B

Hints and solutions

Solutions to Chapter 1

1.1. Setting for simplicity $x = 0$, we have, by Taylor's expansion at 0,

$$f(y) = f(0) + \sum_{i=1}^{n} \frac{\partial f(0)}{\partial y_i} y_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f(0)}{\partial y_i \partial y_j} y_i y_j + \bar{o}\left(|y|^2\right). \tag{B.1}$$

By the symmetry argument,

$$\int_{S_r} y_i d\sigma(y) = 0 \quad \text{and} \quad \int_{S_r} y_i y_j d\sigma(y) = 0 \text{ for } i \neq j,$$

where $S_r \equiv S_r(0)$. To compute the integral of y_i^2, denote

$$I = \int_{S_r} y_i^2 d\sigma(y)$$

and observe that, by symmetry, I does not depend on i. Adding up for all $i = 1, 2, ..., n$, we obtain

$$I = \frac{1}{n} \int_{S_r} |y|^2 d\sigma(y) = \frac{1}{n} r^2 \int_{S_r} d\sigma = \frac{\omega_n}{n} r^{n+1}.$$

Hence, integrating (B.1) over S_r and using

$$\sum_{i=1}^{n} \frac{\partial^2 f}{\partial y_i^2}(0) = \Delta f(0),$$

we obtain

$$\begin{aligned} \frac{1}{\omega_n r^{n-1}} \int_{B_R} f(y) dy &= f(0) + \frac{I}{2\omega_n r^{n-1}} \Delta f(0) + \bar{o}(r^2) \\ &= f(0) + \frac{r^2}{2n} \Delta f(0) + \bar{o}(r^2), \end{aligned}$$

which was to be proved.

Second solution. Let Ω be a bounded region in $\mathbb{R}^n$ with smooth boundary and ν be the unit normal vector field on the boundary $\partial\Omega$ pointing outwards. For any function $u \in C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)$, applying (1.1) to $F = \nabla u$, we obtain

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma = \int_{\Omega} \Delta u \, dx.$$

Set $\Omega = B_r$ where B_r is the ball of radius r centered at the origin 0, that is

$$B_r = \{x \in \mathbb{R}^n : |x| < r\}.$$

Then $\partial\Omega = S_r$ and

$$\frac{1}{\sigma(S_r)} \int_{S_r} \frac{\partial u}{\partial \nu} d\sigma = \frac{d}{dr}\left(\frac{1}{\sigma(S_r)} \int_{S_r} u d\sigma\right).$$

Setting

$$J(r) := \frac{1}{\sigma(S_r)} \int_{S_r} u d\sigma,$$

we obtain from (1.1)

$$\frac{dJ}{dr} = \frac{1}{\omega_n r^{n-1}} \int_{B_r} \Delta u dx$$

whence

$$\frac{d^2 J}{dr^2} = \frac{1}{\omega_n r^{n-1}} \int_{S_r} \Delta u d\sigma - \frac{n-1}{\omega_n r^n} \int_{B_r} \Delta u dx.$$

Since the n-volume of the ball B_r is equal to

$$|B_r| = \int_0^r \sigma(S_t)\, dt = \frac{\omega_n}{n} r^n, \tag{B.2}$$

we obtain that, as $r \to 0+$,

$$\frac{dJ}{dr} = \frac{1}{\omega_n r^{n-1}} \left(\Delta u(0)\,|B_r| + \bar{o}(r^n)\right) = O(r)$$

and

$$\frac{d^2 J}{dr^2} = \Delta u(0) - \frac{n-1}{n}\Delta u(0) + \bar{o}(1) = \frac{1}{n}\Delta u(0) + \bar{o}(1).$$

We obtain that $J(0+) = u(0)$, $J'(0+) = 0$, and $J''(0+) = \frac{1}{n}\Delta u(0)$ whence by the Taylor formula

$$J(r) = J(0) + rJ'(0) + \frac{r^2}{2} J''(0) + \bar{o}\left(r^2\right) = u(0) + \frac{r^2}{2n}\Delta u(0) + \bar{o}\left(r^2\right),$$

which was to be proved.

1.2. Similarly to Exercise 1.1, set $x = 0$ and use the Taylor expansion (B.1). By the symmetry argument, we have

$$\int_{B_R} y_i dy = 0 \quad \text{and} \quad \int_{B_R} y_i y_j dy = 0 \text{ for } i \neq j,$$

where $B_R \equiv B_R(0)$. Set

$$J = \int_{B_R} y_i^2 dy$$

and observe that, by symmetry, J does not depend on i. Adding up for all $i = 1, 2, ..., n$, we obtain

$$nJ = \int_{B_R} |y|^2 dy = \int_0^R \left(\int_{S_r} r^2 d\sigma\right) dr = \int_0^R \omega_n r^{n+1} dr = \frac{\omega_n}{n+2} R^{n+2}.$$

In the same way, we have that the volume of B_R is equal to

$$|B_R| := \int_{B_R} dy = \int_0^R \sigma(S_r)\,dr = \frac{\omega_n}{n} R^n,$$

so that $c_n = \frac{\omega_n}{n}$. Hence, integrating (B.1) over B_R, we obtain

$$\begin{aligned} \frac{1}{|B_R|}\int_{B_R} f(y)dy &= f(0) + \frac{J}{2|B_R|}\Delta f(0) + \bar{o}(R^2) \\ &= f(0) + \frac{R^2}{2(n+2)}\Delta f(0) + \bar{o}(R^2), \end{aligned}$$

which was to be proved.

1.3. (*a*) We have

$$\begin{aligned} \widehat{p_t}(\xi) &= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4t} - ix\xi\right) dx \qquad \text{(B.3)} \\ &= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\left(-\frac{x_1^2+\cdots+x_n^2}{4t} - i(x_1\xi_1+\cdots+x_n\xi_n)\right) dx_1\cdots dx_n. \end{aligned}$$

Consider the integral

$$\begin{aligned} \int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{4t} - is\lambda\right) ds &= \int_{-\infty}^{\infty} \exp\left(-\frac{s^2+4it\lambda s+(2it\lambda)^2-(2it\lambda)^2}{4t}\right) ds \\ &= \exp(-t\lambda^2)\int_{-\infty}^{\infty} \exp\left(-\frac{(s+2i\lambda t)^2}{4t}\right) ds. \end{aligned}$$

By the change $z = s + 2i\lambda t$, the last integral can be treated as a contour integral along the line $\operatorname{Im} z = 2i\lambda t$. Using the standard tools based on the Cauchy integral formula, one reduces the integral to $\operatorname{Im} z = 0$, whence

$$\begin{aligned} \int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{4t} - is\lambda\right) ds &= e^{-t\lambda^2}\int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{4t}\right) ds \\ &= \sqrt{4\pi t}\,e^{-t\lambda^2}. \qquad \text{(B.4)} \end{aligned}$$

Hence, from (B.3) and (B.4), we obtain

$$\widehat{p_t}(\xi) = e^{-t(\xi_1^2+\cdots+\xi_n^2)} = e^{-t|\xi|^2}. \qquad \text{(B.5)}$$

(*b*) Indeed, it follows from (B.5) that

$$\int_{\mathbb{R}^n} p_t(x)\,dx = \widehat{p_t}(0) = 1.$$

(*c*) It is obvious from (B.5) that $\widehat{p_{t+s}} = \widehat{p_t}\,\widehat{p_s}$. Since the (inverse) Fourier transform takes the product of functions to convolution, we obtain

$$p_{t+s} = p_t * p_s.$$

(*d*) We have

$$\widehat{\frac{\partial p_t}{\partial t}} = \frac{\partial}{\partial t}\widehat{p_t} = \frac{\partial}{\partial t}e^{-t|\xi|^2} = -|\xi|^2\,\widehat{p_t}$$

and

$$\widehat{\frac{\partial p_t}{\partial x_k}} = \int_{\mathbb{R}^n} e^{-ix\xi} \frac{\partial}{\partial x_k} p_t(x)dx = -\int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} e^{-ix\xi} p_t(x)dx$$

$$= i\xi_k \int_{\mathbb{R}^n} e^{-ix\xi} p_t(x)dx = i\xi_k \widehat{p}_t.$$

Iterating the last identity, we obtain

$$\widehat{\frac{\partial^2 p_t}{\partial x_k^2}} = (i\xi_k)^2 \widehat{p}_t = -\xi_k^2 \widehat{p}_t$$

and, therefore,

$$\widehat{\Delta p_t} = -(\xi_1^2 + \cdots + \xi_n^2) \widehat{p}_t = -|\xi|^2 \widehat{p}_t = \widehat{\frac{\partial p_t}{\partial t}}.$$

Taking the inverse Fourier transform, we obtain

$$\frac{\partial p_t}{\partial t} = \Delta p_t.$$

1.4. (a) Using (B.5), we obtain

$$\widehat{u}_t(\xi) = \widehat{p}_t \widehat{f}(\xi) = e^{-t|\xi|^2} \widehat{f}(\xi). \tag{B.6}$$

(b) It follows from (B.6) that $\widehat{u}_t \in L^1(\mathbb{R}^n)$ and, moreover,

$$|\xi|^N \widehat{u}_t \in L^1(\mathbb{R}^n) \tag{B.7}$$

for any power N. Indeed, setting

$$E = \{\xi \in \mathbb{R}^n : |\widehat{u}_t(\xi)| > 1\}$$

we obtain that $\widehat{f} \in L^1(E)$ whence

$$|\xi|^N \widehat{u}_t = \left(|\xi|^N e^{-t|\xi|^2}\right) \widehat{f} \in L^1(E),$$

whereas

$$|\xi|^N |\widehat{u}_t| \le |\xi|^N e^{-t|\xi|^2} \in L^1(E^c),$$

whence (B.7) follows.

Formally differentiating the inversion formula

$$u_t(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi} \widehat{u}_t(\xi) d\xi,$$

we obtain, for any multiindex α,

$$\partial^\alpha u_t(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (i\xi)^\alpha e^{ix\xi} \widehat{u}_t(\xi) d\xi. \tag{B.8}$$

Due to (B.7), the integral in (B.8) converges uniformly for all x, which justifies (B.8).

Finally, we obtain

$$\widehat{\Delta u_t} = -|\xi|^2 \widehat{u} = -|\xi|^2 e^{-t|\xi|^2} \widehat{f}(\xi),$$

and

$$\widehat{\frac{\partial}{\partial t}u_t} = \frac{\partial}{\partial t}\widehat{u}_t\left(\xi\right) = -\left|\xi\right|^2 e^{-t|\xi|^2}\widehat{f}\left(\xi\right)$$

whence $\frac{\partial}{\partial t}u_t = \Delta u_t$ follows.

(c) By the Plancherel identity, we have

$$\|u_t\|_{L^2} = c\|\widehat{u}\|_{L^2} \ \text{ and } \ \|f\|_{L^2} = c\|\widehat{f}\|_{L^2}$$

where $c = (2\pi)^{-n/2}$. It is obvious from (B.6) that $|\widehat{u}_t| \leq |\widehat{f}|$, which implies

$$\|\widehat{u}_t\|_{L^2} \leq \|\widehat{f}\|_{L^2},$$

whence the claim follows.

(d) We have

$$\|u_t - f\|_{L^2} = c\|\widehat{u}_t - \widehat{f}\|_{L^2} = c\left[\int_{\mathbb{R}^n}\left|1 - e^{-t|\xi|^2}\right|^2 |\widehat{f}(\xi)|^2 d\xi\right]^{1/2},$$

and the last integral tends to 0 as $t \to 0$ by the dominated convergence theorem, because $|\widehat{f}(\xi)|^2$ is integrable and $1 - e^{-t|\xi|^2} \to 0$ pointwise.

(e) If $\widehat{f} \in L^1\left(\mathbb{R}^n\right)$ then also $\widehat{u}_t \in L^1\left(\mathbb{R}^n\right)$, and the inversion formula yields

$$\begin{aligned} u_t\left(x\right) - f\left(x\right) &= c\int_{\mathbb{R}^n} e^{ix\xi}\left(\widehat{u}_t(\xi) - \widehat{f}(\xi)\right)d\xi \\ &= c\int_{\mathbb{R}^n} e^{ix\xi}\left(e^{-t|\xi|^2} - 1\right)\widehat{f}(\xi)d\xi, \end{aligned}$$

whence

$$|u_t\left(x\right) - f\left(x\right)| \leq c\int_{\mathbb{R}^n}\left|e^{-t|\xi|^2} - 1\right| |\widehat{f}(\xi)|d\xi.$$

By the dominated convergence theorem, the last integral tends to 0 as $t \to 0$, which implies

$$\sup|u_t - f| \to 0.$$

1.5.

(a) This follows from

$$\sup_{|x|>\varepsilon} p_t\left(x\right) \leq \frac{1}{\left(4\pi t\right)^{n/2}}\exp\left(-\frac{\varepsilon^2}{4t}\right) \to 0 \text{ as } t \to 0.$$

(b) We have

$$\begin{aligned} \sup_{t>0} p_t\left(x\right) &= \sup_{t>0}\frac{1}{\left(4\pi t\right)^{n/2}}\exp\left(-\frac{\left|x\right|^2}{4t}\right) \\ &= \sup_{\tau>0}\frac{1}{\left(4\pi\tau\left|x\right|^2\right)^{n/2}}\exp\left(-\frac{1}{4\tau}\right) \\ &= \frac{\text{const}}{\left|x\right|^n}, \end{aligned}$$

where we have changed $\tau = t/|x|^2$. It follows that $p_t(x) \rightrightarrows 0$ as $x \to \infty$.

(*c*) It suffices to prove that $|\nabla p_t(x)|$ is bounded in the set $\{(t,x) : |x| > \varepsilon, t > 0\}$. We have

$$|\nabla p_t(x)| = \frac{1}{(4\pi t)^{n/2}} \frac{|x|}{2t} \exp\left(-\frac{|x|^2}{4t}\right)$$

whence

$$\begin{aligned}\sup_{t>0} |\nabla p_t(x)| &= \sup_{\tau>0} \frac{1}{\left(4\pi\tau |x|^2\right)^{n/2}} \frac{1}{2\tau|x|} \exp\left(-\frac{1}{4\tau}\right) \\ &= \frac{\text{const}}{|x|^{n+1}} \le \frac{\text{const}}{\varepsilon^{n+1}},\end{aligned}$$

which finishes the proof.

1.6. (*a*) Look for v in the form $v(x) = \exp\left(c|x|^2\right)$ for large enough c.

(*b*) Consider first the case when $Lu > 0$ and look at a point $x_0 \in \overline{\Omega}$ where u takes its maximal value. For the general case, consider the function $u + \varepsilon v$ where v is from part (*a*).

1.7. Observe that $f(x) = \pi^{n/2} p_{1/4}(x)$ and use (1.22). The answer is $u(t,x) = \pi^{n/2} p_{t+1/4}(x)$.

Solutions to Chapter 2

2.1. If $q = \infty$ then the embedding $L^\infty_{loc} \hookrightarrow L^p_{loc}$ is obvious because for any compact set $K \subset \Omega$ and any measurable function u in Ω,

$$\int_K |u|^p \, d\mu \le \operatorname*{esup}_K |u|^p \ \mu(K),$$

and $\mu(K) < \infty$. In the case $1 < q < \infty$, we use the Hölder inequality

$$\int_K |fg| \, d\mu \le \left(\int_K |f|^r \, d\mu\right)^{1/r} \left(\int_K |g|^{r'} \, d\mu\right)^{1/r'}, \tag{B.9}$$

which holds for arbitrary measurable functions f, g and exponents $r, r' \in (1, \infty)$ such that $\frac{1}{r} + \frac{1}{r'} = 1$. In particular, applying (B.9) for $f = |u|^p$, $g \equiv 1$, and $r = q/p$, we obtain

$$\int_K |u|^p \, d\mu \le \left(\int_K |u|^q \, d\mu\right)^{1/r} \mu(K)^{1/r'},$$

whence the embedding $L^q_{loc} \hookrightarrow L^p_{loc}$ follows.

2.2. Consider the case $p < \infty$. Assume from the contrary that the set $\{f < 0\}$ has a positive measure. Then, for some $\varepsilon > 0$, the set $E := \{f < -\varepsilon\}$ has also a positive measure. Since $f_k \ge 0$ a.e., we see that $|f - f_k| \ge \varepsilon$ on E whence

$$\|f - f_k\|^p_{L^p} = \int_\Omega |f - f_k|^p \, d\mu \ge \int_E |f - f_k|^p \, d\mu \ge \varepsilon^P \mu(E).$$

In the case $p = \infty$, we have

$$\|f - f_k\|_{L^\infty} = \operatorname*{esup}_{\Omega} |f - f_k| \geq \operatorname*{esup}_{E} |f - f_k| \geq \varepsilon.$$

In the both cases, we conclude that the sequence $\{f_k\}$ cannot converge to f in the norm of $L^P(\Omega)$.

