,y) = \frac{e^t}{(2\pi\sinh 2t)^{1/2}}\exp\left(-\frac{(x-y)^2}{2\sinh 2t} - \frac{x^2+y^2}{2}\tanh t\right). \tag{11.8}$$

REMARK. The function (11.8) is called *the Mehler kernel*.

HINT. Use Example 9.

11.19. Let $f(r)$ be a positive increasing function on $(0,+\infty)$ and assume that there exists a sequence $\{r_k\} \to \infty$ such that

$$f(r_k) \le C r_k^2 \text{ for all } k.$$

Prove that

$$\int^\infty \frac{rdr}{f(r)} = \infty.$$

11.20. Let M be a connected manifold with bounded geometry as in Example 11.

(a) Prove that there is a constant N such that for any $x \in M$, the ball $B(x,\varepsilon)$ can be covered by at most N balls of radius $\varepsilon/2$.
(b) Prove that for any $x \in M$ and integer $k > 1$, the ball $B(x,k\varepsilon/2)$ can be covered by at most N^{k-1} balls of radii $\varepsilon/2$.
(c) Prove that any geodesic ball on M is relatively compact.
(d) Prove that, $V(x,r) \le \exp(Cr)$ for all $x \in M$ and $r \ge 1$. Conclude that M is stochastically complete.

11.21. Let (M,μ) be a complete connected weighted manifold with $\mu(M) < \infty$. Prove that, for all $x,y \in M$,

$$p_t(x,y) \to \frac{1}{\mu(M)} \quad \text{as } t \to \infty. \tag{11.9}$$

11.22. Let (M,μ) be a complete connected weighted manifold and let h be a positive harmonic function on M such that, for some $x_0 \in M$, the function

$$v(r) := \int_{B(x_0,r)} h^2 d\mu$$

satisfies the condition

$$\int^\infty \frac{rdr}{\log v(r)} = \infty. \tag{11.10}$$

Prove that $P_t h = h$.

11.23. Let $f(r)$ be a C^1-function on $(0,+\infty)$ such that $f'(r) > 0$. Prove that

$$\int^\infty \frac{rdr}{f(r)} = \infty \implies \int^\infty \frac{dr}{f'(r)} = \infty.$$

11.24. Prove that any parabolic manifold is stochastically complete.

11.25. Prove that, for any bounded open set $\Omega \subset \mathbb{R}^n$,

$$\lambda_{\min}(\Omega) \geq \frac{1}{n(\operatorname{diam}\Omega)^2}. \tag{11.11}$$

Hence or otherwise show that there exists a constant $c_n > 0$ such that, for any ball $B_r \subset \mathbb{R}^n$,

$$\lambda_{\min}(B_r) = c_n r^{-2}.$$

11.26. Let $(M, \mathbf{g}, \mu)$ be a weighted manifold of dimension $n \geq 2$, and o be a point in M.

(a) Prove that, for any open neighborhood U of o and for any $\varepsilon > 0$, there exists a cutoff function ψ of $\{o\}$ in U such that

$$\int_U |\nabla \psi|^2 \, d\mu < \varepsilon.$$

(b) Prove that

$$\lambda_{\min}(M \setminus \{o\}) = \lambda_{\min}(M). \tag{11.12}$$

(c) Show that (11.12) fails if $n = 1$.

11.27. Let $(M, \mathbf{g}, \mu)$ be a complete weighted manifold. Fix a point $x_0 \in M$ and set

$$\alpha = \limsup_{r \to \infty} \frac{1}{r} \log \mu(B(x_0, r)). \tag{11.13}$$

Prove that

$$\lambda_{\min}(M) \leq \frac{\alpha^2}{4}.$$

11.28. Let $(M, \mathbf{g}, \mu)$ be a weighted model based on $\mathbb{R}^n$ as in Sections 3 and **??**, and let $S(r)$ be the area function of this model. Set

$$\alpha' = \inf_{r>0} \frac{S'(r)}{S(r)} \quad \text{and} \quad \alpha = \limsup_{r \to \infty} \frac{S'(r)}{S(r)}. \tag{11.14}$$

Prove that

$$\frac{(\alpha')^2}{4} \leq \lambda_{\min}(M) \leq \frac{\alpha^2}{4}.$$

CHAPTER 12

Gaussian estimates in the integrated form

12.1. Let Φ be a C^2-function in $I := [0, +\infty)$ such that $\Phi, \Phi', \Phi'' \geq 0$ and

$$\Phi''\Phi \geq \delta \left(\Phi'\right)^2, \tag{12.1}$$

for some $\delta > 0$. Let $\xi(t,x)$ be a continuous function on $I \times M$ and assume that $\xi(t,x)$ is locally Lipschitz in $x \in M$ for any $t \in I$, $\frac{\partial \xi}{\partial t}$ exists and is continuous on $I \times M$, and the following inequality holds on $I \times M$:

$$\frac{\partial \xi}{\partial t} + \frac{1}{4\delta} |\nabla \xi|^2 \leq 0.$$

Prove that the quantity

$$J(t) := \int_M \Phi(P_t f)\, e^{\xi(t,\cdot)} d\mu$$

is non-increasing in $t \in I$ for any non-negative $f \in L^2(M)$.

12.2. Give an alternative proof of (**??**) applying Theorem 12 with the function

$$\xi(t,x) := \alpha d(x,A) - \frac{\alpha^2}{2} t,$$

where α is an arbitrary real parameter.

12.3. The purpose of this question is to prove the following enhanced version of (**??**): if f and g are two functions from $L^2(M)$ such that

$$d(\operatorname{supp} f, \operatorname{supp} g) \geq r,$$

where $r \geq 0$, then, for all $t > 0$,

$$|(P_t f, g)| \leq \|f\|_2 \|g\|_2 \int_r^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) ds. \tag{12.2}$$

(*a*) (*Finite propagation speed for the wave equation*) Let $u(t,x)$ be a C^∞ function on $\mathbb{R} \times M$ that solves in $\mathbb{R} \times M$ the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta_\mu u.$$

Set $K_t = \operatorname{supp} u(t,\cdot)$. Prove that K_t is contained in the closed $|t|$-neighborhood of K_0.

(*b*) Prove (12.2) using part (*a*) and the transmutation formula of Exercise 4.52.

Remark. See Exercise 13.25 concerning the additional factor $e^{-\lambda t}$ in (12.2).

12.4. Using Corollary 12, prove that if the weighted manifold M is geodesically complete and, for some point $x \in M$, a constant $C > 0$, and a sequence $\{r_k\} \to \infty$,

$$\mu\left(B\left(x, r_k\right)\right) \leq \exp\left(C r_k^2\right) \tag{12.3}$$

then M is stochastically complete.

Remark. Of course, this follows from Theorem 11 but the purpose of this Exercise is to give an alternative proof.

12.5. Let A and B be sets as in Theorem 12.

(a) Prove that, for any function $f \in L^{\infty}\left(B^c\right)$,

$$\int_A \left(P_t f\right)^2 d\mu \leq \mu\left(B\right) \|f\|_{L^\infty}^2 \max\left(\frac{R^2}{2t}, 1\right) e^{-\frac{R^2}{2t}+1}. \tag{12.4}$$

(b) Prove that

$$\int_A \int_{B^c} p_t(x,y) d\mu(y) d\mu(x) \leq C\sqrt{\mu(A)\mu(B)} \max\left(\frac{R}{\sqrt{t}}, 1\right) e^{-\frac{R^2}{4t}}, \tag{12.5}$$

where $C = \sqrt{e/2}$.

CHAPTER 13

Green function and Green operator

13.1. Prove that if M is a compact manifold then

(a) $g(x,y) \equiv \infty$;

(b) there is no fundamental solution of the Laplace operator on M.

13.2. Let M be a weighted model (cf. Section 3) and $S(r)$ be the area function of M.

(a) Prove that, for any positive real R that is smaller than the radius of M, the following function

$$h(x) = \int_{|x|}^{R} \frac{dr}{S(r)}$$

is a fundamental solution in B_R of the Laplace operator at the pole o.

(b) Using (a), evaluate the fundamental solutions on $\mathbb{R}^n$ and $\mathbb{H}^n$.

13.3. Prove that if the manifold M is connected then $g(x,y) > 0$ for all $x, y \in M$.

13.4. Prove that if the Green function g is finite then the following identity takes place for all $t > 0$ and $x_0 \in M$:

$$P_t g(x_0, \cdot) = G p_t(x_0, \cdot).$$

13.5. Prove that if $\lambda_{\min}(M) > 0$ then the Green function $g(x,y)$ is C^∞ smooth jointly in x, y in $M \times M \setminus \operatorname{diag}$.

13.6. Prove that if $\lambda_{\min}(M) > 0$ then

$$\|G\|_{L^2 \to L^2} \le \frac{1}{\lambda_{\min}(M)}. \tag{13.1}$$

13.7. Prove that if $\lambda_{\min}(M) > 0$ and $\mu(M) < \infty$ then $g(x,y) \in L^1(M \times M)$.

13.8. Prove that if $\{\Omega_k\}$ is any exhaustion sequence in M then, for all $x, y \in M$,

$$g^{\Omega_k}(x,y) \uparrow g(x,y) \quad \text{as } k \to \infty.$$

13.9. Let Ω be an open subset of a weighted manifold M. Prove that, for any compact set $K \subset \Omega$ and for any non-negative function $f \in L^2(M)$,

$$Gf \le G^\Omega f + \operatorname*{esup}_{M \setminus K} Gf. \tag{13.2}$$

13.10. Let Ω be a non-empty relatively compact open subset of a connected manifold M such that $M \setminus \overline{\Omega}$ is non-empty. Fix a point $x_0 \in \Omega$.

(a) Let φ be a cutoff function of $\{x_0\}$ in Ω. Prove that

$$(1-\varphi)\, g^\Omega(x_0, \cdot) \in W_0^1(\Omega).$$

(b) Prove that for any open set $U \subset \Omega$, containing x_0,

$$g^\Omega(x_0, \cdot) - g^U(x_0, \cdot) \in W_0^1(\Omega).$$

13.11. Assume that $\lambda_{\min}(M) > 0$ and $\mu(M) < \infty$. Prove that, for all $0 \le a < b$ and any $x_0 \in M$, the function

$$v(x) = \begin{cases} g(x_0, x) & \text{if } g(x_0, x) \in [a, b], \\ a, & \text{if } g(x_0, x) < a, \\ b, & \text{if } g(x_0, x) > b, \end{cases}$$

belongs to $W^1(M)$ and

$$\|\nabla v\|_{L^2}^2 \le b - a.$$

13.12. Prove that, for any weighted manifold M and for all $c > 0$, $x_0 \in M$, the function $u = \min(g(x_0, \cdot), c)$ belongs to $W^1_{loc}(M)$ and

$$\|\nabla u\|_{L^2}^2 \le c.$$

13.13. Let Ω be a non-empty relatively compact connected open subset of a weighted manifold M. Prove that

$$\sup_{x \in \Omega} \int_\Omega g^\Omega(x, y)\, d\mu(y) \ge \frac{1}{\lambda_{\min}(\Omega)}. \tag{13.3}$$

13.14. Let M be a connected weighted manifold and Ω be a relatively compact open subset of M such that $M \setminus \overline{\Omega}$ is non-empty. Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis in $L^2(\Omega)$ of eigenfunctions of Ω and $\{\lambda_k\}$ be the corresponding sequence of eigenfunctions. Prove the identity

$$g^\Omega(x, y) = \sum_{k=1}^\infty \frac{1}{\lambda_k} \varphi_k(x)\, \varphi_k(y),$$

where the series converges in $\mathcal{D}'(\Omega \times \Omega)$.

13.15. Prove the following properties of superaveraging functions.

(a) If $\{f_k\}_{k=1}^\infty$ is an increasing sequence of superaveraging functions and $f_k \to f \in L^1_{loc}$ then f is also superaveraging.

(a) If $\{f_i\}_{i \in I}$ is a family of superaveraging functions depending on a parameter i then the function

$$f = \inf_{i \in I} f_i$$

is also superaveraging.

13.16. Let M be a connected, stochastically complete weighted manifold, and let f be a non-negative continuous superaveraging function on M.

(a) Prove that the inequality $P_t f \le f$ is satisfied pointwise and that $P_t f \to f$ as $t \to 0$ pointwise.

(b) (*Strong minimum principle*) Prove that if $f(x) = \inf f$ at some point $x \in M$ then $f \equiv \text{const}$ on M.

(b) (*Minimum principle*) Let Ω be a relatively compact open subset of M with non-empty boundary. Prove that

$$\inf_{\overline{\Omega}} f = \inf_{\partial \Omega} f.$$

13.17. Prove that if the Green function is finite then it is superaveraging with respect to each of its arguments.

13.18. Let Ω be a relatively compact open subset of M such that $\lambda_{\min}(\Omega) > 0$. Let u be a solution of the following weak Dirichlet problem in Ω

$$\begin{cases} \Delta_\mu u = 0, \\ u = f \mod W_0^1(\Omega), \end{cases} \tag{13.4}$$

where $f \in W^1(M)$, and set

$$\widetilde{f} = \begin{cases} f & \text{in } \Omega^c, \\ u & \text{in } \Omega, \end{cases}$$

(see Fig. *13.1*).