2.3. We obviously have

$$|f * g(x)| \leq \int_{\mathbb{R}^n} |f(x-y)| \, |g(y)| \, dy \leq \|f\|_{L^\infty} \|g\|_{L^1},$$

whence the claim follows.

2.4. Use the same argument as in the proof of Lemma 2.4.

2.5. Use Fubini's theorem.

2.6. Use the same argument as in the proof of Lemma 2.1.

2.7. Apply inductively Lemma 2.4.

2.8. If $f \geq 0$ a.e. then we obviously have

$$\int_\Omega f\psi d\mu \geq 0, \tag{B.10}$$

for any non-negative $\psi \in C_0^\infty(\Omega)$. To prove the converse, fix a mollifier φ in $\mathbb{R}^n$ and an open set $\Omega' \Subset \Omega$. If $\varepsilon > 0$ is small enough then, for any $x \in \Omega'$, the function $\varphi_\varepsilon(x - \cdot)$ is supported in $B_\varepsilon(x) \subset \Omega$, which implies by hypothesis (B.10) that

$$f * \varphi_\varepsilon(x) = \int_\Omega f(z) \varphi_\varepsilon(x-z) \, dz \geq 0.$$

By Lemma 2.4, $f * \varphi_\varepsilon \to f$ in $L^1_{loc}(\mathbb{R}^n)$, whence it follows that $f \geq 0$ a.e. in Ω' (cf. Exercise 2.2). Since Ω' was arbitrary, we conclude $f \geq 0$ a.e.in Ω, which was to be proved.

2.9. (a) To prove that a function g is the distributional derivative of f, one has to verify that, for any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\int_{-\infty}^{+\infty} g\varphi \, dx = -\int_{-\infty}^{+\infty} f\varphi' \, dx. \tag{B.11}$$

If f' is continuous then $g = f'$ satisfies (B.11) by the integration by parts formula.

(b) Let $\{t_k\}_{k=-\infty}^{+\infty}$ be an increasing sequence of reals such that $f \in C^1[t_k, t_{k+1}]$ and the intervals $[t_k, t_{k+1}]$ cover all $\mathbb{R}$ (such a sequence exists by the definition of a piecewise continuously differentiable function). For any test function $\varphi \in C^\infty(\mathbb{R})$ we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} f'\varphi dx &= \sum_k \int_{t_k}^{t_{k+1}} f'\varphi dx = \sum_k \left([f\varphi]_{t_k}^{t_{k+1}} - \int_{t_k}^{t_{k+1}} f\varphi' dx \right) \\
&= \sum_k (f\varphi(t_{k+1}) - f\varphi(t_k)) - \int_{-\infty}^{+\infty} f\varphi' dx = -\int_{-\infty}^{+\infty} f\varphi' dx
\end{aligned}$$

where we have used the equality

$$\sum_k \left(f\varphi\left(t_{k+1}\right) - f\varphi\left(t_k\right)\right) = 0,$$

which is true because this sum contains only a finite number of non-zero terms, and they all cancel. Hence, $f'_{dist} = f'$. Note that although f' is not defined at $\{t_k\}$ this does not matter because f'_{dist} is considered as an element of L^2_{loc} and, hence, it is defined up to a set of measure zero, anyway.

(c) Since $f(x) = |x|$ is piecewise continuously differentiable, we obtain by the above

$$f'_{dist} = \operatorname{sign}(x) := \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

(d) For any $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$(f, \varphi') = \int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx = \int_0^\infty \varphi'(x)\,dx = -\varphi'(0) = -(\delta, \varphi),$$

whence we conclude $f'_{dist} = \delta$.

2.10. (a) Denote $U = \cup_\alpha \Omega_\alpha$. We need to show that $(u, \varphi) = (v, \varphi)$ for any $\varphi \in \mathcal{D}(U)$. Let $K = \operatorname{supp}\varphi$. Then the family $\{\Omega_\alpha\}$ covers K, and there is a finite subfamily $\{\Omega_j\}_{j=1}^k$that also covers K. By Theorem 2.2, there is a partition of unity $\psi_1, ..., \psi_k$ associated with this covering, that is, $\psi_j \in \mathcal{D}(\Omega_j)$ and

$$\sum_j \psi_j \equiv 1 \text{ in } K. \tag{B.12}$$

Setting $\varphi_j := \varphi\psi_j$, we obtain that $\varphi_j \in \mathcal{D}(\Omega_j)$ and

$$\varphi \equiv \varphi_1 + ... + \varphi_k \text{ in } \Omega. \tag{B.13}$$

Indeed, this identity holds in K due to (B.12), and in $\Omega \setminus K$ because all the functions involved vanish outside K. Since $u = v$ on Ω_j, we have

$$(u, \varphi_j) = (v, \varphi_j).$$

Adding up all these equalities and using (B.13), we obtain $(u, \varphi) = (v, \varphi)$.

(b) Let $\{\Omega_\alpha\}$ be the family of all open subsets of Ω such that $u = 0$ on Ω_α. By part (a), we have $u = 0$ on $U = \cup_\alpha \Omega_\alpha$. Hence, U is the maximal open set with this property.

2.11. (a) Note that $S = S(u)$ is always a closed set. If $\varphi \in \mathcal{D}(\Omega \setminus S)$ then $u \equiv 0$ on $\operatorname{supp} u$ and

$$(u, \varphi) = \int_\Omega u\varphi d\mu = 0.$$

Hence, u vanishes on $\Omega \setminus S$ in the distributional sense. Let us show that $\Omega \setminus S$ is the maximal open set where u vanishes in this sense. Indeed, let U be an open subset of Ω such that u vanishes in U as a distribution, and assume that U is not contained in $\Omega \setminus S$ so that there exists a point $x_0 \in U \cap S$. By definition of S, in any neighborhood of x_0 there is a point x where $u(x) \neq 0$.

In particular, such a point x can be found in U. Assume that $u(x) > 0$. By the continuity of u, there exists a neighborhood V of x such that $V \subset U$ and $u(y) > 0$ for any $y \in V$. Obviously, there exists a function $\varphi \in \mathcal{D}(V)$ such $(u, \varphi) > 0$, which contradicts the choice of U.

(*b*) Note that the set

$$S_0 := \bigcap_{v=u \text{ a.e.}} S(v) \tag{B.14}$$

is closed as an intersection of closed sets. Let us show that $u = 0$ in $\Omega \setminus S_0$ in the distributional sense, that is,

$$(u, \varphi) = 0 \text{ for any } \varphi \in \mathcal{D}(\Omega \setminus S_0).$$

Indeed, $\operatorname{supp} \varphi$ is covered by the union of all open sets $\Omega \setminus S(v)$, which implies by the compactness of $\operatorname{supp} \varphi$ that it is covered by some *finite* union of sets $\Omega \setminus S(v_i)$. Since v_i vanishes outside $S(v_i)$ and $v_i = u$ a.e., we obtain that $u = 0$ a.e. in $\Omega \setminus S(v_i)$. This implies that $u = 0$ a.e. also in the union of the sets $\Omega \setminus S(v_i)$, whence it follows that $(u, \varphi) = 0$.

We are left to show that if $U \subset \Omega$ is an open set where u vanishes in the distributional sense then $U \subset \Omega \setminus S_0$. Corollary 2.5 yields that $u = 0$ a.e. in U. Define function $v(x)$ in Ω by

$$v(x) = \begin{cases} u(x), & x \in \Omega \setminus U, \\ 0, & x \in U. \end{cases}$$

Then we have $v = u$ a.e. and $S(v) \subset \Omega \setminus U$, which implies $S_0 \subset \Omega \setminus U$ and, hence, $U \subset \Omega \setminus S_0$.

2.12. Let us prove first the product rule for the first derivative:

$$\partial_j (fu) = (\partial_j f) u + f (\partial_j u). \tag{B.15}$$

Indeed, for any $\varphi \in \mathcal{D}$, we have

$$\begin{aligned} (\partial_j (fu), \varphi) &= -(fu, \partial_j \varphi) = -(u, f \partial_j \varphi) = (u, (\partial_j f) \varphi - \partial_j (f\varphi)) \\ &= (u, (\partial_j f) \varphi) + (\partial_j u, f\varphi) = ((\partial_j f) u, \varphi) + (f \partial_j u, \varphi), \end{aligned}$$

whence (B.15) follows. By induction, (B.15) implies that

$$\partial_j^m (fu) = \sum_{k=1}^{m} \binom{m}{k} \partial_j^{m-k} f \, \partial_j^k u.$$

Using one more induction, one obtains (2.19) for the operator $\partial^\alpha = \partial_1^{\alpha_1} ... \partial_n^{\alpha_n}$.

2.13. (*a*) We need to show that

$$(\partial^\alpha u_k, \varphi) \to (\partial^\alpha u, \varphi) \text{ as } k \to \infty,$$

for any $\varphi \in \mathcal{D}$. Indeed, we have

$$(\partial^\alpha u_k, \varphi) = (-1)^{|\alpha|} (u_k, \partial^\alpha \varphi) \to (-1)^{|\alpha|} (u, \partial^\alpha \varphi) = (\partial^\alpha u, \varphi),$$

which was to be proved.

(b) The proof is similar to part (a):

$$(fu_k, \varphi) = (u_k, f\varphi) \to (u, f) = (fu, \varphi) \, .$$

2.14. If $x_k \to x$ then any subsequence of $\{x_k\}$ also converges to x. Assume that any subsequence of $\{x_k\}$ contains a sub-subsequence that converges to x, but $\{x_k\}$ does not converge to x. The latter means that there exists an open neighborhood U of x such that outside U there are infinitely many terms of the sequence $\{x_k\}$. In other words, a subsequence of $\{x_k\}$ lies outside U, which implies that it cannot have any sub-subsequence converging to x. This contradiction proves the claim.

2.15. Construct a sequence that does not converges a.e. but each subsequence has a sub-subsequence that converges to 0 a.e.. Then use Exercise 2.14.

2.16. Left to the reader

2.17. If $v \in \mathcal{D}(\mathbb{R}^n)$ then (2.27) is just definition of $\partial_i u$. Assume now $v \in L^2 \cap C^\infty$. Let ψ be a cutoff function of the unit ball $B_1(0)$ in $\mathbb{R}^n$ so that $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\psi \equiv 1$ on $B_1(0)$. Set

$$v_l(x) = \psi\left(\frac{x}{l}\right) v(x)$$

so that $v_l \in \mathcal{D}$. Therefore, we have

$$(\partial_i u, v_l)_{L^2} = -(u, \partial_i v_l)_{L^2} \, . \tag{B.16}$$

Letting $l \to \infty$, we obtain $v_l \stackrel{L^2}{\to} v$ but also $\partial_i v_l \stackrel{L^2}{\to} \partial_i v$, where the latter follows from

$$\partial_i v_l = \partial_i \left[\psi\left(\frac{x}{l}\right) v(x)\right] = \psi\left(\frac{x}{l}\right) \partial_l v(x) + \frac{1}{l}(\partial_i \psi)\left(\frac{x}{l}\right) v(x) \, .$$

Passing to the limit in (B.16), we obtain (2.27).

Finally, consider the general case $v \in L^2$ and $\partial_i v \in L^2$. Let φ be a mollifier and set

$$v_k = v * \varphi_{1/k}.$$

By Lemma 2.9, $v_k \in C^\infty$, and by Theorem 2.11 $v_k \in L^2$. Therefore, by the previous part of the proof, we have

$$(\partial_i u, v_k)_{L^2} = -(u, \partial_i v_k)_{L^2} \, . \tag{B.17}$$

By Lemma 2.9, we also have $\partial_i v_k = (\partial_i v) * \varphi_{1/k}$, and by Theorem 2.11, $v_k \stackrel{L^2}{\to} v$ and $\partial_i v_k \stackrel{L^2}{\to} \partial_i v$. Hence, letting $k \to \infty$ in (B.17), we obtain (2.27).

2.18. Use the same argument as in Theorem 2.11.

2.19. Use the same argument as in Theorem 2.11 (see also Exercise 2.20).

2.20. Assume first that $f \geq 0$. By Fubini's theorem, the function Qf is measurable. To prove $Qf \in L^r(M)$ and the estimate (2.30), consider first the case $r = \infty$. It obviously follows from (2.28) that

$$Qf(x) \leq \|f\|_\infty \int_M q(x,y)\, d\mu(y) \leq K\|f\|_\infty,$$

whence the claim follows. In the case $r = 1$, we obtain by Fubini's theorem and (2.29)

$$\begin{aligned}\|Qf\|_{L^1} &= \int_M \left[\int_M q(x,y) f(y)\, d\mu(y)\right] d\mu(x) \\ &= \int_M \left[\int_M q(x,y)\, d\mu(x)\right] f(y)\, d\mu(y) \\ &\leq K \int_M f(y)\, d\mu(y),\end{aligned}$$

whence (2.30) follows.

Let $1 < r < \infty$ and let $r' = \frac{r}{r-1}$ be the Hölder conjugate to r. Using the Hölder inequality and (2.28), we obtain, for almost all $x \in M$,

$$\begin{aligned}Qf(x) &\leq \int_M q(x,y)^{1/r'} \left[q(x,y)^{1/r} f(y)\right] d\mu(y) \\ &\leq \left(\int_M q(x,y)\, d\mu(y)\right)^{1/r'} \left(\int_M q(x,y) f^r(y)\, d\mu(y)\right)^{1/r} \\ &\leq K^{1/r'} \left(\int_M q(x,y) f^r(y)\, d\mu(y)\right)^{1/r},\end{aligned}$$

that is

$$|Qf(x)|^r \leq K^{r/r'} \int_M q(x,y) f^r(y)\, d\mu(y).$$

Using Fubini's theorem once again and (2.29), we obtain

$$\begin{aligned}\int_M |Qf(x)|^r\, d\mu(x) &\leq K^{r/r'} \int_M \left[\int_M q(x,y) f^r(y)\, d\mu(y)\right] d\mu(x) \\ &= K^{r/r'} \int_M \left[\int_M q(x,y)\, d\mu(x)\right] f^r(y)\, d\mu(y) \\ &\leq K^{r/r'+1} \int_M f^r(y)\, d\mu(y),\end{aligned}$$

which together with $1/r' + 1/r = 1$ implies (2.30).

Finally, if f is an arbitrary function from L^r then also f_+ and f_- belong to L^r, and by the first part of the proof, we have (2.30) for f_+ and f_-. Then $Qf = Qf_+ - Qf_-$ is measurable and

$$|Qf| \leq Q|f| = Qf_+ + Qf_-$$

whence

$$\|Qf\|_{L^p} \leq \|Qf_+\|_{L^r} + \|Qf_-\|_{L^r} \leq K(\|f_+\|_{L^r} + \|f_-\|_{L^r}) = K\|f\|_{L^r},$$

which was to be proved.

2.21. By the Hölder inequality,

$$\begin{aligned} |Qf(x)| &\leq \|q(x,\cdot)\|_{L^{r'}}\|f\|_{L^r} \\ &\leq \|q(x,\cdot)\|_{L^\infty}^{1-1/r'}\|q(x,\cdot)\|_{L^1}^{1/r'}\|f\|_{L^r} \\ &\leq C^{1/r}K^{1/r'}\|f\|_{L^r}, \end{aligned}$$

whence

$$\|Qf\|_{L^\infty} \leq C^{1/r}K^{1/r'}\|f\|_{L^r}, \tag{B.18}$$

which matches (2.31) for $s = \infty$.

If $s < \infty$ then, using the interpolation inequality

$$\|F\|_{L^s} \leq \|F\|_{L^\infty}^{1-r/s}\|F\|_{L^r}^{r/s},$$

we obtain from (2.30) and (B.18)

$$\begin{aligned} \|Qf\|_{L^s} &\leq \left(C^{1/r}K^{1/r'}\|f\|_{L^r}\right)^{1-r/s}(K\|f\|_{L^r})^{r/s} \\ &= C^{1/r-1/s}K^{1/r'+1/s}\|f\|_{L^r}. \end{aligned}$$

2.22. (a) If $\overline{B_\varepsilon(x)} \subset U$ then for some $\varepsilon' > \varepsilon$, we also have $B_{\varepsilon'}(x) \subset U$ which implies that, any $y \in B_{\varepsilon'-\varepsilon}(x)$ belongs to U. Hence, U is open. For any point $x \in U$ there is $\varepsilon > 0$ such that $\overline{B_\varepsilon(x)} \subset U$. Taking $k \geq 1/\varepsilon$ we obtain $x \in U_{1/k}$, which proves (2.32).