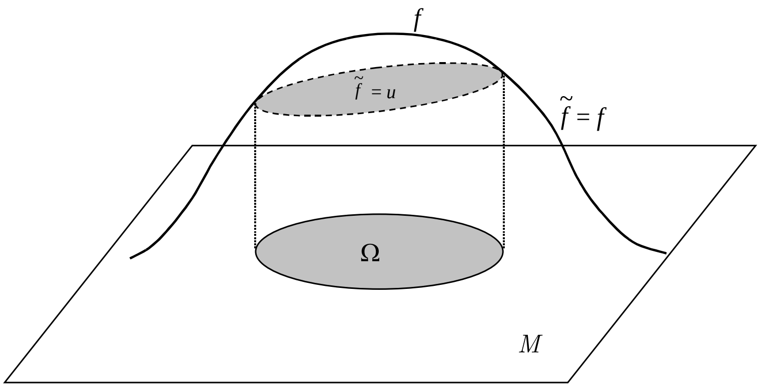

FIGURE 13.1. Function $\widetilde{f}$ in Exercise 13.18

(*a*) Prove that if $f \in W_0^1(M)$ then also $\widetilde{f} \in W_0^1(M)$.
(*b*) Prove that if f is superaveraging then also $\widetilde{f}$ is superaveraging and $0 \le \widetilde{f} \le f$.

13.19. Let f and h be two superaveraging functions from $W_0^1(M)$. Then, for any $t > 0$,

$$(-\Delta_\mu P_t f, h) \le (\nabla f, \nabla h). \tag{13.5}$$

13.20. Let $f \in W_0^1(M)$ and $\{\Omega_k\}$ be a compact exhaustion sequence in M. Let $u_k \in W^1(\Omega_k)$ solve in Ω_k the weak Dirichlet problem problem

$$\begin{cases} \Delta_\mu u_k = 0, \\ u_k = f \mod W_0^1(\Omega_k). \end{cases}$$

Then

$$\|\nabla u_k\|_{L^2} \to 0 \text{ as } k \to \infty.$$

13.21. Let f and h be two superaveraging functions from $W_0^1(M)$. If $\{\Omega_k\}_{k=1}^\infty$ is a compact exhaustion sequence such that $\lambda_{\min}(\Omega_k) > 0$ for any k, then

$$\sup_{t>0} \int_{M \setminus \Omega_k} (-\Delta_\mu P_t f)\, h \, d\mu \to 0 \text{ as } k \to \infty.$$

13.22. Prove the classical Harnack inequality: if $f(x)$ is a positive harmonic function in a ball $B(x,r)$ in $\mathbb{R}^n$ then

$$\sup_{B(x,r/2)} f \le C_n \inf_{B(x,r/2)} f\,, \tag{13.6}$$

where the constant C_n depends only on n.

13.23. (*The Liouville theorem*) Prove that any positive harmonic function in $\mathbb{R}^n$ is identical constant.

13.24. Let M be a connected weighted manifold. Prove that if $g(x,y) < \infty$ for some couple $x, y \in M$ then $g(x,y)$ is finite, that is, $g(x,y) < \infty$ for all distinct points $x, y \in M$.
REMARK. Hence, the following dichotomy takes places: either $g(x,y) \equiv \infty$ for all $x, y \in M$ or $g(x,y) < \infty$ for all distinct $x, y \in M$.

13.25. Prove the following improved version of (12.2): if f and g are two functions from $L^2(M)$ such that

$$d(\operatorname{supp} f, \operatorname{supp} g) \geq r,$$

where $r \geq 0$, then, for all $t > 0$,

$$|(P_t f, g)| \leq \|f\|_2 \|g\|_2 e^{-\lambda_{\min}(M)t} \int_r^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) ds. \tag{13.7}$$

13.26. Let M be a connected non-compact manifold and Ω be a relatively compact open subset of M.

(a) Prove that, for any $p \in [1, +\infty]$, G^Ω is a bounded operator from $L^p(\Omega)$ to $L^p(\Omega)$.

(b) Prove that the function $u = G^\Omega f$ satisfies the equation $-\Delta_\mu u = f$ for any $f \in L^p(\Omega)$.

13.27. Let M be a connected weighted manifold and let $f \in L^1_{loc}(M)$ and $f \geq 0$. Prove that if $Gf(x)$ is finite then Gf belongs to L^1_{loc} and $-\Delta_\mu(Gf) = f$.

13.28. Let M be a connected weighted manifold with a finite Green function $g(x,y)$. Fix a point $x_0 \in M$ and a compact set $K \subset M$. Prove that if u is a harmonic function on M and

$$u(x) \leq g(x, x_0) \text{ for all } x \in M \setminus K,$$

then $u(x) \leq 0$ for all $x \in M$.

13.29. Let M be a connected weighted manifold. Prove that if $h(x)$ is a fundamental solution of the Laplace operator at a point $x_0 \in M$ such that $h(x) \to 0$ as $x \to \infty$, then $h(x) = g(x, x_0)$.

13.30. Prove that, on an arbitrary connected weighted manifold M, the following conditions are equivalent:

(i) the Green function is finite;

(ii) there exists a positive non-constant superharmonic function (that is, M is non-parabolic);

(iii) there exists a positive non-constant superaveraging function.

13.31. Let M be a connected weighted manifold and Ω be a non-empty relatively compact open subset of M such that $M \setminus \overline{\Omega}$ is non-empty. Prove that, for all $x \in M$, $y \in \Omega$,

$$g(x,y) \leq g^\Omega(x,y) + \sup_{z \in \partial\Omega} g(z,y). \tag{13.8}$$

Here we set $g^\Omega(x,y) = 0$ if $x \notin \Omega$ or $y \notin \Omega$.

13.32. Prove that a fundamental solution of the Laplace operator exists on any non-compact connected weighted manifold.

13.33. Prove that if, for some $x \in M$ and a compact set $K \subset M$,

$$\int_{M \setminus K} g(x,y)\, d\mu(y) < \infty \tag{13.9}$$

then M is stochastically incomplete.

13.34. Let M be a weighted model of dimension $n \geq 2$, and $S(r)$ be its boundary area function (cf. Section 3). Prove that the Green function of the central ball B_R satisfies the identity

$$g^{B_R}(x,o) = \int_r^R \frac{ds}{S(s)}, \tag{13.10}$$

where $r = |x|$. Deduce that the Green function of M satisfies the identity

$$g(x,o) = \int_r^\infty \frac{ds}{S(s)}. \tag{13.11}$$

Hence or otherwise give an example of a complete manifold M where the Green function belongs to $L^1(M)$.