Denote for simplicity $f_\varepsilon = f * \varphi_\varepsilon$. Note that $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ by Lemma 2.1. Since $\operatorname{supp}\varphi_\varepsilon \subset B_\varepsilon(0)$, we obtain, for any two points $x, y \in \mathbb{R}^n$,

$$f_\varepsilon(x) - f_\varepsilon(y) = \int_{B_\varepsilon(0)} (f(x-z) - f(y-z))\varphi_\varepsilon(z)\,dz.$$

Assume that $x, y \in U_\varepsilon$. Then $x - z$ and $y - z$ belong to U, whence

$$|f(x-z) - f(y-z)| \leq L|x-y|$$

and

$$|f_\varepsilon(x) - f_\varepsilon(y)| \leq \int_{\mathbb{R}^n} L|x-y|\varphi_\varepsilon(z)\,dz = L|x-y|,$$

which proves that f_ε is Lipschitz in U_ε.

(b) As in the proof of Lemma 2.4, we have

$$f_\varepsilon(x) - f(x) = \int_{B_\varepsilon(0)} (f(x-z) - f(x))\varphi_\varepsilon(z)\,dz.$$

Then, for any $x \in U_\delta$ and $\varepsilon < \delta$, we have $x - z \in U$ and, hence,

$$|f(x-z) - f(x)| \leq L|z| \leq L\varepsilon.$$

This implies

$$\sup_{x\in U_\delta} |f_\varepsilon(x) - f(x)| \leq L\varepsilon$$

and $f_\varepsilon \rightrightarrows f$ in U_δ as $\varepsilon \to 0$.

2.23. Let us use the notation of Exercise 2.22. The fact that f_ε is Lipschitz in U_ε with a Lipschitz constant L implies that, for any vector $\xi \in \mathbb{R}^n$,

$$\left|\frac{\partial f_\varepsilon}{\partial \xi}\right| \le L\,|\xi|\,, \tag{B.19}$$

which easily follows from

$$|f_\varepsilon(x+t\xi) - f_\varepsilon(x)| \le L\,|t\xi|\,.$$

In particular, we have also $|\partial_j f_\varepsilon| \le L$ in U_ε. Therefore, for any index j, the sequence of functions $\left\{\partial_j f_{1/k}\right\}_{k=1}^{\infty}$ is bounded in $L^2(\Omega)$ for any open set $\Omega \Subset U$. By the weak compactness of a ball in L^2, there exists a sequence of integers $k_i \to \infty$ such that the subsequence $\partial_j f_{1/k_i}$ converges weakly in $L^2_{loc}(U)$ for any index $j = 1, ..., n$.

Rename for simplicity f_{1/k_i} by f_i, and let the weak limit of $\partial_j f_i$ be v_j. Let us show that $v_j = \partial_j f$. Indeed, for any $\varphi \in \mathcal{D}(U)$, we have, by the weak convergence,

$$(\partial_j f_i, \varphi) \to (v_j, \varphi) \quad \text{as } i \to \infty, \tag{B.20}$$

and by part (b)

$$(\partial_j f_i, \varphi) = -(f_i, \partial_j \varphi) \to -(f, \partial_j \varphi)\,, \quad \text{as } i \to \infty,$$

whence

$$(v_j, \varphi) = -(f, \partial_j \varphi)$$

and, hence, $v_j = \partial_j f$.

In particular, we have $\partial_j f \in L^2_{loc}(U)$. Let us prove that

$$|\nabla f| \le L \ \text{ a.e.}, \tag{B.21}$$

which will also imply that $\partial_j f \in L^\infty(U)$.

For any smooth compactly supported vector field ξ in U, we have by (B.20) and (B.19)

$$\int_U v_j \xi^j d\mu = \lim_{i\to\infty} \int_U \partial_j f_i\, \xi^j d\mu = \lim_{i\to\infty} \int_{\mathbb{R}^n} \frac{\partial f_i}{\partial \xi} d\mu \le L \int_{\mathbb{R}^n} |\xi|\, d\mu,$$

whence it follows that

$$\operatorname*{esup}_U |v| = \sup_{\xi \ne 0} \frac{\int_U v_j \xi^j d\mu}{\int_U |\xi|\, d\mu} \le L,$$

which proves (B.21).

2.24. The fact that fg is Lipschitz follows from the estimate

$$\begin{aligned} |fg(x) - fg(y)| &= |(f(x) - f(y))\,g(x) + (g(x) - g(y))\,f(y)| \\ &\le \sup|g|\, C_f\, |x-y| + \sup|f|\, C_g\, |x-y| \end{aligned}$$

where C_f and C_g are the Lipschitz constants of f and g, respectively.

It suffices to prove the product rule assuming that U is bounded. Choose a mollifier φ. By Exercise 2.23(b), (c) there is a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of positive numbers such that $\varepsilon_k \to 0$ and the sequence $f_k := f * \varphi_{\varepsilon_k}$ has the following properties: $f_k \to f$ locally uniformly in U, and $\partial_j f_k \to \partial_j f$ weakly in

$L^2_{loc}(U)$. The same applies to $g_k := g * \varphi_{\varepsilon_k}$. Since f_k and g_k are smooth functions, the product rule for them holds trivially:

$$\partial_j (f_k g_k) = (\partial_j f_k) g_k + f_k (\partial_j g_k).$$

Passing to the limit in $\mathcal{D}'(U)$ and using the fact that the norms $\|\partial_j f_k\|_\infty$ and $\|\partial_j g_k\|_\infty$ are uniformly bounded, we finish the proof.

2.25. Assume first that $\operatorname{supp} f \in (a,b)$. Let φ be a smooth cutoff function of $\operatorname{supp} f$ in (a,b). Then, by the definition of the distributional derivative,

$$\int_a^b f'\varphi dx = -\int_a^b f\varphi' dx.$$

However, the integral in the left hand side is equal to $\int_a^b f' dx$ because $\varphi \equiv 1$ on $\operatorname{supp} f$, and the integral in the right hand side is 0 because $\varphi' \equiv 0$ on $\operatorname{supp} f$. Hence, in this case we have

$$\int_a^b f'(x)\, dx = 0.$$

Assume now that $f(a) = f(b) = 0$. Then we extend f to $\mathbb{R}$ by setting $f(x) = 0$ outside $[a,b]$, and f is a Lipschitz function in $\mathbb{R}$. Hence, by the previous case,

$$\int_a^b f' dx = \int_{a-1}^{b+1} f' dx = 0.$$

Now let f be any Lipschitz function on $[a,b]$ and φ be a Lipschitz function on $[a,b]$ such that $\varphi(a) = \varphi(b) = 0$. Then $f\varphi$ is a Lipschitz function that vanishes at a and b, and by the previous case we obtain

$$\int_a^b (f\varphi)'\, dx = 0.$$

Using the product rule for Lipschitz functions (see Exercise 2.24), we obtain

$$\int_a^b f'\varphi dx = -\int_a^b f\varphi' dx. \tag{B.22}$$

Now apply this formula with the function

$$\varphi_\varepsilon(x) = \begin{cases} x/\varepsilon, & a \le x \le a+\varepsilon \\ 1, & a+\varepsilon \le x \le b-\varepsilon \\ \frac{1-x}{\varepsilon}, & b-\varepsilon \le x \le b, \end{cases}$$

where $0 < \varepsilon < \frac{1}{2}(b-a)$. Since

$$\varphi'_\varepsilon(x) = \begin{cases} 1/\varepsilon, & a < x < a+\varepsilon, \\ 0, & a+\varepsilon < x < b-\varepsilon, \\ -1/\varepsilon, & b-\varepsilon < x < b, \end{cases}$$

we obtain from (B.22)

$$\int_a^b f'\varphi_\varepsilon dx = -\frac{1}{\varepsilon}\int_a^{a+\varepsilon} f dx + \frac{1}{\varepsilon}\int_{b-\varepsilon}^b f dx.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{B.23}$$

The integration by parts formula (2.33) follows from (B.23) applied to fg and from the product rule.

2.26. (a) The case $k=0$ is trivial: if f is a bounded continuous function and $u \in L^2(\Omega)$ then $fu \in L^2(\Omega)$ and

$$\|fu\|_{L^2} \le \sup|f|\,\|u\|_{L^2}.$$

Let $k \ge 1$. Using the product rule (2.19) from Exercise 2.12, we obtain that, for any $|\alpha| \le k$, $\partial^\alpha(fu)$ is a linear combination of functions $\partial^{\alpha-\beta} f \partial^\beta u$ which all are in $L^2(\Omega)$ because $\partial^{\alpha-\beta} f$ is bounded continuous and $\partial^\beta u \in L^2$. Hence, $\partial^{\alpha-\beta} f \partial^\beta u \in L^2$ and $\partial^\alpha(fu) \in L^2$, which implies $fu \in W^k$.

It follow from (2.19) that

$$\begin{aligned}\|\partial^\alpha(fu)\|_{L^2} &\le C\sum_{\beta\le\alpha}\|\partial^{\alpha-\beta} f\,\partial^\beta u\|_{L^2}\\ &\le C\sum_{\beta\le\alpha}\sup\left|\partial^{\alpha-\beta} f\right|\|\partial^\beta u\|_{L^2}\\ &\le C\|f\|_{C^k}\|u\|_{W^k},\end{aligned}$$

where C is a constant depending on k, n. Adding up for all $|\alpha| \le k$, we obtain (2.38).

(b) Let $\Omega' \Subset \Omega$ be an open set. Since $u \in W^k(\Omega')$ and

$$\|f\|_{C^k(\Omega')} < \infty,$$

we obtain by part (b) that $fu \in W^k(\Omega')$, whence it follows that $u \in W^k_{loc}(\Omega)$.

2.27. The convergence in W^k obviously implies that in $\mathcal{D}'$. Therefore, $f_k \to f$ in $\mathcal{D}'$ whence by Exercise 2.13 we have $\partial^\alpha f_k \to \partial^\alpha f$ in $\mathcal{D}'$. Since also $\partial^\alpha f_k \to g$ in $\mathcal{D}'$, we conclude that $g = \partial^\alpha f$.

2.28. Let $\{f_j\}$ be a Cauchy sequence in W^k. Then for any multiindex α such that $|\alpha| \le k$, the sequence $\{\partial^\alpha f_j\}$ is Cauchy in L^2. Since L^2 is complete, any such sequence has a limit in L^2, say h^α. Setting $h \equiv h^0$, we obtain by Exercise 2.27 that $h^\alpha = \partial^\alpha h$ and, hence,

$$\|f_j - h\|^2_{W^k} = \sum_{|\alpha|\le k}\|\partial^\alpha f_j - \partial^\alpha h\|^2_{L^2} = \sum_{|\alpha|\le k}\|\partial^\alpha f_j - h^\alpha\|^2_{L^2} \to 0$$

as $j \to \infty$. Therefore, $f_j \to h \in W^k$, which was to be proved.

2.29. If $u \in W^k_c(\Omega)$ then extend u by 0 outside Ω. It follows from Lemma 2.6 that the derivatives of u are also extended by 0 and, hence, $u \in W^k(\mathbb{R}^n)$. Let φ be any mollifier. Then, by Lemma 2.9, $u * \varphi_\varepsilon$ is in $\mathcal{D}(\Omega)$ and, by Theorem 2.13, $u * \varphi_\varepsilon \to u$ in $W^k(\mathbb{R}^n)$, which finishes the proof.

2.30. In the view of Exercise 2.29, it suffices to show that any function $u \in W^k(\mathbb{R}^n)$ can be approximated by a sequence $\{u_l\}_{l=1}^{\infty}$ of functions $u_l \in W^k(\mathbb{R}^n)$ with compact supports. Let ψ be a cutoff function of the unit ball $B_1(0)$ in $\mathbb{R}^n$ so that $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\psi \equiv 1$ on $B_1(0)$. Set

$$u_l(x) = \psi\left(\frac{x}{l}\right) u(x).$$

Clearly, u_l has a compact support. By Exercise 2.26, $u_l \in W^k(\mathbb{R}^n)$. Let us show that $u_l \to u$ in $W^k(\mathbb{R}^n)$ as $l \to \infty$. Observe that $\psi(x/l) = 1$ if $|x| < l$ and, hence, $u_l = u$ on the ball $B_l \equiv B_l(0)$. Noticing that

$$u_l - u = (\psi(x/l) - 1)\, u$$

and using the estimate (2.38) of Exercise 2.26, we obtain

$$\|u_l - u\|_{W^k(B_l^c)} \leq C\|\psi(x/l) - 1\|_{C^k(B_l^c)}\|u\|_{W^k(B_l^c)},$$

where $B_l^c \equiv \mathbb{R}^n \setminus B_l$. Since

$$\|u\|_{W^k(B_l^c)} \to 0 \text{ as } l \to \infty$$

and the norm $\|\psi(x/l) - 1\|_{C^k}$ remains bounded, we conclude that $u_l \to u$ in $W^k(\mathbb{R}^n)$.

2.31. The identity (2.39) is trivial for $v \in \mathcal{D}(\Omega)$. Since the both parts of (2.39) are continuous in v with respect to the W^1-norm, it is extended to all $v \in W_0^1(\Omega)$.

2.32. (a) Let us first prove (2.40) for functions from $\mathcal{D}(\mathbb{R}^n)$ and for the classical derivatives, that is,

$$\widehat{\partial^\alpha \varphi} = (i\xi)^\alpha\, \widehat{\varphi}(\xi), \tag{B.24}$$

where $\varphi \in \mathcal{D}(\mathbb{R}^n)$ (cf. Exercise 1.3). Indeed, we have

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)\, dx,$$

whence, for any $k = 1, ..., n$,

$$\begin{aligned}(i\xi_k)\,\widehat{\varphi}(\xi) &= \int_{\mathbb{R}^n}(i\xi_k)\, e^{-ix\xi}\varphi(x)\, dx = -\int_{\mathbb{R}^n}\frac{\partial}{\partial x^k}e^{-ix\xi}\varphi(x)\, dx \\ &= \int_{\mathbb{R}^n} e^{-ix\xi}\frac{\partial}{\partial x^k}\varphi(x)\, dx = \widehat{\partial_k\varphi}(\xi).\end{aligned}$$

Iterating this identity, we obtain (B.24).

Next, let us use the Plancherel identity: if $u_1, u_2 \in L^2(\mathbb{R}^n)$ then

$$(u_1, u_2)_{L^2} = c\,(\widehat{u}_1, \widehat{u}_2)_{L^2}\ , \tag{B.25}$$

where $c = (2\pi)^{-n}$. Recall that, for complex valued functions, the inner product is defined by

$$(u_1, u_2)_{L^2} = \int_{\mathbb{R}^n} u_1\overline{u}_2 dx\ ,$$

where the bar stands for complex conjugation.

Let $v = \partial^\alpha u$ and note that, by definition,

$$(v, \varphi) = (-1)^{|\alpha|} (u, \partial^\alpha \varphi) \tag{B.26}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We have by (B.25)

$$(v, \varphi) = (v, \overline{\varphi})_{L^2} = c(\widehat{v}, \widehat{\overline{\varphi}})_{L^2} \tag{B.27}$$

and, similarly,

$$(u, \partial^\alpha \varphi) = c(\widehat{u}, \widehat{\partial^\alpha \overline{\varphi}})_{L^2}.$$

By (B.24), we have

$$\widehat{\partial^\alpha \overline{\varphi}} = (i\xi)^\alpha \widehat{\overline{\varphi}},$$

Denoting for simplicity $\psi = \widehat{\overline{\varphi}}$, we obtain

$$(\widehat{v}, \psi)_{L^2} = (-1)^\alpha (\widehat{u}, (i\xi)^\alpha \psi)_{L^2} = (-1)^\alpha ((-i\xi)^\alpha \widehat{u}, \psi)_{L^2} = ((i\xi)^\alpha \widehat{u}, \psi)_{L^2}. \tag{B.28}$$

Note that ψ ranges in the Fourier image of $\mathcal{D}(\mathbb{R}^n)$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (see Theorem 2.3) and the Fourier transform is an isometry of L^2 (up to a constant factor), the Fourier image of $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, too. Hence, (B.28) implies $\widehat{v} = (i\xi)^\alpha \widehat{u}$, which was to be proved.