13.35. Prove that the Green function of the ball $B = B_R(0)$ in $\mathbb{R}^n$ is given by the following formulas, for all $x, y \in B$:

(a) If $n > 2$ then

$$g^B(x,y) = \frac{1}{\omega_n(n-2)}\left(\frac{1}{|x-y|^{n-2}} - \left(\frac{R}{|y|}\right)^{n-2} \frac{1}{|x-y^*|^{n-2}}\right), \tag{13.12}$$

where y^* is the inversion of y with respect to the ball B, that is

$$y^* = \frac{y}{|y|^2} R^2.$$

(b) If $n = 2$ then

$$g^B(x,y) = \frac{1}{2\pi} \log \frac{|x-y^*|\,|y|}{|x-y|\, R}.$$

(c) If $n = 1$ then

$$g^B(x,y) = \frac{1}{2}|x-y| - \frac{1}{2R} xy + \frac{R}{2}.$$

13.36. Let $F(t)$ be a positive monotone increasing function on $\mathbb{R}_+$ and assume that

$$p_t(x,y) \leq \frac{1}{F(\sqrt{t})} \exp\left(-c\frac{r^2}{t}\right)$$

for some $x, y \in M$ and all $t > 0$, where $r = d(x,y)$ and $c > 0$. Prove that if F satisfies the doubling property

$$F(2s) \leq AF(s) \quad \text{for all } s > 0, \tag{13.13}$$

then

$$g(x,y) \leq C \int_r^\infty \frac{s\,ds}{F(s)}, \tag{13.14}$$

where $C = C(A, c)$.

If in addition F satisfies the condition

$$\frac{F(s)}{F(s')} \geq a\left(\frac{s}{s'}\right)^\alpha, \quad \text{for all } s > s' > 0, \tag{13.15}$$

where $a > 0$ and $\alpha > 2$ then

$$g(x, y) \leq C \frac{r^2}{F(r)}, \tag{13.16}$$

where $C = C(A, a, \alpha, c)$.

CHAPTER 14

Ultracontractive estimates and eigenvalues

14.1. Prove that if the heat semigroup $\{P_t\}$ is $L^p \to L^2$ ultracontractive with the rate function $\theta(t)$ where $1 \le p < 2$ then $\{P_t\}$ is also $L^p \to L^{p^*}$ ultracontractive with the rate function $\theta^2(t/2)$.

14.2. Prove that if (**??**) holds for all relatively compact open sets Ω then it holds also for all open sets Ω with $\mu(\Omega) < \infty$.

14.3. Assume that the following Nash inequality holds:

$$\int_M |\nabla u|^2 \, d\mu \ge \|u\|_2^2 \Lambda \left(\frac{\|u\|_1^2}{\|u\|_2^2} \right),$$

for any non-zero function $u \in C_0^\infty(M)$, where Λ is a decreasing function on $[0, +\infty)$. Prove the Faber-Krahn inequality

$$\lambda_{\min}(\Omega) \ge \Lambda(\mu(\Omega)),$$

for any open set $\Omega \subset M$ with finite measure.

14.4. Give an example of a manifold where the Faber-Krahn inequality can holds only with function $\Lambda(v) \equiv 0$.

14.5. Prove that the Faber-Krahn inequality with function

$$\Lambda(v) = a v^{-2/\nu} \tag{14.1}$$

where a and ν are positive constants, implies that, for any relatively compact ball $B(x,r)$,

$$\mu(B(x,r)) \ge c a^{\nu/2} r^\nu, \tag{14.2}$$

where $c = c(\nu) > 0$.

Hint. First prove that

$$\mu(B(x,r)) \ge c \left(ar^2\right)^{\frac{\nu}{\nu+2}} \mu(B(x,r/2))^{\frac{\nu}{\nu+2}}$$

and then iterate this inequality.

14.6. Prove that the Faber-Krahn inequality with function (14.1) with $\nu > 2$ is equivalent to the Sobolev inequality:

$$\int_M |\nabla u|^2 \, d\mu \ge c \left(\int_M |u|^{\frac{2\nu}{\nu-2}} \, d\mu \right)^{\frac{\nu-2}{\nu}} \tag{14.3}$$

for any $u \in W_0^1(M)$, where $c = c(a, \nu) > 0$.

14.7. Prove that the Sobolev inequality (14.3) implies the following inequality, for any $u \in C_0^\infty(M)$:

$$\int_M |\nabla u|^2 \, d\mu \geq c \left(\int_M |u|^\alpha \, d\mu \right)^{-a} \left(\int_M |u|^\beta \, d\mu \right)^b \tag{14.4}$$

for any set of positive reals α, β, a, b that satisfy the following conditions:

$$\alpha < \beta < \frac{2\nu}{\nu - 2} \tag{14.5}$$

and

$$\begin{cases} b - a = 1 - \frac{2}{\nu}, \\ \beta b - \alpha a = 2. \end{cases} \tag{14.6}$$

Remark. Under the conditions (14.5), the numbers a, b solving (14.6) always exist and are positive. For example, if $\alpha = 1$ and $\beta = 2$ then $a = 4/\nu$ and $b = 1 + 2/\nu$, so that (14.4) coincides with the Nash inequality (**??**). If $\alpha = 2$ and $\beta = 2 + 4/\nu$ then $a = 2/\nu$ and $b = 1$, and we obtain the *Moser inequality*

$$\int_M |\nabla u|^2 \, d\mu \geq c \left(\int_M |u|^2 \, d\mu \right)^{-2/\nu} \left(\int_M |u|^{2+4/\nu} \, d\mu \right).$$

14.8. Prove that if Λ_1, Λ_2 are two functions of class $\mathbf{L}$ then also $\Lambda_1 + \Lambda_1$ and $\max(\Lambda_1, \Lambda_2)$ belong to $\mathbf{L}$.

14.9. Let Λ be a function of class $\mathbf{L}$ such that

$$\Lambda(v) = \begin{cases} c_1 v^{-\alpha_1}, & v \leq v_1, \\ c_2 v^{-\alpha_2}, & v \geq v_2, \end{cases}$$

where $\alpha_1, c_1, v_1 > 0$, $\alpha_2, c_2 \geq 0$, and $v_2 > v_1$. Prove that $\Lambda \in \mathbf{L}_\delta$ for some $\delta > 0$.

14.10. For any function $\gamma \in \Gamma$, denote by Λ_γ the $\mathbf{L}$-transform of γ, and for any function $\Lambda \in \mathbf{L}$, denote by γ_Λ the Γ-transform of Λ. Let a, b be positive constants.

(a) Set $\widetilde{\Lambda}(v) = a\Lambda(bv)$. Prove that

$$\gamma_{\widetilde{\Lambda}}(t) = b^{-1}\gamma_\Lambda(at).$$

(b) Set $\widetilde{\gamma}(t) = a\gamma(bt)$. Prove that

$$\Lambda_{\widetilde{\gamma}}(v) = b\Lambda_\gamma\left(a^{-1}v\right).$$

(c) Prove that if Λ_1 and Λ_2 are two functions from $\mathbf{L}$ and $\Lambda_1 \leq \Lambda_2$ then $\gamma_{\Lambda_1} \leq \gamma_{\Lambda_2}$.

14.11. Prove that the product of two functions from $\widetilde{\Gamma}_\delta$ belongs to $\widetilde{\Gamma}_\delta$, and the product of two functions from Γ_δ belongs to $\Gamma_{\delta/2}$.

14.12. Show that there is a function $\gamma \in \Gamma$ that does not belong to any class Γ_δ.

14.13. Let $F(s)$ be a positive function of class C^2 on $[0, +\infty)$ such that $F'(s)$ does not vanish for large s. Assume that

$$\int_0^\infty \frac{ds}{F(s)} = \infty$$

and

$$c := \lim_{s \to \infty} \frac{F'' F}{(F')^2}(s) \neq 0.$$

Prove that
$$\int_0^t \frac{ds}{F(s)} \sim -\frac{c^{-1}}{F'(t)} \text{ as } t \to \infty.$$

14.14. Let Λ be a function of class $\mathbf{L}$ such that
$$\Lambda(v) = \exp\left(-v^{\beta}\right) \quad \text{for } v \geq 1,$$
where $\beta > 0$. Evaluate the asymptotic of its Γ-transform $\gamma(t)$ as $t \to \infty$.

14.15. Prove that the claim of Theorem 14 remains true for any $f \in L^1(M)$.

CHAPTER 15

Pointwise Gaussian estimates I

15.1. Fix $x_0 \in M$, $R > r > 0$ and let the ball $B\left(x_0, R\right)$ be relatively compact. Assume also that, for some $a, n > 0$, the Faber-Krahn inequality

$$\lambda_{\min}(U) \geq a\mu\left(U\right)^{-2/n}, \tag{15.1}$$

holds for any open set $U \subset B(x_0, r)$. Let $u\left(t,x\right)$ be a non-negative bounded C^2-function $(0,T) \times B\left(x_0, R\right)$, where $T > 0$, such that

(i) $\frac{\partial u}{\partial t} - \Delta_\mu u \leq 0$,

(ii) $u\left(t,\cdot\right) \to 0$ as $t \to 0$ in $L^2\left(B\left(x_0, R\right)\right)$.

Prove that, for all $x \in B\left(x_0, r/2\right)$ and $t \in (0,T)$,

$$u\left(t,x\right) \leq C\|u\|_{L^\infty} \frac{\mu\left(B\left(x_0,R\right)\right)^{\frac{1}{2}}}{\left(at\right)^{n/4}} \max\left(1, \frac{\sqrt{t}}{r}\right)^{\frac{n}{2}+1} \max\left(1, \frac{\delta}{\sqrt{t}}\right) e^{-\frac{\delta^2}{4t}} \tag{15.2}$$

where $\delta = R - r$ and $C = C\left(n\right)$.

15.2. Prove that the Faber-Krahn inequality holds on a weighted n-dimensional manifold M with function

$$\Lambda\left(v\right) = \begin{cases} cv^{-2/n}, & v < v_0, \\ 0, & v \geq v_0, \end{cases}$$

where c, v_0 are some positive constants, provided M belongs to one of the following classes:

(a) M is compact;

(b) M has bounded geometry (see Example 11).

Remark. If M is non-compact and has bounded geometry then the Faber-Krahn function Λ can be improved by setting $\Lambda\left(v\right) = cv^{-2}$ for $v \geq v_0$ – see [?].

15.3. Prove that, on any weighted manifold M there is a positive continuous function $F\left(x,s\right)$ on $M \times \mathbb{R}_+$, which is monotone increasing in s and such that the heat kernel on M satisfies the following estimate

$$p_t(x,y) \leq \frac{C\left(1+\frac{\rho^2}{t}\right)^{n/2}}{F\left(x,\sqrt{t}\right)^{1/2} F\left(y,\sqrt{t}\right)^{1/2}} \exp\left(-\frac{\rho^2}{4t}\right), \tag{15.3}$$

for all $x, y \in M$ and $t > 0$, where $n = \dim M$ and $C = C\left(n\right)$ (cf. Exercise 16.3).

15.4. Prove that if M has bounded geometry then, for some constant C,

$$p_t(x,y) \leq \frac{C\left(1+\frac{\rho^2}{t}\right)^{n/2}}{\min\left(1,t\right)^{n/2}} \exp\left(-\frac{\rho^2}{4t}\right), \tag{15.4}$$

for all $x, y \in M$ and $t > 0$.

15.5. Under the hypotheses of Corollary 15, assume in addition that $n > 2$ and

$$\mu(B(x,r)) \leq Cr^n$$

for all $r > 0$. Prove that each of the conditions $(a) - (c)$ is equivalent to the following estimate of the Green function:

$$g(x,y) \leq Cd(x,y)^{2-n},$$

for all distinct $x, y \in M$.

REMARK. Note for comparison that the Faber-Krahn inequality of Corollary 15 implies $\mu(B(x,r)) \geq \text{const}\, r^n$ – see Exercise 14.5.

15.6. Under conditions of Corollary 15, let $n \geq 2$ and $\lambda := \lambda_{\min}(M) > 0$. Prove that, for any $\varepsilon \in (0,1)$, the Green function of M satisfies the estimate

$$g(x,y) \leq Ce^{-(1-\varepsilon)\sqrt{\lambda}\rho} \begin{cases} \rho^{2-n}, & n > 2, \\ \left(1 + \log_+ \frac{1}{\rho}\right), & n = 2, \end{cases} \tag{15.5}$$

for all $x \neq y$, where $C = C(n, \varepsilon, \lambda, c)$.

15.7. Let M be an arbitrary weighted manifold of dimension $n \geq 2$. Prove that if the Green function of M is finite then, for any $x \in M$ and for all y close enough to x,

$$g(x,y) \leq C \begin{cases} \rho^{2-n}, & n > 2, \\ \log \frac{1}{\rho}, & n = 2, \end{cases} \tag{15.6}$$

where $C = C(n)$.