(b) It follows from (B.25) and the result of part (a) that

$$\|\partial^\alpha u\|_{L^2}^2 = c\|\widehat{\partial^\alpha u}\|_{L^2}^2 = c\|(i\xi)^\alpha \widehat{u}\|_{L^2}^2 = c\int_{\mathbb{R}^n} |\xi^\alpha|^2 |\widehat{u}(\xi)|^2 d\xi,$$

which was to be proved.

2.33. Since the function $(i\xi)^\alpha \widehat{u}(\xi)$ is in $L^2(\mathbb{R}^n)$, it is the Fourier image of a function from $L^2(\mathbb{R}^n)$, say $v(x)$. Let us show that $\partial^\alpha u = v$. For that, we need to verify the identity (B.26) for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Using $\widehat{v} = (i\xi)^\alpha \widehat{u}$, (B.27), and (B.25), we obtain

$$\begin{aligned}(v, \varphi) &= c((i\xi)^\alpha \widehat{u}, \widehat{\overline{\varphi}})_{L^2} = c(\widehat{u}, (-i\xi)^\alpha \widehat{\overline{\varphi}})_{L^2} = c(-1)^{|\alpha|}(\widehat{u}, \widehat{\partial^\alpha \overline{\varphi}})_{L^2} \\ &= (-1)^{|\alpha|}(u, \partial^\alpha \overline{\varphi})_{L^2} = (-1)^{|\alpha|}(u, \partial^\alpha \varphi),\end{aligned}$$

which was to be proved.

2.34. We obtain from Exercise 2.32 that if $u \in W^k$ then

$$\|u\|_{W^k}^2 = \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^2}^2 = c\int_{\mathbb{R}^n} \left[\sum_{|\alpha|\le k} |\xi^\alpha|^2\right] |\widehat{u}(\xi)|^2 d\xi.$$

Since the sum in the bracket is comparable to $\left(1 + |\xi|^2\right)^k$, we obtain

$$\|u\|_{W^k}^2 \simeq \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^k d\xi, \tag{B.29}$$

whence it follows that the right hand side of (B.29) is finite.

Conversely, if the right hand side of (B.29) is finite then, by Exercise 2.33, $\partial^\alpha u \in L^2$ for any $|\alpha| \le k$ and, hence, $u \in W^k$.

2.35. By Lemma 2.8, we have, for any $\psi \in \mathcal{D}(\mathbb{R}^n)$,

$$(u * \varphi, \psi) = \left(u, \varphi' * \psi\right) \leq \|u\|_{W^{-k}} \|\varphi' * \psi\|_{W^k},$$

where $\varphi'(x) = \varphi(-x)$. Since by Theorem 2.13

$$\|\varphi' * \psi\|_{W^k} \leq \|\psi\|_{W^k},$$

we obtain

$$(u * \varphi, \psi) \leq \|u\|_{W^{-k}} \|\psi\|_{W^k},$$

whence (2.49) follows.

2.36. Let $W_0^k(\Omega)$ denote the closure of $\mathcal{D}(\Omega)$ in $W^k(\Omega)$. Then it follows from the definition of $W^{-k}(\Omega)$ that the space $W^{-k}(\Omega)$ is dual to $W_0^k(\Omega)$ (the dual space consists of all bounded linear functional). However, $W_0^k(\Omega)$ is a Hilbert space as a closed subspace of a Hilbert space. Therefore, by Riesz representation theorem, the dual space to $W_0^k(\Omega)$ is isometric to $W_0^k(\Omega)$, whence the both claims follow.

2.37. By Exercise 1.7, we have $P_t f(x) = \pi^{n/2} p_{t+1/4}(x)$. Hence,

$$\begin{aligned}
\varphi(t) &= (P_t f, f) = \pi^{n/2}\left(p_{t+1/4}, p_{1/4}\right) \\
&= \pi^{n/2} p_{t+1/4} * p_{1/4}(0) \\
&= \pi^{n/2} p_{t+1/2}(0) \\
&= 4^{-n/2}(t+1/2)^{-n/2}.
\end{aligned}$$

2.38. The proof of Lemma 2.17 goes through except that one needs to verify (2.55) without Lemma 2.1. Indeed, (2.55) follows directly from

$$\partial_j(P_t f) = \frac{\partial}{\partial x^j} \int_{\mathbb{R}^n} p_t(y) f(x-y)\, dy = \int_{\mathbb{R}^n} p_t(y) \frac{\partial}{\partial x^j} f(x-y)\, dy,$$

and this is true because the last integral converges locally uniformly in x, thanks to the boundedness of the derivative $\partial_j f$.

2.39. By Exercise 1.4

$$\widehat{P_t f}(\xi) = \widehat{p}_t(\xi) \widehat{f}(\xi) = e^{-t|\xi|^2} \widehat{f}(\xi).$$

By the Plancherel identity,

$$\|f - P_t f\|_{L^2}^2 = c\|\widehat{f} - \widehat{P_t f}\|_{L^2}^2 = c\|(1 - e^{-t|\xi|^2})\widehat{f}\|_{L^2}^2,$$

where $c = (2\pi)^{-n}$. Using the inequality

$$1 - e^{-t|\xi|^2} \leq \sqrt{t|\xi|^2}, \tag{B.30}$$

we obtain

$$\|f - P_t f\|_{L^2}^2 \leq ct\| |\xi| \widehat{f}\|_{L^2}^2 = ct \int_{\mathbb{R}^n} |\xi|^2 \left|\widehat{f}(\xi)\right|^2 d\xi.$$

If $f \in W^1$ then, by Exercise 2.32,

$$\|\nabla f\|_{L^2}^2 = \sum_j \|\partial_j f\|_{L^2}^2 = c \sum_j \|\widehat{\partial_j f}\|_{L^2}^2 = c \int_{\mathbb{R}^n} \sum_j |\xi_j|^2 \, |\widehat{f}(\xi)|^2 d\xi = c \int_{\mathbb{R}^n} |\xi|^2 \, |\widehat{f}(\xi)|^2 d\xi,$$

which together with the previous line yields (2.66).

If $f \in W^2$ then we use instead of (B.30)

$$1 - e^{-t|\xi|^2} \leq t \, |\xi|^2,$$

which yields

$$\|f - P_t f\|_{L^2}^2 \leq ct^2 \| \, |\xi|^2 \, \widehat{f} \|_{L^2}^2 = ct^2 \int_{\mathbb{R}^n} |\xi|^4 \, |\widehat{f}(\xi)|^2 d\xi.$$

Using Exercise 2.32, we have

$$\|\Delta f\|_{L^2}^2 = c\|\widehat{\Delta f}\|_{L^2}^2 = c\| \sum_j \xi_j^2 \widehat{f}(\xi) \|_{L^2}^2 = c \int_{\mathbb{R}^n} |\xi|^4 \, |\widehat{f}(\xi)|^2 d\xi,$$

which together with the previous line finishes the proof.

Solutions to Chapter 3

3.1. By Lemma 3.4, there exists a countable family $\{U_i\}_{i=1}^{\infty}$ of relatively compact charts covering all M. Set

$$\Omega_k = \bigcup_{j=1}^{k} U_j \tag{B.31}$$

so that $\{\Omega_k\}_{k=1}^{\infty}$ is an increasing sequence of relatively compact open sets covering M. However, we may not have yet the inclusion $\overline{\Omega}_k \subset \Omega_{k+1}$. To achieve that, we will select a subsequence of $\{\Omega_k\}$. The first term to be selected is Ω_1. If we have already selected Ω_i then observe that $\overline{\Omega}_i$ is a compact set and, hence is covered by a finitely many of sets $\{\Omega_k\}$. Since this family is increasing, $\overline{\Omega}_i$ is covered by one of Ω_k. Hence, select this Ω_k as the next term in the subsequence.

Let M be connected. The sets U_j considered above are always connected (cf. the proof of Lemma 3.4). All we need is to renumber the sequence $\{U_j\}$ in an appropriate order so that each set Ω_k defined by (B.31) is connected. We will do this by means of an inductive construction. At each step, some of the sets $\{U_j\}$ will be declared *selected* and denoted by $V_1, V_2, \ldots$. Set $V_1 = U_1$ and declare U_1 selected. Choose a non-selected set U_j with the minimal j that intersects V_1, denote it by V_2 and declare selected, etc. If $V_1, \ldots, V_i$ are already defined then choose a non-selected set U_j with minimal j that intersects $V_1 \cup V_2 \ldots \cup V_i$, denote it by V_{i+1} and declare selected. The process stops if we cannot choose V_{i+1}, and continues countably many times otherwise. By construction, all the unions $V_1 \cup V_2 \ldots \cup V_i$ are connected, so we need only to verify that the sequence $\{V_i\}$ covers all M.

Assume first that the sequence $\{V_i\}$ is finite. Then, at some step i, any non-selected U_j is disjoint with $V := V_1 \cup V_2 ... \cup V_i$. Let U be the union of all non-selected U_j. All selected U_j are contained in $V_1, ..., V_i$ and, hence, their union is V. Since U and V are two disjoint open sets covering M, one of them must be empty, which can be only U, whence $V = M$.

Assume now that the sequence $\{V_i\}$ is infinite, and show that it covers M. If this is not the case then there exists U_j which is not covered by $V = \bigcup_i V_i$. If U_j intersects V then it should have been selected at some step because there are selected sets $U_{j'}$ with $j' > j$. Hence, any U_j that is not covered by V is actually disjoint with V. Let U be the union of all such sets U_j. Clearly, U and V cover M and are disjoint, which implies by the connectedness of M that $U = \emptyset$ and, hence, $V = M$.

3.2. First of all, there exists countable family $\{U_i\}$ of locally compact charts covering M (see Lemma 3.4). Let $\{\Omega_k\}$ be a sequence from Exercise 3.1. Let us construct inductively a locally finite family $\mathcal{F}$ of relatively compact charts which will also cover M. At step 0, set $\mathcal{F} = \emptyset$. At step $k \geq 1$, consider the compact set $\overline{\Omega}_k \setminus \Omega_{k-1}$ (where $\Omega_0 := \emptyset$). This set is covered by a finite number of charts from the family $\{U_i\}$; say $U_1, ..., U_m$. Then add to $\mathcal{F}$ the charts $U_i \setminus \overline{\Omega}_{k-1}$, $i = 1, ..., m$. Clearly, the newly added charts cover $\overline{\Omega}_k \setminus \overline{\Omega}_{k-1}$ and do not intersect $\overline{\Omega}_{k-1}$.

The family of charts $\mathcal{F}$ obtained in this way covers all sets $\overline{\Omega}_k \setminus \overline{\Omega}_{k-1}$ and hence M. Let us verify that it is locally finite. Indeed, any compact set K is contained in one of the sets Ω_k. Up to the step k of the above construction, family $\mathcal{F}$ contains a finite number of chart. From step $k+1$ onwards, each added chart does not intersect Ω_k. Hence, there is only a finite number of charts in $\mathcal{F}$ intersecting Ω_k and hence K, which finishes the proof.

3.3. Use d and ∇ in the local coordinates.

3.4. The same hint as above.

3.5. Let $x^1, ..., x^n$ and $y^1, ..., y^n$ be two coordinate systems and let g^x and g^y be the matrices of $\mathbf{g}$ in these systems, respectively. By Lemma 3.12, we have

$$g^y = J^T g^x J$$

where J is the Jacobian matrix of the change $y = y(x)$. It follows that

$$\det g^y = (\det J)^2 \det g^x. \tag{B.32}$$

The same identity holds for the metric $\widetilde{\mathbf{g}}$. Dividing it by (B.32) and noticing that $(\det J)^2$ cancels out, we obtain

$$\frac{\det \widetilde{g}^y}{\det g^y} = \frac{\det \widetilde{g}^x}{\det g^x},$$

which was to be proved.

3.6. (a) Fix a point $x \in M$ and choose an orthonormal basis $e = \{e_1, ..., e_n\}$ in $T_x M$ with respect to the inner product $\langle , \rangle_{\mathbf{g}}$ where the quadratic form $\widetilde{\mathbf{g}}(x)$ is diagonal, say $\widetilde{g}_{ii} = \lambda_i$ and $\widetilde{g}_{ij} = 0$ if $i \neq j$. Then we have in this basis

$$\det g\left(x\right)=1 \text{ and } \quad \det \widetilde{g}\left(x\right)=\lambda_1...\lambda_n.$$

On the other hand,

$$\lambda_i=\widetilde{g}_{ii}=\langle e_i,e_i\rangle_{\widetilde{\mathbf{g}}}\leq C\langle e_i,e_i\rangle_{\mathbf{g}}=Cg_{ii}=C,$$

whence it follows that

$$\frac{\det \widetilde{g}\left(x\right)}{\det g\left(x\right)}\leq C^n.$$

By Exercise 3.5, the ratio of the determinants is independent on the choice of the coordinate system. Hence, we obtain from (3.21) that $\frac{d\widetilde{\nu}}{d\nu}\leq C^{n/2}$.

(b) It follows from (3.32) that

$$\frac{\widetilde{\mathbf{g}}^{-1}}{\mathbf{g}^{-1}}\geq C^{-1},$$

where $\mathbf{g}^{-1}$ is the metric tensor on covectors, whose matrix in the local coordinates is $\left(g^{ij}\right)$ (cf. Section 3.3). Indeed, in the basis e as in part (b), the matrix of $\mathbf{g}^{-1}$ is the identity matrix, while that of $\widetilde{\mathbf{g}}^{-1}$ is the diagonal matrix with the diagonal entries $\lambda_i^{-1}\geq C^{-1}$, whence the claim follows. Using (3.19), we obtain

$$\left|\nabla f\right|_{\mathbf{g}}^2=\langle df,df\rangle_{\mathbf{g}^{-1}}\leq C\langle df,df\rangle_{\widetilde{\mathbf{g}}^{-1}}=C\left|\nabla f\right|_{\widetilde{\mathbf{g}}}^2,$$

which finishes the proof.

3.7. For any $\varphi\in C_0^\infty\left(M\right)$, we obtain using the divergence theorem and (3.20)

$$\begin{aligned}\int_M \operatorname{div}_\mu\left(u\omega\right)\varphi d\mu &= -\int_M\langle u\omega,\nabla\varphi\rangle d\mu=-\int_M\langle\omega,u\nabla\varphi\rangle d\mu\\ &= -\int_M\langle\omega,\nabla\left(u\varphi\right)-\varphi\nabla u\rangle d\mu\\ &= -\int_M\langle\omega,\nabla\left(u\varphi\right)\rangle d\mu+\int_M\langle\omega,\varphi\nabla u\rangle d\mu\\ &= \int_M\left(\operatorname{div}_\mu\omega\right)u\varphi\,d\mu+\int_M\langle\omega,\nabla u\rangle\varphi d\mu,\end{aligned}$$

whence (3.48) follows.

3.8. Using the identity $\Delta_\mu=\operatorname{div}_\mu\nabla$ and the product rules for ∇ and div_μ (cf. Exercises 3.3 and 3.7), we obtain

$$\begin{aligned}\Delta_\mu\left(uv\right) &= \operatorname{div}_\mu\left(\nabla\left(uv\right)\right)=\operatorname{div}_\mu\left(u\nabla v+v\nabla u\right)\\ &= \langle\nabla u,\nabla v\rangle+u\Delta_\mu v+\langle\nabla v,\nabla u\rangle+v\Delta_\mu u\\ &= u\Delta_\mu v+2\langle\nabla u,\nabla v\rangle+\left(\Delta_\mu u\right)v.\end{aligned}$$

3.9. Use the chain rule for ∇ of Exercise 3.4 and the product rule for div_μ of Exercise 3.7.

3.10. The Hermite polynomials satisfy the equation

$$h_k'' - 2xh_k' + 2kh_k = 0,$$

which can be obtained directly from the definition. Writing it in the form

$$\Delta_\mu h_k + 2kh_k = 0,$$

we obtain that h_k is an eigenfunction of the weighted Laplace operator Δ_μ with the eigenvalue $2k$.

3.11. Using the obvious identity $\widetilde{\nabla} = \frac{1}{a}\nabla$ where $\widetilde{\nabla}$ is the gradient of $\widetilde{\mathbf{g}}$, the Green formula (3.43) and (3.17), we obtain, for all $u, v \in C_0^\infty(M)$,

$$\begin{aligned}\int u\widetilde{\Delta}_{\widetilde{\mu}} v\, d\widetilde{\mu} &= -\int \langle \widetilde{\nabla} u, \widetilde{\nabla} v\rangle_{\widetilde{\mathbf{g}}} d\widetilde{\mu} = -\int \langle du, \widetilde{\nabla} v\rangle\, b d\mu \\ &= -\int \langle \nabla u, \frac{b}{a}\nabla v\rangle_{\mathbf{g}}\, d\mu = \int u \operatorname{div}_\mu\left(\frac{b}{a}\nabla v\right)\, d\mu = \int u\left(\frac{1}{b}\operatorname{div}_\mu\left(\frac{b}{a}\nabla v\right)\right) d\widetilde{\mu},\end{aligned}$$

whence the claim follows.

3.12. For all $u, v \in C_0^\infty(M)$, we have

$$\int uLv\ d\widetilde{\mu} = \int u \operatorname{div}_\mu(A\nabla v)\, d\mu = -\int \langle \nabla u, A\nabla v\rangle_{\mathbf{g}} d\mu = -\int \langle du, A\nabla v\rangle d\mu$$

where we have used the divergence theorem on $(M, \mathbf{g}, \mu)$ and the identity (3.17). On the other hand, using the Green formula on $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$, we obtain

$$\int u\widetilde{\Delta}_{\widetilde{\mu}} v\, d\widetilde{\mu} = -\int \langle \widetilde{\nabla} u, \widetilde{\nabla} v\rangle_{\widetilde{\mathbf{g}}} d\widetilde{\mu} = -\int b\langle \widetilde{\nabla} u, \widetilde{\nabla} v\rangle_{\widetilde{\mathbf{g}}} d\mu = -\int \langle du, b\widetilde{\nabla} v\rangle d\mu.$$

Hence, the identity $L = \widetilde{\Delta}_{\widetilde{\mu}}$ amounts to

$$A\nabla v = b\widetilde{\nabla} v.$$

Since $\nabla v = \mathbf{g}^{-1}dv$ and $\widetilde{\nabla} v = \widetilde{\mathbf{g}}^{-1}dv$ (see (3.17)), the above equation is equivalent to

$$A\mathbf{g}^{-1} = b\widetilde{\mathbf{g}}^{-1},$$

whence $\widetilde{\mathbf{g}} = b\mathbf{g}A^{-1}$.

3.13. If $d\widetilde{\mu} = bd\mu$ then, by Exercise 3.11 and the product rule (3.48) of Exercise 3.7,

$$\Delta_{\widetilde{\mu}} u = \frac{1}{b}\operatorname{div}_\mu(b\nabla u) = \operatorname{div}_\mu(\nabla u) + \frac{1}{b}\langle \nabla b, \nabla u\rangle = \Delta_\mu u + \langle \nabla \log b, \nabla u\rangle.$$

Hence, $L = \Delta_{\widetilde{\mu}}$ provided $\log b = v$ that is, $b = e^v$.

3.14. Note that dF is a non-zero covector, that is, a linear functional in T_xM, and the equation $\langle dF, \xi\rangle = 0$, indeed, defines a $(n-1)$-dimensional subspace of T_xM. Since $\dim T_xN = n - 1$, it suffices to verify that every vector from T_xN satisfies equation (3.52). Indeed, if $\xi \in T_xN$ then, by definitions (3.9) and (3.50),

$$\langle dF, \xi\rangle = \xi(F) = \xi(F|_N) = \xi(0) = 0.$$

In the case $M = \mathbb{R}^n$, rewrite the above identity in the form $\langle \nabla F, \xi \rangle_{\mathbf{g}} = 0$, whence all the rest claims follows.

3.15. Using the notation of Section 3.8, we have

$$g = \begin{pmatrix} \boxed{g_X} & 0 \\ 0 & \psi^2(x) \boxed{g_Y} \end{pmatrix}. \tag{B.33}$$

In particular, we see that

$$\det g = \psi^{2m}(x) \det g_X \det g_Y,$$

which implies (3.56). It also follows from (B.33) that

$$g^{-1} = \begin{pmatrix} \boxed{g_X^{-1}} & 0 \\ 0 & \psi^{-2}(x) \boxed{g_Y^{-1}} \end{pmatrix}.$$

Using (3.18) we see that the gradient ∇ on M is given by the column-vector

$$\nabla u = \begin{bmatrix} \nabla_X u \\ \psi^{-2} \nabla_Y u \end{bmatrix}.$$

Consider a vector field $v = \begin{bmatrix} v_X \\ v_Y \end{bmatrix}$ on M. By (3.35), we obtain the following formula for the divergence div on M

$$\begin{aligned} \operatorname{div} v &= \frac{1}{\psi^m \sqrt{\det g_Y}} \operatorname{div}_X \left(\psi^m \sqrt{\det g_Y} v_X \right) + \frac{1}{\psi^m \sqrt{\det g_X}} \operatorname{div}_Y \left(\psi^m \sqrt{\det g_X} v_Y \right) \\ &= \operatorname{div}_X v_X + \frac{1}{\psi^m} \langle \nabla_X \psi^m, v_X \rangle + \operatorname{div}_Y v_Y. \end{aligned}$$

Finally, applying this to $v = \nabla u$ we obtain

$$\Delta u = \operatorname{div} \nabla u = \operatorname{div}_X (\nabla_X u) + \frac{1}{\psi^m} \langle \nabla_X \psi^m, \nabla_X u \rangle + \operatorname{div}_Y (\psi^{-2} \nabla_Y u),$$

whence (3.57) follows.

3.16. Let us simplify the notation by renaming x^{n+1} to t and x' to x. Then the equation of the sphere $\mathbb{S}^n$ is

$$|x|^2 + t^2 = 1, \tag{B.34}$$

and the Euclidean metric in $\mathbb{R}^{n+1}$ is given by

$$\mathbf{g}_{\mathbb{R}^{n+1}} = \left(dx^1\right)^2 + ... + (dx^n)^2 + dt^2. \tag{B.35}$$

Since the spherical metric is obtained by restricting $\mathbf{g}_{\mathbb{R}^{n+1}}$ to $\mathbb{S}^n$, all we need is to rewrite (B.35) via the coordinates $y^1, ..., y^n$ taking into account the equation (B.34). The point y is obtained from by scaling x by the factor

$1+t$, which arises from comparison of the segments $[-1,t]$ and $[-1,0]$ of the axis x^{n+1}. Hence, we have

$$y^i = \frac{x^i}{1+t}. \tag{B.36}$$

The equation (B.34) implies that on $\mathbb{S}^n$

$$tdt = -x^i dx^i.$$

Equation (B.36) yields

$$dy^i = \frac{(1+t)\,dx^i - x^i dt}{(1+t)^2},$$

whence

$$\begin{aligned}
\sum_i \left(dy^i\right)^2 &= (1+t)^{-4} \sum_i \left((1+t)^2 \left(dx^i\right)^2 + \left(x^i\right)^2 dt^2\right) \\
&\quad - (1+t)^{-4} \sum_i \left(x^i dx^i\,(1+t)\,dt + (1+t)\,dt\,x^i dx^i\right) \\
&= (1+t)^{-4} \left(\sum_i (1+t)^2 \left(dx^i\right)^2 + \left(1-t^2\right) dt^2 + 2t\,(1+t)\,dt^2\right) \\
&= (1+t)^{-2} \left(\sum_i \left(dx^i\right)^2 + dt^2\right)
\end{aligned}$$

and

$$\left(dx^1\right)^2 + ... + (dx^n)^2 + dt^2 = (1+t)^2 \left(\left(dy^1\right)^2 + ... + (dy^n)^2\right). \tag{B.37}$$

It follows from (B.34) and (B.36) that

$$|y|^2 = \frac{|x|^2}{(1+t)^2} = \frac{1-t}{1+t}$$

whence

$$1+t = \frac{2}{1+|y|^2}.$$

Substituting into (B.37), we obtain

$$\left(dx^1\right)^2 + ... + (dx^n)^2 + dt^2 = \frac{4}{\left(1+|y|^2\right)^2} \left(\left(dy^1\right)^2 + ... + (dy^n)^2\right),$$

which was to be proved.

3.17. Left to the reader

3.18. This is similar to Exercise 3.16. Renaming x^{n+1} to t and x' to x, we obtain the equation of the hyperboloid $\mathbb{H}^n$

$$t^2 - |x|^2 = 1. \tag{B.38}$$

Since $|x|^2 = t^2 - 1 < (1+t)^2$, the identity

$$y = \frac{x}{1+t} \tag{B.39}$$

defines a point $y \in \mathbb{R}^n$ with $|y| < 1$, that is, $y \in \mathbb{B}^n$. Conversely, any point $y \in \mathbb{B}^n$ uniquely determines a pair (x,t) satisfying (B.38) by

$$t = \frac{1+|y|^2}{1-|y|^2} \quad \text{and} \quad x = \frac{2y}{1-|y|^2}. \tag{B.40}$$

The hyperbolic metric is obtained by restricting to $\mathbb{H}^n$ the Minkowski metric

$$\mathbf{g}_{Mink} = \left(dx^1\right)^2 + ... + (dx^n)^2 - dt^2.$$

The equation (B.38) implies that on $\mathbb{H}^n$

$$tdt = x^i dx^i.$$

Equation (B.39) yields

$$dy^i = \frac{(1+t)\,dx^i - x^i dt}{(1+t)^2},$$

whence

$$\begin{aligned}
\sum_i \left(dy^i\right)^2 &= (1+t)^{-4} \sum_i \left((1+t)^2 \left(dx^i\right)^2 + \left(x^i\right)^2 dt^2\right) \\
&\quad - (1+t)^{-4} \sum_i \left(x^i dx^i\,(1+t)\,dt + (1+t)\,dt\,x^i dx^i\right) \\
&= (1+t)^{-4} \left(\sum_i (1+t)^2 \left(dx^i\right)^2 + \left(t^2-1\right) dt^2 - 2t\,(1+t)\,dt^2\right) \\
&= (1+t)^{-2} \left(\sum_i \left(dx^i\right)^2 - dt^2\right)
\end{aligned}$$

and

$$\left(dx^1\right)^2 + ... + (dx^n)^2 - dt^2 = (1+t)^2 \left(\left(dy^1\right)^2 + ... + (dy^n)^2\right).$$

It follows from (B.40) that

$$1+t = \frac{2}{1-|y|^2},$$

which implies

$$\left(dx^1\right)^2 + ... + (dx^n)^2 - dt^2 = \frac{4}{\left(1-|y|^2\right)^2} \left(\left(dy^1\right)^2 + ... + (dy^n)^2\right),$$

which was to be proved.

3.19. The hyperbolic space $\mathbb{H}^n$ is represented as a hyperboloid in $\mathbb{R}^{n+1}$ by the equation

$$(x^{n+1})^2 - |x'|^2 = 1.$$

The polar coordinates (r,θ) in $\mathbb{H}^n$ are related to $x^1,...,x^{n+1}$ by (3.69), that is,

$$\cosh r = x^{n+1} \quad \text{and} \quad \theta = \frac{x'}{|x'|}.$$

The coordinates $y^1,...,y^n$ in the Poincaré model are related to $x^1,...,x^{n+1}$ by (3.71), that is

$$y = \frac{x'}{x^{n+1}+1}.$$

This implies

$$\frac{y}{|y|} = \frac{x'}{|x'|} = \theta,$$

$$1+|y|^2 = \frac{\left(x^{n+1}\right)^2 + 2x^{n+1} + 1 + |x'|^2}{\left(x^{n+1}+1\right)^2} = \frac{2\left(x^{n+1}\right)^2 + 2x^{n+1}}{\left(x^{n+1}+1\right)^2} = \frac{2x^{n+1}}{x^{n+1}+1},$$

and

$$1-|y|^2 = \frac{\left(x^{n+1}\right)^2 + 2x^{n+1} + 1 - |x'|^2}{\left(x^{n+1}+1\right)^2} = \frac{2x^{n+1}+2}{\left(x^{n+1}+1\right)^2} = \frac{2}{x^{n+1}+1},$$

whence

$$\frac{1+|y|^2}{1-|y|^2} = x^{n+1} = \cosh r.$$

3.20. (*a*) The canonical metric of the sphere $\mathbb{S}^n$ in the polar coordinates has the form

$$\mathbf{g}_{\mathbb{S}^n} = dr^2 + \sin^2 r\, \mathbf{g}_{\mathbb{S}^{n-1}},$$

where $0<r<\pi$ and $\theta \in \mathbb{S}^{n-1}$. Then the Riemannian volume ω_{n+1} of $\mathbb{S}^n$ coincides with the volume of the ball of radius π. Computing the latter by means of (3.89), we obtain the recursive formula

$$\omega_{n+1} = \omega_n \int_0^\pi \sin^{n-1} r\, dr. \tag{B.41}$$

(*b*) Using $\omega_2 = 2\pi$, we obtain from (B.41)

$$\omega_3 = \omega_2 \int_0^\pi \sin r\, dr = 4\pi,$$

and

$$\omega_4 = \omega_3 \int_0^\pi \sin^2 r dr = 2\pi^2.$$

Denote by $S_M(r)$ and $V_M(r)$ respectively the area function and the volume function of a manifold M. Using (3.91) and (3.92), we obtain

$$S_{\mathbb{R}^n}(r) = \omega_n r^{n-1} \text{ and } V_{\mathbb{R}^n}(r) = \omega_n \int_0^r r^{n-1} dr = \frac{\omega_n}{n} r^n,$$

$$S_{\mathbb{S}^n}(r) = \omega_n \sin^{n-1} r \text{ and } V_{\mathbb{S}^n}(r) = \omega_n \int_0^r \sin^{n-1} dr,$$

$$S_{\mathbb{H}^n}(r) = \omega_n \sinh^{n-1} r \text{ and } V_{\mathbb{H}^n}(r) = \omega_n \int_0^r \sinh^{n-1} dr.$$

It follows that

$$\begin{aligned} V_{\mathbb{S}^2}(r) &= 2\pi(1-\cos r), \\ V_{\mathbb{S}^3}(r) &= \pi(2r - \sin 2r) \\ V_{\mathbb{S}^4}(r) &= \frac{\pi^2}{6}(\cos 3r - 9\cos r + 8) \end{aligned}$$

and

$$\begin{aligned} V_{\mathbb{H}^2}(r) &= 2\pi(\cosh r - 1), \\ V_{\mathbb{H}^3}(r) &= \pi(\sinh 2r - 2r), \\ V_{\mathbb{H}^4}(r) &= \frac{\pi^2}{6}(\cosh 3r - 9\cosh r + 8). \end{aligned}$$

3.21. Let us first evaluate the integral

$$I_n = \int_0^\pi \sin^n r dr,$$

where n is a non-negative integer. Integrating by parts as $\sin^{n-1} r d\cos r$, we obtain the following recursive relation, for all $n \geq 2$:

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{B.42}$$

Let us prove by induction that, for all $n \geq 0$,

$$I_n = \frac{\sqrt{\pi}\Gamma((n+1)/2)}{\Gamma((n+2)/2)} \tag{B.43}$$

For $n = 0$ we have $I_0 = \pi$, which matches the right hand side of (B.43) because $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$. For $n = 1$ we have $I_1 = 2$, which again matches the right hand side of (B.43) because $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$.For $n \geq 2$ we obtain, using the inductive hypothesis for I_{n-2}, (B.42), and the identity $z\Gamma(z) = \Gamma(z+1)$, that

$$I_n = \frac{n-1}{n}\sqrt{\pi}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)} = \sqrt{\pi}\frac{\Gamma((n+1)/2)}{\Gamma((n+2)/2)},$$

which proves (B.43).

Combining (B.43) with (B.41) in the form $\omega_{n+1} = \omega_n I_{n-1}$, we obtain, for all $n \geq 1$,

$$\omega_{n+1} = \omega_n \frac{\sqrt{\pi}\Gamma(n/2)}{\Gamma((n+1)/2)},$$

which easily implies (3.94) by induction in n.