15.8. Let M be a complete manifold satisfying the relative Faber-Krahn inequality. Prove that the Green function $g(x,y)$ is finite if and only if, for all $x \in M$,

$$\int^{\infty} \frac{rdr}{V(x,r)} < \infty.$$

Prove also the estimate for all $x, y \in M$:

$$g(x,y) \leq C \int_{d(x,y)}^{\infty} \frac{rdr}{V(x,r)}.$$

15.9. Under conditions of Theorem 15, prove that the relative Faber-Krahn inequality (??) implies the following enhanced version of (??):

$$p_t(x,y) \leq \frac{C\left(1 + \frac{\rho^2}{t}\right)^{\frac{\nu-1}{2}}}{V\left(x, \sqrt{t}\right)^{1/2} V\left(y, \sqrt{t}\right)^{1/2}} \exp\left(-\frac{\rho^2}{4t}\right). \tag{15.7}$$

HINT. *Use the mean-value inequality of Theorem 15 and (12.2).*

CHAPTER 16

Pointwise Gaussian estimates II

16.1. Let for some $x \in M$ and all $t \in (0, T)$

$$p_t(x,x) \le \frac{1}{\gamma(t)}, \tag{16.1}$$

where $T \in (0, +\infty]$ and γ is a monotone increasing function on $(0, T)$ satisfying the doubling property

$$\gamma(2t) \le A\gamma(t), \tag{16.2}$$

for some $A \ge 1$ and all $t < T/2$. Prove that, for all $D > 2$ and $t > 0$,

$$E_D(t,x) \le \frac{C}{\gamma\,(t \wedge T)}, \tag{16.3}$$

where $C = C(A)$.

16.2. Using Exercise 16.1, give an alternative proof of Corollary 15: on any weighted manifold M,

$$E_D(t,x) < \infty$$

for all $D > 2$, $x \in M$, $t > 0$.

16.3. Using Lemma 15, prove that on any weighted manifold M, for any $D > 2$ there exists a function $\Phi(t, x)$ that is decreasing in t and such that the following inequality holds

$$p_t(x,y) \le \Phi(t,x)\Phi(t,y)\exp\left(-\frac{d^2(x,y)}{2Dt} - \lambda_{\min}(M)\,t\right), \tag{16.4}$$

for all $x, y \in M$ and $t > 0$ (cf. Exercise 15.3).

16.4. Assume that a weighted manifold M admits the Faber-Krahn inequality with a function $\Lambda \in \mathbf{L}$ and let γ be its $\mathbf{L}$-transform. Assume that γ is regular in the sense of Definition 16. Prove that, for any $D > 2$ and for all $t > 0$ and $x, y \in M$,

$$p_t\,(x,y) \le \frac{C}{\gamma\,(ct)}\exp\left(-\frac{d^2\,(x,y)}{2Dt}\right),$$

where C depends on D and on the regularity constants of γ.

16.5. Assume that the volume function $V\,(x,r) = \mu\,(B\,(x,r))$ of a weighted manifold M is doubling and that the heat kernel of M admits the estimate

$$p_t\,(x,x) \le \frac{C}{V\left(x,\sqrt{t}\right)},$$

for all $x \in M$ and $t \in (0, T)$, where $T \in (0, +\infty]$ and C is a constant. Prove that

$$p_t\,(x,y) \le \frac{C}{V\left(x,\sqrt{t}\right)}\exp\left(-\frac{d^2\,(x,y)}{2Dt}\right),$$

for all $D > 2$, $x, y \in M$, $t \in (0, T)$ and some constant C.

Remark. If $T = +\infty$ and the manifold M is complete and non-compact, then this follows from Theorem 15.

CHAPTER 17

Reference material

17.1. Prove that if $\{x_k\}$ and $\{y_k\}$ are two sequences in $\mathcal{H}$ such that $x_k \to x$ and $y_k \to y$ then

$$(x_k, y_k) \to (x, y)\,.$$

17.2. Prove that if $x_k \rightharpoonup x$ then

$$\|x\| \le \liminf_{k\to\infty} \|x_k\|.$$

17.3. Let $\{x_k\}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$.

(*a*) Prove that $x_k \to x$ if and only of

$$x_k \rightharpoonup x \text{ and } \|x_k\| \to \|x\|.$$

That is, the strong convergence is equivalent to the weak convergence and the convergence of the norms.

(*b*) Prove that $x_k \rightharpoonup x$ if and only if the numerical sequence $\{\|x_k\|\}$ is bounded and, for a dense subset $\mathcal{D}$ of $\mathcal{H}$,

$$(x_k, y) \to (x, y) \text{ for any } y \in \mathcal{D}.$$

That is, the weak convergence is equivalent to the convergence "in distribution" and the boundedness of the norms.

17.4. Let $\{v_k\}_{k=1}^{\infty}$ be an orthonormal sequence in $\mathcal{H}$.

(*a*) Prove that $v_k \rightharpoonup 0$ as $k \to \infty$.

(*b*) Prove that, for any sequence of reals c_k, the series

$$\sum_{k=1}^{\infty} c_k v_k$$

converges weakly if and only if it converges strongly.

17.5. A subset S of a Hilbert space $\mathcal{H}$ if called weakly closed if it contains all weak limits of all sequences from S. Prove that any closed subspace of $\mathcal{H}$ is also weakly closed.

17.6. Let $\{f_k\}$ be a sequence of functions from $L^2(M, \mu)$ such that $f_k \overset{L^2}{\rightharpoonup} f$. Prove that

$$\operatorname{esup} f \le \liminf_{k\to\infty} (\operatorname{esup} f_k) \tag{17.1}$$

and

$$\operatorname{einf} f \ge \limsup_{k\to\infty} (\operatorname{einf} f_k)\,. \tag{17.2}$$

17.7. Prove that if $f_k \overset{L^2}{\to} f$ then $f_k^2 \overset{L^1}{\to} f^2$.Hence or otherwise show that, for any function $g \in L^\infty$,

$$\int_M f_k^2 g\, d\mu \to \int_M f^2 g\, d\mu.$$

17.8. If an operator A in $\mathcal{H}$ is injective and surjective then one defines the inverse operator A^{-1} such that, for any $x \in \mathcal{H}$, $A^{-1}x$ is the unique vector $y \in \operatorname{dom} A$ such that $Ay = x$.

(a) Prove that if A^{-1} exists then $AA^{-1} = \mathrm{id}$ and $A^{-1}A \subset \mathrm{id}$.

(b) Prove that if A and B are two operators such that

$$AB = \mathrm{id} \text{ and } BA \subset \mathrm{id}$$

then A^{-1} exists and $A^{-1} = B$.