3.22. We obviously have

$$\Delta_{\mathbb{S}^1} = \frac{d^2}{d\theta^2},$$

where θ is the angle on $\mathbb{S}^1$. If (ρ, θ) are the polar coordinates on $\mathbb{S}^2$ then by (3.84)

$$\Delta_{\mathbb{S}^2} = \frac{\partial^2}{\partial \rho^2} + \cot \rho \frac{\partial}{\partial \rho} + \frac{1}{\sin^2 \rho} \frac{\partial^2}{\partial \theta^2}.$$

If (r, ρ, θ) are the spherical coordinates on $\mathbb{S}^3$ then we obtain

$$\begin{aligned} \Delta_{\mathbb{S}^3} &= \frac{\partial^2}{\partial r^2} + 2 \cot r \frac{\partial}{\partial r} + \frac{1}{\sin^2 r} \Delta_{\mathbb{S}^2} \\ &= \frac{\partial^2}{\partial r^2} + 2 \cot r \frac{\partial}{\partial r} + \frac{1}{\sin^2 r} \left(\frac{\partial^2}{\partial \rho^2} + \cot \rho \frac{\partial}{\partial \rho} + \frac{1}{\sin^2 \rho} \frac{\partial^2}{\partial \theta^2} \right). \end{aligned}$$

Similarly, it follows from (3.83) that

$$\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2}$$

and

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial^2}{\partial \rho^2} + \cot \rho \frac{\partial}{\partial \rho} + \frac{1}{\sin^2 \rho} \frac{\partial^2}{\partial \theta^2} \right).$$

Finally, (3.85) yields

$$\Delta_{\mathbb{H}^2} = \frac{\partial^2}{\partial \rho^2} + \coth \rho \frac{\partial}{\partial \rho} + \frac{1}{\sinh^2 \rho} \frac{\partial^2}{\partial \theta^2}$$

and

$$\Delta_{\mathbb{H}^3} = \frac{\partial^2}{\partial r^2} + 2 \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \left(\frac{\partial^2}{\partial \rho^2} + \cot \rho \frac{\partial}{\partial \rho} + \frac{1}{\sin^2 \rho} \frac{\partial^2}{\partial \theta^2} \right). \quad \text{(B.44)}$$

3.23. (a) Since the function u depends only on r, we have by (B.44)

$$\Delta_{\mathbb{H}^3} u = u'' + 2 (\coth r) u',$$

and a direct computation yields $\Delta_{\mathbb{H}^3} u = -u$.

(b) By Exercise 3.19, the relation between the polar radius r and the coordinates $y \in \mathbb{B}^n$ is given by

$$\cosh r = \frac{1 + |y|^2}{1 - |y|^2}.$$

This implies

$$\sinh r = \frac{2 |y|}{1 - |y|^2}$$

and

$$e^r = \cosh r + \sinh r = \frac{1 + |y|}{1 - |y|}.$$

Hence, we obtain

$$u = \frac{r}{\sinh r} = \frac{1 - |y|^2}{2 |y|} \log \frac{1 + |y|}{1 - |y|}. \quad \text{(B.45)}$$

We need to prove that this function extends smoothly to $r = 0$, that is, to $y = 0$. For that, let us just expand the right hand side of (B.45) to a Taylor series in a neighborhood of $|y| = 0$. We have

$$\begin{aligned} \log \frac{1+|y|}{1-|y|} &= \left(|y| - \frac{|y|^2}{2} + \frac{|y|^3}{3} - ...\right) - \left(-|y| - \frac{|y|^2}{2} - \frac{|y|^3}{3} - ...\right) \\ &= 2|y| + 2\frac{|y|^3}{3} + 2\frac{|y|^5}{5} + ... \end{aligned}$$

whence

$$u = \left(1 - |y|^2\right)\left(2 + 2\frac{|y|^2}{3} + 2\frac{|y|^4}{5} + ...\right).$$

Hence, u is a smooth function of $|y|^2$ in a neighborhood of 0. Since $|y|^2$ is a smooth function of y, the smoothness of u in $\mathbb{H}^3$ follows.

3.24. Since u does not depend on the polar angle, the equation $\Delta_\mu u = 0$ becomes

$$u'' + \frac{S'}{S} u = 0$$

(cf. (3.93). This equation equivalent to

$$\left(Su'\right)' = 0,$$

and solving it we obtain

$$u(r) = C \int_{r_1}^{r} \frac{dr}{S(r)} + C_1. \tag{B.46}$$

In $\mathbb{R}^n$ we have $S(r) = \omega_n r^{n-1}$ and (B.46) yields

$$u(r) = C_1 + C \begin{cases} \log \frac{1}{r}, & n = 2, \\ r^{2-n}, & n > 2. \end{cases}$$

Since in $\mathbb{S}^n$ we have $S(r) = \omega_n \sin^{n-1} r$, we obtain from (B.46)

$$u(r) = C_1 + C \begin{cases} \log \tan \frac{r}{2}, & \text{in } \mathbb{S}^2, \\ \cot r, & \text{in } \mathbb{S}^3. \end{cases}$$

Similarly,

$$u(r) = C_1 + C \begin{cases} \log \tanh \frac{r}{2}, & \text{in } \mathbb{H}^2, \\ \coth r, & \text{in } \mathbb{H}^3. \end{cases}$$

3.25. Using the notation (3.86) and (3.87) from Section 3.10, and writing $u' = \frac{\partial u}{\partial r}$, we obtain

$$\begin{aligned}\int_A (\Delta_\mu u)\, v\, d\mu &= \int_a^b \int_{\mathbb{S}^{n-1}} \left(u'' + \frac{\sigma'}{\sigma} u' + \frac{1}{\psi^2} \Delta_{\mathbb{S}^{n-1}} u \right) v \sigma (r)\, d\theta dr \\ &= \int_{\mathbb{S}^{n-1}} \left(\int_a^b (u'\sigma)'\, v dr \right) d\theta \\ &\quad + \int_a^b \left(\int_{\mathbb{S}^{n-1}} (\Delta_{\mathbb{S}^{n-1}} u)\, v d\theta \right) \psi^{-2} \sigma dr.\end{aligned}$$

Integrating by parts in the interval (a, b) yields

$$\int_a^b (u'\sigma)'\, v dr = \left[u' v \sigma \right]_a^b - \int_a^b u' v' \sigma dr.$$

By the Green formula (3.41) on $\mathbb{S}^{n-1}$,

$$\int_{\mathbb{S}^{n-1}} (\Delta_{\mathbb{S}^{n-1}} u)\, v d\theta = - \int_{\mathbb{S}^{n-1}} \langle \nabla_\theta u, \nabla_\theta v \rangle d\theta.$$

Combining together the above lines and using a consequence of (3.78) that

$$\langle \nabla u, \nabla v \rangle = u' v' + \psi^{-2} (r) \langle \nabla_\theta u, \nabla_\theta v \rangle,$$

we obtain

$$\begin{aligned}\int_A (\Delta_\mu u)\, v\, d\mu &= \left[\int_{\mathbb{S}^{n-1}} u' v \sigma d\theta \right]_a^b - \int_{\mathbb{S}^{n-1}} \int_a^b u' v' \sigma dr d\theta \\ &\quad - \int_a^b \int_{\mathbb{S}^{n-1}} \psi^{-2} (r) \langle \nabla_\theta u, \nabla_\theta v \rangle \sigma d\theta dr \\ &= \int_{S_b} u' v d\mu_{S_b} - \int_{S_a} u' v d\mu_{S_a} - \int_A \langle \nabla u, \nabla v \rangle\, d\mu,\end{aligned}$$

which proves (3.96). Switching in (3.96) u and v and subtracting the resulting identity from (3.96), we obtain (3.97).

3.26. (a) This follows from Lemma 3.19.

(b) The change of coordinates on S is given by

$$\begin{aligned} x^i &= \left| x' \right| f^i (\theta) = \Phi (t)\, f^i (\theta)\,, \; i = 1, ..., n \\ x^{n+1} &= t, \end{aligned}$$

where f^i are the same functions as in (3.61). Therefore, the metric $\mathbf{g}_S$ is given by

$$\begin{aligned} \left(dx^1\right)^2 + ... + (dx^n)^2 + \left(dx^{n+1}\right)^2 &= \sum_{i=1}^n \left(f^i \Phi' dt + \Phi df^i \right)^2 + dt^2 \\ &= \left(1 + \left(\Phi'\right)^2 \right) dt^2 + \Phi^2 \mathbf{g}_{\mathbb{S}^{n-1}}. \end{aligned}$$

(c) The change $\rho = \int \sqrt{1+\Phi'(t)^2}dt$ obviously implies

$$\left(1+(\Phi')^2\right)dt^2 = d\rho^2.$$

The function $\Psi(\rho)$ is obtained by the condition $\Psi(\rho) = \Phi(t)$.

3.27. $\mathbf{g}_{Cyl} = d\rho^2 + \mathbf{g}_{\mathbb{S}^{n-1}}$ and $\mathbf{g}_{Cone} = d\rho^2 + \frac{\rho^2}{2}\mathbf{g}_{\mathbb{S}^{n-1}}$.

3.28. The changes

$$x^{n+1} = t \text{ and } |x'| = \frac{1}{\cosh s} =: u(s)$$

(where $s > 0$) bring the equation of the pseudo-sphere to the form

$$t = s - \tanh s.$$

The function Φ (cf. Exercise 3.26) is defined by the condition $u = \Phi(t)$. Therefore,

$$\rho = \int \sqrt{1+\Phi'(t)^2}dt = \int \sqrt{\left(\frac{du}{ds}\right)^2 + \left(\frac{dt}{ds}\right)^2} ds = \int \tanh s\, ds = \log\cosh s.$$

The function Ψ is defined by the condition $\Psi(\rho) = \Phi(t)$ whence $\Psi(\rho) = u = e^{-\rho}$, $\rho > 0$.

3.29. (a) The metric of $\mathbb{R}^2$ has the form (3.99) with $f \equiv 1$, and (3.100) clearly gives $K_{\mathbb{R}^2} \equiv 0$. To apply (3.100) to the other metrics, let us first notice that by the chain rule

$$\Delta \log f = \frac{f\Delta f - |\nabla f|^2}{f^2} = \frac{\Delta f}{f} - \left(\frac{|\nabla f|}{f}\right)^2.$$

where ∇f is the gradient of f in the Euclidean metric $\left(dx^1\right)^2 + \left(dx^2\right)^2$. In particular, if the function f depends only on the polar radius r then $|\nabla f| = |f'|$ and $\Delta f = f'' + \frac{1}{r}f'$ whence

$$K_{M,\mathbf{g}} = f^2 \Delta \log f = ff'' + \frac{1}{r}f'f - (f')^2. \tag{B.47}$$

The canonical metric of $\mathbb{S}^2$ in the stereographic projection has the form (3.99) with

$$f(x) = \frac{1}{2}\left(1+|x|^2\right) = \frac{1}{2}\left(1+r^2\right).$$

Hence, $f' = r$ and $f'' = 1$, whence (B.47) yields $K_{\mathbb{S}^2} \equiv 1$. Similarly, the canonical metric of $\mathbb{H}^2$ in the Poincaré model has the form (3.100) with $f(x) = \frac{1}{2}\left(1-|x|^2\right)$ whence in the same way we obtain $K_{\mathbb{H}^2} \equiv -1$.

(b) $K_{\mathbb{R}^2_+,\mathbf{g}} \equiv -1$.

3.30. Let us write down the Laplace operator $\Delta_{\mathbf{g}}$ in the coordinates x^1, x^2 using the fact that the matrix $g = (g_{ij})$ of the metric $\mathbf{g}$ has the form

$$g = \begin{pmatrix} f^{-2} & 0 \\ 0 & f^{-2} \end{pmatrix}.$$

Since $\det g = f^{-4}$ and

$$g^{-1} = \begin{pmatrix} f^2 & 0 \\ 0 & f^2 \end{pmatrix},$$

we obtain

$$\begin{aligned} \Delta_{\mathbf{g}} &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^1} \left(\sqrt{\det g} g^{11} \frac{\partial}{\partial x^1} \right) + \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^2} \left(\sqrt{\det g} g^{22} \frac{\partial}{\partial x^2} \right) \\ &= f^2 \frac{\partial^2}{(\partial x^1)^2} + f^2 \frac{\partial^2}{(\partial x^2)^2} \\ &= f^2 \Delta, \end{aligned}$$

that is

$$\Delta_{\mathbf{g}} = f^2 \Delta.$$

Since

$$\widetilde{\mathbf{g}} = \frac{\left(dx^1\right)^2 + \left(dx^2\right)^2}{(fh)^2},$$

the formula (3.100) gives for this metric

$$K_{M,\widetilde{\mathbf{g}}} = (fh)^2 \Delta \log (fh) = h^2 \left(f^2 \Delta \log f + f^2 \Delta \log h\right) = h^2 \left(K_{M,\mathbf{g}} + \Delta_{\mathbf{g}} \log h\right),$$

which was to be proved.

3.31. Let us change the variable

$$\rho = \int \frac{dr}{\psi (r)}$$

so that $d\rho = \frac{dr}{\psi(r)}$. Clearly, in the coordinates ρ, θ the metric has the form

$$\mathbf{g} = \psi^2 (r) \left(d\rho^2 + d\theta^2\right),$$

which matches (3.99) with $f(\rho) = \frac{1}{\psi(r)}$. Since $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2}$ and ψ does not depend on θ, we obtain by (3.100)

$$K_{M,\mathbf{g}} = -\frac{1}{\psi^2} \frac{d^2}{d\rho^2} \log \psi.$$

We have

$$\frac{d}{d\rho} \log \psi = \frac{1}{\psi} \frac{dr}{d\rho} \frac{d\psi}{dr} = \psi'$$

and

$$\frac{d^2}{d\rho^2} \log \psi = \frac{d}{d\rho} \psi' = \frac{dr}{d\rho} \psi'' = \psi \psi'',$$

whence the result follows.

3.32. We have:

For $\mathbb{R}^2$: $\psi (r) = r$, $K \equiv 0$.
For $\mathbb{S}^2$: $\psi (r) = \sin r$, $K \equiv 1$.
For $\mathbb{H}^2$: $\psi (r) = \sinh r$, $K \equiv -1$.
For Cyl: $\psi (r) = 1$, $K \equiv 0$.
For $Cone$: $\psi (r) = \mathrm{const}\, r$, $K \equiv 0$.

For PS: $\psi(r) = e^{-r}$, $K \equiv -1$.

3.33. Left to the reader

3.34. Left to the reader

3.35. Fix a point $x \in M$, denote by M_x the connected component of M containing x, and consider the set $N_x = \{y : d(x,y) < \infty\}$. We need to show that $N_x = M_x$. The inclusion $N_x \subset M_x$ is obvious: if $d(x,y) < \infty$ then there is a continuous path connecting x and y, which implies that $y \in M_x$. To prove the converse, it suffices to show that N_x is open and closed as a subset of M. Indeed, N_x is open as the union of balls $B(x,r)$ when $r \to \infty$, and its complement

$$N_x^c = \{y \in M : d(x,y) = \infty\}$$

is open because $y \in N_x^c$ implies that, for any $\varepsilon > 0$, also $B(y,\varepsilon) \subset N_x^c$, which follows from the triangle inequality.

3.36. (a) Using (3.77), we obtain

$$|\dot\gamma|^2 = \sum_{i,j=0}^{n-1} g_{ij}\dot\gamma^i\dot\gamma^j = \left|\dot\gamma^0\right|^2 + \sum_{i,j=1}^{n-1} g_{ij}\dot\gamma^i\dot\gamma^j \geq \left|\dot\gamma^0\right|^2,$$

whence it follows that

$$|\gamma| \geq \int_a^b \left|\dot\gamma^0\right| dt \geq \left|\int_a^b \dot\gamma^0 dt\right| = \left|\gamma^0(b) - \gamma^0(a)\right| = \left|r'' - r'\right|.$$

(b) If $\theta' = \theta'' =: \theta$ then the path

$$\gamma(t) = \left(r'(1-t) + r''t, \theta\right)$$

connects x' and x'', because $\gamma(0) = x'$ and $\gamma(1) = x''$. Since $\dot\gamma(t) = (r'' - r', 0)$ and $|\dot\gamma| = |r'' - r'|$, we obtain by (3.104) $\ell(\gamma) = r'' - r'$.

3.37. Let us show that any smooth path γ connecting the points 0 and $x = (r,\theta)$ has the length at least r. If $x' \neq 0$ is a point on the image of γ and $r' = |x'|$ then, by Exercise 3.36, $\ell(\gamma) \geq r - r'$. Since such a point x' exists with arbitrarily small r', we conclude that $\ell(\gamma) \geq r$ and hence $d(0,x) \geq r$. The path $\gamma(t) = (tr, \theta)$ defined for $t \in [0,1]$, connects 0 and x, and it is easy to see that $\ell(\gamma) = r$. Hence, $d(0,x) = r$, which was to be proved.

In $\mathbb{R}^n$, the above argument proves that $d(0,x) = |x|$. Since the origin of the polar coordinates in $\mathbb{R}^n$ may be at any point, setting it to y we obtain that $d(x,y) = |x - y|$.