17.9. Prove that, for any operator A in a Hilbert space,

$$\|A\| = \sup_{x \in \operatorname{dom} A, \|x\| \leq 1, \|y\| \leq 1} (Ax, y). \tag{17.3}$$

17.10. Prove that, for any bounded operator A, the adjoint operator A^* is also bounded and

$$\|A\| = \|A^*\| \quad \text{and} \quad \|A^*A\| = \|A\|^2.$$

17.11. Let A be a densely defined symmetric non-negative definite operator.

(a) Prove that, for all $x, y \in \operatorname{dom} A$,

$$(Ax, y)^2 \leq (Ax, x)(Ay, y). \tag{17.4}$$

(b) Prove that

$$\|A\| = \sup_{x \in \operatorname{dom} A, \|x\| \leq 1} (Ax, x).$$

17.12. Let A be a densely defined self-adjoint operator.

(a) Prove that $(\operatorname{ran} A)^\perp = \ker A$ and $(\ker A)^\perp = \overline{\operatorname{ran} A}$.

(b) Prove that A is invertible and the inverse A^{-1} is bounded if and only if there exists $c > 0$ such that

$$\|Ax\| \geq c\|x\| \text{ for all } x \in \operatorname{dom} A. \tag{17.5}$$

17.13. A densely defined operator A in a Hilbert space $\mathcal{H}$ is called closed if, for any sequence $\{x_k\} \subset \operatorname{dom} A$, the conditions $x_k \to x$ and $Ax_k \to y$ imply $x \in \operatorname{dom} A$ and $Ax = y$.

(a) Prove that any self-adjoint operator is closed.

(b) Prove that if A is a non-negative definite self-adjoint operator then $\operatorname{dom} A$ is a Hilbert space with respect to the following inner product:

$$(x, y) + (Ax, Ay).$$

17.14. Let F be a function satisfying (**??**), and let F_U be the associated Lebesgue-Stieltjes measure on $\mathbb{R}$. Set $F(a+) := \lim_{\lambda \to a+} F(\lambda)$ and prove that, for all $a < b$,

$$\begin{aligned} F_{(a,b)} &= F(b) - F(a+), \\ F_{[a,b]} &= F(b+) - F(a), \\ F_{\{a\}} &= F(a+) - F(a), \\ F_{(a,b]} &= F(b+) - F(a+). \end{aligned}$$

17.15. Let $\{s_k\}_{k=-\infty}^{\infty}$ be a double sequence of reals and let $\{t_k\}_{k=-\infty}^{\infty}$ be a double sequence of positive reals such that $\sum_k t_k < \infty$. Define function F by

$$F(\lambda) = \sum_{\{k: s_k < \lambda\}} t_k. \tag{17.6}$$

(a) Prove that F satisfies the conditions (**??**).

(b) Prove that, for any Borel set U,

$$F_U = \sum_{\{k:s_k\in U\}} t_k. \tag{17.7}$$

(c) Prove that, for any non-negative Borel function φ on $\mathbb{R}$,

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda) = \sum_{k\in\mathbb{Z}} t_k \varphi(s_k). \tag{17.8}$$

(d) Prove that a Borel function φ on $\mathbb{R}$ is integrable against F if and only if

$$\sum_{k\in\mathbb{Z}} t_k |\varphi(s_k)| < \infty,$$

and its integral against F is given by (17.8).

17.16. Prove that if function F satisfies (**??**) and F is continuously differentiable then

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda) = \int_{-\infty}^{+\infty} \varphi(\lambda)\, F'(\lambda)\, d\lambda, \tag{17.9}$$

for any non-negative Borel function φ.

17.17. For any function F on $\mathbb{R}$, defined its total variation on $\mathbb{R}$ by

$$\operatorname{var} F := \sup_{\{\lambda_k\}} \sum_{k\in\mathbb{Z}} |F(\lambda_{k+1}) - F(\lambda_k)| \tag{17.10}$$

where the supremum is taken over all increasing double sequences $\{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_k \to -\infty$ as $k \to -\infty$ and $\lambda_k \to +\infty$ as $k \to +\infty$.

(a) Show that F is the difference of two bounded monotone increasing functions if and only if $\operatorname{var} F < \infty$.

(b) Show that F is the difference of two functions satisfying (**??**) if and only if F is left-continuous and $\operatorname{var} F < \infty$.

(c) Let F be a left-continuous function on $\mathbb{R}$ such that $\operatorname{var} F < \infty$. Prove that

$$\operatorname{var} F = \sup_{|\varphi|\le 1} \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda),$$

where the supremum is taken over all continuous functions φ on $\mathbb{R}$ such that $|\varphi(\lambda)| \le 1$ for all λ.

(d) Show that if $F \in C^1(\mathbb{R})$ then

$$\operatorname{var} F = \int_{-\infty}^{+\infty} |F'(\lambda)|\, d\lambda.$$

17.18. Let F be any function on $\mathbb{R}$. We say that a Borel function φ is integrable against F if there are two functions $F^{(1)}$ and $F^{(2)}$ satisfying (**??**) such that $F = F^{(1)} - F^{(2)}$ and φ is integrable against $F^{(1)}$ and $F^{(2)}$. In this case, set

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda) := \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF^{(1)}(\lambda) - \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF^{(2)}(\lambda). \tag{17.11}$$

Prove that the value of the right hand side of (17.11) does not depend on the choice of $F^{(1)}$ and $F^{(2)}$.

17.19. Prove the following properties of projectors in a Hilbert space.

(a) Any projector P is a linear bounded self-adjoint operator and $P^2 = P$.
(b) For any bounded self-adjoint operator A such that $A^2 = A$, its range $\operatorname{ran} A$ is a closed subspace and A is the projector onto $\operatorname{ran} A$.
(c) Any projector P is non-negative definite, and $\|P\| = 1$ unless $P = 0$.

17.20. Let P be a projector and let $\{v_k\}$ be an orthonormal basis in $\mathcal{H}$. Prove that

$$\sum_k \|Pv_k\|^2 = \dim \operatorname{ran} P.$$

17.21. Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be a spectral resolution in a Hilbert space $\mathcal{H}$.

(a) Prove that if $a \le b$ then

$$E_a E_b = E_b E_a = E_a. \tag{17.12}$$

(b) Prove that $E_b - E_a$ is a projector for all $a \le b$. Hence or otherwise prove that the function $\lambda \mapsto \|E_\lambda x\|$ is monotone increasing, for any $x \in \mathcal{H}$.
(c) For a Borel set $U \subset \mathbb{R}$, define the operator

$$E_U := \int_U dE_\lambda.$$

Prove that, for all $-\infty < a < b < +\infty$,

$$E_{[a,b)} = E_b - E_a. \tag{17.13}$$

(d) Prove that if the intervals $[a_1, b_1)$ and $[a_2, b_2)$ are disjoint then the subspaces $\operatorname{ran} E_{[a_1,b_1)}$ and $\operatorname{ran} E_{[a_2,b_2)}$ are orthogonal.