3.38. Let γ_1 be the part of γ connecting x and z, and γ_2 be the part of γ connecting z and y. Then we have $\ell(\gamma_1) \geq d(x,z)$ and $\ell(\gamma_2) \geq d(z,y)$. whence

$$d(x,y) = \ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2) \geq d(x,z) + d(z,y). \tag{B.48}$$

On the other hand, by the triangle inequality, we have

$$d(x,y) \leq d(x,z) + d(z,y).$$

Therefore, all inequalities in (B.48) are, in fact, equalities, whence $\ell(\gamma_1) = d(x,z)$.

3.39. It suffices to prove that $\langle \nabla f(x), \xi \rangle \le |\xi|$, for any tangent vector $\xi \in T_x M$; by the definition of ∇f, this means that $\xi(f) \le |\xi|$. Consider a smooth path $\gamma : [0, \varepsilon] \to M$ for some $\varepsilon > 0$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. Then

$$\xi(f) = (f \circ \gamma)'(0) = \frac{d}{dt}(f(\gamma(t)))\Big|_{t=0},$$

so that we need to prove that

$$\frac{d}{dt}(f(\gamma(t)))\Big|_{t=0} \le |\dot{\gamma}(0)|. \tag{B.49}$$

Using the definition of the geodesic distance and the triangle inequality, we obtain, for any $t \in (0, \varepsilon)$,

$$f(\gamma(t)) - f(\gamma(0)) = d(\gamma(t), p) - d(\gamma(0), p) \le d(\gamma(t), \gamma(0)) \le \ell\left(\gamma|_{[0,t]}\right) \tag{B.50}$$

Since

$$\ell\left(\gamma|_{[0,t]}\right) = \int_0^t |\dot{\gamma}(s)|\, ds,$$

dividing (B.50) by t and letting $t \to 0$, we obtain (B.49).

3.40. (a) Since M as a smooth manifold can be identified with $\mathbb{R}^n$, we can assume that r is a polar radius in $\mathbb{R}^n$, that is,

$$r = \sqrt{(x^1)^2 + \dots + (x^n)^2}.$$

Clearly, $r = r(x)$ is a smooth function away from the origin o, so the only problem is to show that $a \circ r$ is infinitely many times differentiable at o. Note that r^2 is a smooth function on the entire $\mathbb{R}^n$. Using that a is an even function, we will prove that $a \circ r$ can be represented as a composition of a smooth function with r^2, which will settle the problem. In other words, it suffices to prove that the function $a(t)$ is a smooth function of t^2.

Observe that

$$a(t) = a(0) + \int_0^t a'(s)\, ds = a(0) + t \int_0^1 a'(tu)\, du.$$

Applying the same formula to a' and noticing that $a'(0) = 0$, we obtain

$$a(t) = a(0) + t^2 b(t) \tag{B.51}$$

where

$$b(t) := \int_0^1 \int_0^1 a''(tuv)\, du dv \tag{B.52}$$

is again a smooth even function on $\mathbb{R}$.

To proceed further, we need the following claim.

Claim. *Let $f(x)$ be a function on $[0, +\infty)$, which is infinitely smooth in $(0, +\infty)$. We say that $f \in D^k$, where k is a non-negative integer, if f is k*

times differentiable at 0 *(if* $k = 0$ *then this means that* f *is continuous at* 0*). If* $f(x) \in D^0$ *then* $x^k f(x) \in D^k$ *for any* k*.*

Inductive basis for $k = 0$ and 1. The case $k = 0$ is trivial. If $k = 1$ then differentiating the function $xf(x)$ at 0, we obtain

$$(xf(x))'_{x=0} = \lim_{x\to 0} \frac{xf(x)}{x} = \lim_{x\to 0} f(x) = f(0),$$

whence $xf(x) \in D^1$.

Inductive step from $k-1$ and k to $k+1$. Since $x^{k+1}f(x) = x\left(x^k f(x)\right)$ and $x^k f(x) \in D^k$, it follows that also $x^{k+1} f(x) \in D^k$. By the product rule, we have

$$\left(x^{k+1} f(x)\right)' = x^k f(x) + x\left(x^k f(x)\right)'.$$

Since $\left(x^k f(x)\right)' \in D^{k-1}$, we obtain by the inductive hypothesis that $x\left(x^k f(x)\right)' \in D^k$ whence it follows that also $\left(x^{k+1} f(x)\right)' \in D^k$. Hence, $x^{k+1} f(x) \in D^{k+1}$, which was to be proved.

Now we can prove by induction in k that, for any smooth even function a on $\mathbb{R}$, we have $a(\sqrt{x}) \in D^k$, where x is a variable in $[0, +\infty)$. The case $k = 0$ follows from the continuity of a. For the inductive step from k to $k+1$, apply the inductive hypothesis to the function b from (B.51), (B.52), so that $b(\sqrt{x}) \in D^k$. By (B.51), we have

$$a\left(\sqrt{x}\right) = a(0) + xb\left(\sqrt{x}\right),$$

and by the above Claim we conclude that $a(\sqrt{x}) \in D^{k+1}$.

Consequently, we see that the function $a(\sqrt{x})$ is infinitely many times differentiable in $[0, +\infty)$, which was to be proved.

3.41. (a) Set $\widetilde{\mathbf{g}} := C_a \mathbf{g} = a^2 \mathbf{g}$. Consider the path $\gamma(t) = tx$ where $t \in [0,1]$ and $x \in \mathbb{R}^n$. Then the polar radius r of the point x in the metric $\mathbf{g}$ is given by

$$r = \ell_{\mathbf{g}}(\gamma) = \int_0^1 |\dot\gamma(t)|_{\mathbf{g}}\, dt = \int_0^1 |x|\, dt = |x|,$$

while the polar radius $\widetilde{r}$ of x in $\widetilde{\mathbf{g}}$ is given by

$$\widetilde{r} = \ell_{\widetilde{\mathbf{g}}}(\gamma) = \int_0^1 |\dot\gamma(t)|_{\widetilde{\mathbf{g}}}\, dt = \int_0^1 a(t|x|)\,|x|\, dt = \int_0^{|x|} a(s)\, ds,$$

which was to be proved. It follows from (3.109) that the radius of $(M, \widetilde{\mathbf{g}})$ is infinity.

(b) It is obvious that $a \star b$ is smooth, positive and even. It satisfies the condition (3.109) because

$$\begin{aligned} \int_0^\infty (a \star b)(t)\, dt &= \int_0^\infty a\left(\int_0^t b(s)\, ds\right) b(t)\, dt \\ &= \int_0^\infty a(\tau)\, d\tau = \infty, \end{aligned}$$

where we have made the change $\tau = \int_0^t b(s)\,ds$. Hence, $a \star b \in \mathcal{S}$.

It follows from part (a) that

$$C_a \circ C_b = C_{a\star b}.$$

Indeed, setting $\widetilde{\mathbf{g}} = C_b\mathbf{g}$, we obtain

$$(C_a \circ C_b)\,\mathbf{g} = C_a\widetilde{\mathbf{g}} = a^2(\widetilde{r})\,\widetilde{\mathbf{g}} = a^2(\widetilde{r})\,b^2(r)\,\mathbf{g} = (a\star b)^2\,\mathbf{g},$$

which was claimed. Hence, the family $\{C_a\}_{a\in\mathcal{S}}$ of conformal changes of metric is closed under composition. Obviously, $(C_a)^{-1}$ also belongs to this family. Since the composition is always associative, the family $\{C_a\}_{a\in\mathcal{S}}$ forms a group with respect to composition, which implies that $\mathcal{S}$ is a group with respect to $\star$. Alternatively, the latter can be verified directly by computation.

(c) Noticing that

$$\int_0^r b(s)\,ds \simeq \int_0^r s^{\beta-1}\log^{[v]} s\; ds \simeq r^{\beta}\log^{[v]} r \quad \text{as } r\to\infty$$

and

$$\log^{[u]}\left(\int_0^r b(s)\,ds\right) \simeq \log^{[u]} r,$$

we obtain

$$a\left(\int_0^r b(s)\,ds\right)b(r) \simeq r^{\beta(\alpha-1)}\log^{[(\alpha-1)v+u]} r\ \left(r^{\beta-1}\log^{[v]} r\right) \simeq r^{\alpha\beta-1}\log^{[\alpha v+u]} r,$$

which was to be proved.

3.42. For any $x\in S$ and $\xi\in T_xS$, the vector ξ can be considered as an element of T_xM by

$$\xi(f) = \xi(f|_S),$$

for $f\in C^\infty(M)$. Denoting $\widetilde{J} = J|_S$, we have by definition (3.112) of $d\widetilde{J}$, for any $f\in C^\infty(S)$,

$$d\widetilde{J}\xi(f) = \xi(f\circ\widetilde{J}).$$

In particular, applying this to $f\in C^\infty(M)$, we obtain

$$d\widetilde{J}\xi(f) = \xi(f\circ J|_S) = \xi(f\circ J) = dJ\xi(f)$$

The fact that J is an isometry of $(M,\mathbf{g})$ implies that, for any $x\in M$ and $\xi\in T_xM$,

$$|\xi|_{\mathbf{g}} = |dJ\xi|_{\mathbf{g}}.$$

Hence, for all $x\in S$ and $\xi\in T_xS$,

$$|\xi|_{\mathbf{g}|_S} = |\xi|_{\mathbf{g}} = |dJ\xi|_{\mathbf{g}} = |d\widetilde{J}\xi|_{\mathbf{g}} = |d\widetilde{J}\xi|_{\mathbf{g}|_S},$$

whence it follows that $\widetilde{J}$ is an isometry of $(S,\mathbf{g}|_S)$.

3.43. (a) Let $\gamma:[a,b]\to M$ be a smooth curve. By the definition of length, we have

$$\ell_{\mathbf{g}_M}(\gamma) = \int_a^b |\dot{\gamma}(t)|_{\mathbf{g}_M}\,dt \tag{B.53}$$

and

$$\ell_{\mathbf{g}_N}(\gamma) = \int_a^b |\dot{\gamma}(t)|_{\mathbf{g}_N} dt. \tag{B.54}$$

Set $x = \gamma(t) \in M$, $\xi = \dot{\gamma}(t) \in T_x M$ and recall that ξ is defined by

$$\xi(f) = \frac{d}{dt} f(\gamma(t)),$$

for $f \in C^\infty(M)$. Hence, by (3.112), $dJ\xi$ is a tangent vector at Jx such that

$$dJ\xi(f) = \xi(J_* f) = \frac{d}{dt} f(J \circ \gamma(t)) = (J \circ \gamma)^{\cdot}(f).$$

Therefore,

$$dJ\dot{\gamma} = (J \circ \gamma)^{\cdot}.$$

On the other hand, the hypothesis $\mathbf{g}_M = J_* \mathbf{g}_N$ and the definition (3.115) of $J_* \mathbf{g}_N$ imply that

$$|\dot{\gamma}|_{\mathbf{g}_M} = |\dot{\gamma}|_{J_* \mathbf{g}_N} = |dJ\dot{\gamma}|_{\mathbf{g}_N}. \tag{B.55}$$

whence

$$|\dot{\gamma}|_{\mathbf{g}_M} = |(J \circ \gamma)^{\cdot}|_{\mathbf{g}_N}.$$

Combining with (B.53) and (B.54), we obtain

$$\ell_{\mathbf{g}_M}(\gamma) = \ell_{\mathbf{g}_N}(J \circ \gamma),$$

which was to be proved.

(b) The geodesic distance is defined as the infimum of the length of a smooth path connecting x, y. By part (a), a Riemannian isometry preserves the length, whence the claim follows.

3.44. (a) Following the same line of arguments as in solution of Exercise 3.43(a), we only need to replace (B.55) by

$$|\dot{\gamma}|_{\mathbf{g}_M} \simeq |\dot{\gamma}|_{J_* \mathbf{g}_N} = |dJ\dot{\gamma}|_{\mathbf{g}_N}, \tag{B.56}$$

whence $\ell_{\mathbf{g}_M}(\gamma) \simeq \ell_{\mathbf{g}_N}(J \circ \gamma)$ follows.

(b) Since the geodesic distance is defined using the length of smooth paths, the claim follows from (a).

(c) For simplicity of notation, let us identify M and N as smooth manifolds using the quasi-isometric diffeomorphism $J : M \to N$. Let ν_M and ν_N be the Riemannian measures of $\mathbf{g}_M$ and $\mathbf{g}_N$, respectively. By Exercise 3.5, we have $d\nu_N \simeq d\nu_M$. Since also $\Upsilon_N \simeq \Upsilon_M$, we obtain $d\mu_N \simeq d\mu_M$, whence (3.123) follows.

(d) By Exercise 3.5, we have $|\nabla f|^2_{\mathbf{g}_M} \simeq |\nabla f|^2_{\mathbf{g}_N}$. Using also the comparison $d\mu_M \simeq d\mu_N$ from part (c), we obtain (3.124).

3.45. (*a*) Using (3.114), we obtain

$$\begin{aligned}
& \left(dy^1\right)^2+\ldots+\left(dy^{n-1}\right)^2+\left(dy^n\right)^2-\left(dy^{n+1}\right)^2 \\
= \;& \left(dx^1\right)^2+\ldots+\left(dx^{n-1}\right)^2 \\
& +\cosh^2\alpha\left(dx^n\right)^2+\sinh^2\alpha\left(dx^{n+1}\right)^2+\cosh\alpha\sinh\alpha\left(dx^n dx^{n+1}+dx^{n+1}dx^n\right) \\
& -\sinh^2\alpha\left(dx^n\right)^2-\cosh\alpha^2\alpha\left(dx^{n+1}\right)^2-\cosh\alpha\sinh\alpha\left(dx^n dx^{n+1}+dx^{n+1}dx^n\right) \\
= \;& \left(dx^1\right)^2+\ldots+\left(dx^{n-1}\right)^2+\left(dx^n\right)^2-\left(dx^{n+1}\right)^2.
\end{aligned}$$

(*b*) Recall that $\mathbb{H}^n$ is a hyperboloid in $\mathbb{R}^{n+1}$, given by the equation

$$\left(x^{n+1}\right)^2-\left(x^1\right)^2-\ldots-\left(x^n\right)^2=1,\quad x^{n+1}>0.$$

Similarly to the above computation, the mapping J, defined by (3.125), maps $\mathbb{H}^n$ onto itself. By the same argument as in Exercise 3.42, $J|_{\mathbb{H}^n}$ preserves the induced metric of $\mathbb{H}^n$, which is $\mathbf{g}_{\mathbb{H}^n}$.

3.46. Let us first show that, for any point $p\in\mathbb{H}^n$, there exists an isometry J of $\mathbb{H}^n$ such that $Jp=o$ where $o=(0,\ldots 0,1)$ is the origin of $\mathbb{H}^n$. First, by rotation in the subspace $\mathbb{R}^n$ with coordinates $\left(x^1,\ldots,x^n\right)$, we can assume that the projection of p onto $\mathbb{R}^n$ lies on the axis x^n, that is,

$$p=(0,\ldots,0,a,b),$$

where

$$b^2-a^2=1.$$

Then there exists real α such that

$$b=\cosh\alpha \text{ and } a=-\sinh\alpha,$$

and, setting J to be the hyperbolic rotation (3.125) of Exercise 3.45 with this parameter α, we obtain from (3.125) $Jp=o$.

If $q,q'\in\mathbb{H}^n$ are two points such that $d(o,q)=d(o,q')$ then, in the polar coordinates, the points q and q' have the same polar radius (cf. Exercise 3.37). Therefore, for a suitable rotation J of the polar angle, we obtain $Jq=q'$, while $Jo=o$.

Assume now that points p,q,p',q' satisfy (3.126). Let J and J' be isometries that bring, respectively, the points p and p' to the origin o. Then

$$d\left(o,J'q'\right)=d\left(p',q'\right)=d(p,q)=d(o,Jq)$$

and, hence, there exists an isometry J'' such that

$$J''J'q'=Jq \quad\text{and}\quad J''o=o.$$

Since $o=Jp=J'p'$, we obtain

$$J''J'p'=Jp,$$

which together with the previous line implies that $J^{-1}J''J'$ is an isometry that maps p' to p and q' to q.

Solutions to Chapter 4

4.1. Left to the reader

4.2. (a) For any $\varphi \in \mathcal{D}(M)$, we have $f\varphi \in \mathcal{D}(M)$ whence

$$(fu_k, \varphi) = (u_k, f\varphi) \to (u, f\varphi) = (fu, \varphi),$$

whence $fu_k \overset{\mathcal{D}'}{\to} fu$.