17.22. Let $P_1, ..., P_k$ be projectors in $\mathcal{H}$ such that $\operatorname{ran} P_i \perp \operatorname{ran} P_j$ for $i \ne j$. Consider the operator

$$A = \sum_{i=1}^{k} \lambda_i P_i,$$

where λ_i are reals. Let $\varphi(\lambda) = \alpha_0 + \alpha_1 \lambda + ... \alpha_n \lambda^n$ be a polynomial with real coefficients, and define the operator

$$\varphi(A) := \alpha_0 \operatorname{id} + \alpha_1 A + ... + \alpha_n A^n.$$

Prove that

$$\varphi(A) = \sum_{i=1}^{k} \varphi(\lambda_i) P_i \tag{17.14}$$

and, for any $x \in \mathcal{H}$,

$$\|\varphi(A) x\|^2 = \sum_{i=1}^{k} \varphi(\lambda_i)^2 \|P_i x\|^2.$$

Prove also that if φ and ψ are two polynomials then

$$\varphi(A) + \psi(A) = (\varphi + \psi)(A) \tag{17.15}$$

and

$$\varphi(A) \psi(A) = (\varphi\psi)(A). \tag{17.16}$$

17.23. Let A be self-adjoint operator, and φ and ψ be Borel functions on $\operatorname{spec} A$.

(a) Prove that
$$\varphi(A) + \psi(A) = (\varphi + \psi)(A),$$
provided either both functions φ, ψ are non-negative or one of them is bounded.

(b) Prove that
$$\varphi(A)\,\psi(A) = (\varphi\psi)(A),$$
provided ψ is bounded.

17.24. Let A be a densely defined self-adjoint operator.

(a) Prove that if the inverse A^{-1} exists and is a bounded operator then $A^{-1} = \frac{1}{A}$. Here the operator $\frac{1}{A}$ is defined by $\frac{1}{A} := \psi(A)$ where $\psi(\lambda) = \frac{1}{\lambda}$ on $\operatorname{spec} A$.

(b) Prove that if $\operatorname{spec} A \subset [0, +\infty)$ then there exists a non-negative definite self-adjoint operator X such that $X^2 = A$.

(c) Prove that if $\operatorname{spec} A \subset [0, +\infty)$ then $\operatorname{ran} e^{-A} \subset \operatorname{dom} A$.

17.25. Let A be a compact self-adjoint operator, and let $\{v_k\}$ be an orthonormal basis in $(\ker A)^{\perp}$ of the eigenvectors of A with the eigenvalues $\{\lambda_k\}$, which is guaranteed by the Hilbert-Schmidt theorem. Prove that $\operatorname{spec} A$ consists of the sequence $\{\lambda_k\}$ and, possibly, 0.

17.26. Let A be a densely defined self-adjoint operator.

(a) Prove that A is non-negative definite if and only if $\operatorname{spec} A \subset [0, +\infty)$.

(b) Set
$$a = \inf_{\substack{x \in \operatorname{dom} A \\ \|x\|=1}} (Ax, x) \quad \text{and} \quad b = \sup_{\substack{x \in \operatorname{dom} A \\ \|x\|=1}} (Ax, x).$$
Prove that
$$\inf \operatorname{spec} A = a \quad \text{and} \quad \sup \operatorname{spec} A = b.$$

17.27. Let $\{E_\lambda\}$ be a spectral resolution of a self-adjoint operator A. For any Borel set $U \subset \mathbb{R}$, define the operator E_U by
$$E_U := 1_U(A) = \int_U dE_\lambda.$$
The mapping $U \mapsto E_U$ is called a spectral measure.

(a) Prove that E_U is a projector. Show that if $U = [a, b)$ where $a < b$ then $E_U = E_b - E_a$. In particular, $E_{(-\infty, b)} = E_b$.

(b) Prove that if $U_1 \subset U_2$ then $\operatorname{ran} E_{U_1} \subset \operatorname{ran} E_{U_2}$.

(c) Prove that if U_1 and U_2 are disjoint then $\operatorname{ran} E_{U_1} \perp \operatorname{ran} E_{U_2}$.

(d) Prove that if $\{U_i\}_{i=1}^{\infty}$ is an increasing sequence of Borel sets in $\mathbb{R}$ and $U = \bigcup_{i=1}^{\infty} U_i$ then $E_{U_i} \to E_U$ in the strong topology. Prove that the same is true if the sequence $\{U_i\}$ is decreasing and $U = \bigcap_{i=1}^{\infty} U_i$.

17.28. Let A be a densely defined self-adjoint operator and $\{E_\lambda\}$ be its spectral resolution.

(a) Prove that if a is an eigenvalue of A then $E_{\{a\}} := 1_{\{a\}}(A)$ is the projector onto the eigenspace of a.

(b) Prove that if a is an eigenvalue of A with an eigenvector x then, for any Borel function φ on $\operatorname{spec} A$, $x \in \operatorname{dom} \varphi(A)$ and
$$\varphi(A)\, x = \varphi(\alpha)\, x.$$

17.29. Let A be a self-adjoint operator whose spectrum consists of a finite sequence $\lambda_1, ..., \lambda_k$. Let P_i be the projector onto the eigenspace of λ_i, that is, $\operatorname{ran} P_i = \ker (A - \lambda_i \operatorname{id})$. Prove that $A = \sum_{i=1}^{k} \lambda_i P_i$.

17.30. Let A be a densely defined non-negative definite self-adjoint operator in $\mathcal{H}$ and $\{E_\lambda\}$ be its spectral resolution. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of Borel functions on $[0, +\infty)$ such that, for all n and $\lambda \in [0, +\infty)$,

$$|\varphi_n (\lambda)| \leq \Phi (\lambda),$$

where Φ is a non-negative Borel function on $[0, +\infty)$ such that

$$\int_0^{\infty} \Phi^2 (\lambda) \, d\|E_\lambda x\|^2 < \infty, \tag{17.17}$$

for some $x \in \mathcal{H}$. Prove that if $\varphi_n (\lambda) \to \varphi (\lambda)$ for any $\lambda \in [0, +\infty)$ then $x \in \operatorname{dom} \varphi (A) \cap \operatorname{dom} \varphi_n (A)$ and

$$\varphi_n (A) \, x \to \varphi (A) \, x.$$