(b) Let $\varphi \in \mathcal{D}(M)$ be supported away from $\operatorname{supp} f \cap \operatorname{supp} u$, which implies that $\operatorname{supp}\varphi \cap \operatorname{supp} f$ is disjoint from $\operatorname{supp} u$. Since

$$\operatorname{supp}(f\varphi) \subset \operatorname{supp} f \cap \operatorname{supp}\varphi,$$

we obtain that $\operatorname{supp}(f\varphi)$ is disjoint from $\operatorname{supp} u$ and, hence $(u, f\varphi) = 0$. By (4.8), we obtain $(fu, \varphi) = 0$, whence the claim follows.

(c) To prove

$$\nabla(fu) = f\nabla u + (\nabla f)\,u, \tag{B.57}$$

we need to verify that, for any $\omega \in \vec{\mathcal{D}}(M)$,

$$(\nabla(fu), \omega) = (f\nabla u, \omega) + ((\nabla f)\,u, \omega). \tag{B.58}$$

Using the definition of ∇ in $\mathcal{D}'$ and the definitions of the products fu, $f\nabla u$ and $(\nabla f)\,u$, we obtain

$$(\nabla(fu), \omega) = -(fu, \operatorname{div}_\mu \omega) = -(u, f \operatorname{div}_\mu \omega),$$
$$(f\nabla u, \omega) = (\nabla u, f\omega) = -(u, \operatorname{div}_\mu(f\omega)),$$
$$((\nabla f)\,u, \omega) = (u, \langle \nabla f, \omega\rangle).$$

We are left to notice that, by Exercise 3.7,

$$\operatorname{div}_\mu(f\omega) = \langle \nabla f, \omega\rangle + f \operatorname{div}_\mu \omega,$$

which together with the previous three lines implies (B.58).

4.3. Set

$$S = \max(\sup|f|, \sup|\nabla f|).$$

Obviously, we have

$$\|fu\|_{L^2} \le S\|u\|_{L^2},$$

and it follows from (B.57) that

$$\|\nabla(fu)\|_{L^2} \le S\|u\|_{L^2} + S\|\nabla u\|_{L^2}$$

whence

$$\begin{aligned}\|fu\|^2_{W^1} &= \|fu\|^2_{L^2} + \|\nabla(fu)\|^2_{L^2}\\ &\le 3S^2\|u\|^2_{L^2} + 2S^2\|\nabla u\|^2_{L^2}\\ &\le 3S^2\|u\|^2_{W^1},\end{aligned}$$

whence the claim follows.

4.4. Let us show that any $f \in L^p(M, \mu)$ can be approximated in L^p norm by a sequence of functions from $C_0^\infty(M)$. Obviously, we can assume $f \ge 0$. Let $\{\Omega_k\}$ be an increasing sequence of relatively compact open sets in M covering all M (see Exercise 3.1). Since the sequence $\{1_{\Omega_k} f\}$ increases and converges to f pointwise, it converges to f also in L^p norm. Switching from f to $1_{\Omega_k} f$, we can assume in the sequel that the support $K = \operatorname{supp} f$ is compact.

By Lemma 3.4, there is a countable family $\{U_i\}_{i=1}^{\infty}$ of relatively compact charts covering M and such that the closure $\overline{U_i}$ is contained in a chart. Then select a finite number of charts U_i, say $U_1, ..., U_k$ covering K, and let $\{\varphi_i\}$ be the associated partition of unity. By Theorem 2.3, $C_0^{\infty}(U_i)$ is dense in $L^p(U_i, \lambda)$ where λ is the Lebesgue measure in U_i. Since the measures μ and λ are comparable in U_i (that is, the density $\frac{d\mu}{d\lambda}$ is bounded between two positive constants), we obtain that $C_0^{\infty}(U_i)$ is dense also in $L^p(U_i, \mu)$. Hence, for any $\varepsilon > 0$ there exists $f_i \in C_0^{\infty}(U_i)$ such that

$$\|f\varphi_i - f_i\|_{L^p(U_i,\mu)} < \varepsilon.$$

Adding up all such inequalities, using the triangle inequality and observing that

$$\sum_{i=1}^{k} f\varphi_i \equiv f \text{ on } M,$$

we obtain

$$\|f - \sum_{i=1}^{k} f_i\|_{L^p(M,\mu)} < \varepsilon k.$$

Since $\sum_{i=1}^{k} f_i \in C_0^{\infty}(M)$, this proves that $C_0^{\infty}(M)$ is dense in $L^p(M, \mu)$.

To prove the separability of $L^p(M, \mu)$, observe that by Theorem 2.3, $L^p(U_i, \lambda)$ is separable, where $\{U_i\}_{i=1}^{\infty}$ is the same family as above. Let $\mathcal{F}_i$ be a countable dense family in $L^p(U_i, \lambda)$. Then it is also dense in $L^p(U_i, \mu)$. Consider the family $\mathcal{F}$ that consists of all finite sums of functions from $\mathcal{F}_i$ across all i. Obviously, $\mathcal{F}$ is countable. The fact that $\mathcal{F}$ is dense in $L^p(M, \mu)$ is proved exactly in the same way as in the first part of the proof, replacing $f_i \in C_0^{\infty}(U_i)$ by $f_i \in \mathcal{F}_i$ and noticing that the sum $\sum_{i=1}^{k} f_i$ belongs to $\mathcal{F}$.

4.5. In $\mathbb{R}^n$, this follows from Lemmas 2.1 and 2.4. The partition of unity (Theorem 3.5) allows to extend this result to an arbitrary manifold.

4.6. We need to prove that $(u, \varphi) = 0$ for any function $\varphi \in \mathcal{D}(M)$. It suffices to show that any function $\varphi \in \mathcal{D}(M)$ can be represented in the form $\varphi = \psi_1 - \psi_2$ where ψ_1 and ψ_2 are non-negative functions from $\mathcal{D}(M)$. Let ψ be a cutoff function of $\operatorname{supp}\varphi$ in M (cf. Theorem 3.5) and $C = \sup|\varphi|$. Then the function $\psi_1 = C\psi$ is non-negative, belongs to $\mathcal{D}(M)$, and $\psi_1 \geq \varphi$. Setting $\psi_2 = \psi_1 - \varphi$ we complete the proof.

4.7. (a) We need to prove that if $u \in L^1_{loc}(M)$ and

$$\int_M u\varphi d\mu \geq 0 \tag{B.59}$$

for all non-negative $\varphi \in \mathcal{D}(M)$ then $u \geq 0$ a.e.. Let us use Exercise 2.8, where the same fact was proved in $\mathbb{R}^n$. Let $U \subset M$ be any chart and λ

be the Lebesgue measure in U. Since the density $\frac{d\mu}{d\lambda}$ is a smooth positive function, the condition (B.59) implies that

$$\int_U u\varphi \, d\lambda \geq 0$$

for all non-negative $\varphi \in \mathcal{D}(U)$. By Exercise 2.8 we conclude that $u \geq 0$ a.e. in U. Since M can be covered by a countable family of charts (cf. Lemma 3.4), we obtain $u \geq 0$ a.e. in M.

(b) This trivially follows from (a). Alternatively, one can use Exercise 4.6 to conclude that $u = 0$ in the distributional sense, and then deduce from Corollary 2.5 that $u = 0$ a.e..

4.8. (a) Select first a subsequence from $\{u_k\}$ which realizes the lim inf of the norms $\|u_k\|_{L^2}$. By the weak compactness of a ball in a Hilbert space, we can select further a subsequence from $\{u_k\}$ that convergences in L^2 weakly. Renumber this subsequence again by $\{u_k\}$ and let v be its weak limit in L^2. This means that, for any $\varphi \in L^2$,

$$(u_k, \varphi)_{L^2} \to (v, \varphi)_{L^2} \,. \tag{B.60}$$

Obviously, this implies that $u_k \xrightarrow{\mathcal{D}'} v$ whence it follows that $u = v$ and hence $u \in L^2$. We are left to show that

$$\|v\|_{L^2} \leq \lim_{k\to\infty} \|u_k\|_{L^2}. \tag{B.61}$$

Setting in (B.60) $\varphi = v$, we obtain

$$\|v\|_{L^2}^2 = \lim_{k\to\infty} (u_k, v) \leq \lim_{k\to\infty} \|u_k\|_{L^2} \|v\|_{L^2},$$

whence (B.61) follows.

(b) As in part (a), select first a subsequence from $\{u_k\}$ that realizes the lim inf of the norms $\|\nabla u_k\|_{L^2}$. Select further a subsequence along which $\{\nabla u_k\}$ converges in $\vec{L}^2$ weakly. Renumber this subsequence again by $\{u_k\}$, and let ω be the weak limit of ∇u_k in $\vec{L}^2$. Since $u_k \xrightarrow{\mathcal{D}'} u$ and $\nabla u_k \xrightarrow{\vec{\mathcal{D}}'} \omega$, we conclude by Lemma 4.2 that $\nabla u = \omega$ and, hence, $\nabla u \in \vec{L}^2$ and $u \in W^1$. The estimate of $\|\omega\|_{L^2}$ follows in the same way as (B.61).

4.9. Left to the reader

4.10. Left to the reader

4.11. (a) Set

$$v^i = g^{ij} \frac{\partial f}{\partial x^j}. \tag{B.62}$$

To prove that v is the distributional gradient $\nabla_{\mathbf{g}} f$ in U, we must verify that, for any smooth vector field w compactly supported in U,

$$\int_U \langle v, w \rangle_{\mathbf{g}} d\mu = -\int_U f \operatorname{div}_\mu w \, d\mu. \tag{B.63}$$

Indeed, we have

$$\langle v,w\rangle_{\mathbf{g}}=g_{ik}v^{i}w^{k}=g_{ik}g^{ij}\frac{\partial f}{\partial x^{i}}w^{k}=\frac{\partial f}{\partial x^{i}}w^{i} \tag{B.64}$$

and by (3.44)

$$\operatorname{div}_{\mu}w=\rho^{-1}\frac{\partial}{\partial x^{i}}\left(\rho w^{i}\right)$$

where $\rho=\frac{d\mu}{d\lambda}$ and λ is the Lebesgue measure in U. Using the definition of weak derivative $\frac{\partial f}{\partial x^{i}}$ and the fact that $w^{i}\rho\in\mathcal{D}\left(U\right)$, we obtain

$$\begin{aligned}\int_{U}\langle v,w\rangle_{\mathbf{g}}d\mu &= \int_{U}\frac{\partial f}{\partial x^{i}}w^{i}\rho\,d\lambda=-\int_{U}f\frac{\partial}{\partial x^{i}}\left(w^{i}\rho\right)d\lambda\\ &= -\int_{U}f\rho^{-1}\frac{\partial}{\partial x^{i}}\left(\rho w^{i}\right)d\mu=-\int_{U}f\operatorname{div}_{\mu}w\,d\mu,\end{aligned}$$

whence the claim follows.

It follows from (B.62) that

$$|\nabla_{\mathbf{g}}f|_{\mathbf{g}}^{2}=|v|_{\mathbf{g}}^{2}=g_{kl}v^{k}v^{l}=g_{kl}g^{ki}\frac{\partial f}{\partial x^{i}}g^{kj}\frac{\partial f}{\partial x^{j}}=g^{ij}\frac{\partial f}{\partial x^{i}}\frac{\partial f}{\partial x^{j}},$$

which proves (4.9) and implies that $|\nabla_{\mathbf{g}}f|_{\mathbf{g}}^{2}$ is locally integrable, that is, $\nabla_{\mathbf{g}}f\in\vec{L}_{loc}^{2}\left(U\right)$.

(*b*) Set $v=\nabla_{\mathbf{g}}f$ so that v satisfies the identity (B.63). To prove that

$$\frac{\partial f}{\partial x^{j}}=g_{ij}v^{i} \tag{B.65}$$

we need to verify that, for any $\varphi\in\mathcal{D}\left(U\right)$,

$$\int_{U}g_{ij}v^{i}\varphi\,d\lambda=-\int_{U}f\frac{\partial\varphi}{\partial x^{j}}d\lambda,$$

which is equivalent to

$$\int_{U}g_{ij}v^{i}\varphi\rho^{-1}d\mu=-\int_{U}f\frac{\partial\varphi}{\partial x^{j}}\rho^{-1}d\mu,$$

Consider a vector field

$$w=\left(0,...\varphi\rho^{-1},...0\right)$$

where $\varphi\rho^{-1}$ is the j-th component; that is, $w^{k}=\delta_{j}^{k}\varphi\rho^{-1}$. Then we have

$$\langle v,w\rangle_{\mathbf{g}}=g_{ik}v^{i}w^{k}=g_{ik}v^{i}\delta_{j}^{k}\varphi\rho^{-1}=g_{ij}v^{i}\varphi\rho^{-1}$$

and

$$\operatorname{div}_{\mu}w=\rho^{-1}\frac{\partial}{\partial x^{k}}\left(\rho w^{k}\right)=\rho^{-1}\frac{\partial\varphi}{\partial x^{j}}.$$

Hence, by (B.63),

$$\int_{U}g_{ij}v^{i}\varphi\rho^{-1}d\mu=\int_{U}\langle v,w\rangle_{\mathbf{g}}d\mu=-\int_{U}f\operatorname{div}_{\mu}w\,d\mu=-\int_{U}f\frac{\partial\varphi}{\partial x^{j}}\rho^{-1}d\mu,$$

which was to be proved.

The identity (B.65) implies (B.62), and then (4.9) follows in the same way as in part (a). Since the matrix $\left(g^{ij}\right)$ is positive definite, (4.9) and $\nabla_{\mathbf{g}} f \in \vec{L}^2_{loc}$ imply $\frac{\partial f}{\partial x^j} \in L^2_{loc}$.

4.12. By definition, we have $f \in W^1(\Omega)$ if $f \in L^2(\Omega)$ and $\frac{\partial f}{\partial x^j} \in L^2(\Omega)$ for all j. By Exercise 4.11, this implies that

$$(\nabla_{\mathbf{g}} f)^i = g^{ij} \frac{\partial f}{\partial x^j} = \frac{\partial f}{\partial x^i}$$

because $\left(g^{ij}\right) = \mathrm{id}$, and

$$|\nabla_{\mathbf{g}} f|^2_{\mathbf{g}} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i}\right)^2,$$

whence

$$\|\nabla_{\mathbf{g}} f\|^2_{L^2} = \sum_{i=1}^n \left\|\frac{\partial f}{\partial x^i}\right\|^2_{L^2}.$$

This implies that $\nabla_{\mathbf{g}} f \in \vec{L}^2(\Omega)$, $f \in W^1(\Omega, \mathbf{g}, \lambda)$, and

$$\begin{aligned} \|f\|^2_{W^1(\Omega,\mathbf{g},\lambda)} &= \|f\|^2_{L^2} + \|\nabla_{\mathbf{g}} f\|^2_{L^2} \\ &= \|f\|^2_{L^2} + \sum_{i=1}^n \left\|\frac{\partial f}{\partial x^i}\right\|^2_{L^2} \\ &= \|f\|^2_{W^1(\Omega)}. \end{aligned}$$

Conversely, if $f \in W^1(\Omega, \mathbf{g}, \lambda)$ then by Exercise 4.11 we have the same identities as above and $f \in W^1(\Omega)$, whence the identity of the spaces $W^1(\Omega)$ and $W^1(\Omega, \mathbf{g}, \lambda)$ follows.

4.13. Set for simplicity $\Omega = \mathbb{R}^n \setminus \{o\}$ and let μ be the Lebesgue measure in $\mathbb{R}^n$.

(a) We need to prove that, for any smooth compactly supported vector field w on $\mathbb{R}^n$,

$$\int_\Omega \langle \nabla f, w \rangle d\mu = -\int_{\mathbb{R}^n} f \operatorname{div} w d\mu. \tag{B.66}$$

If w is supported in Ω then this is just the Divergence Theorem. In general, set

$$B_r = \{x \in \mathbb{R}^n : |x| < r\},$$

$$S_r = \partial B_r = \{x \in \mathbb{R}^n : |x| = r\},$$

and observe that, by $f, \nabla f \in L^2_{loc} \subset L^1_{loc}$,

$$\begin{aligned} \int_\Omega \langle \nabla f, w \rangle d\mu &= \lim_{r\to 0} \int_{\mathbb{R}^n \setminus B_r} \langle \nabla f, w \rangle d\mu = \lim_{r\to 0} \left(\int_{S_r} \langle \nu, w \rangle f d\sigma - \int_{\mathbb{R}^n \setminus B_r} f \operatorname{div} w d\mu \right) \\ &= \lim_{r\to 0} \int_{S_r} \langle \nu, w \rangle f d\sigma - \int_{\mathbb{R}^n} f \operatorname{div} w d\mu, \end{aligned} \tag{B.67}$$

where σ is the area on the sphere S_r and ν is the unit normal vector fields on S_r pointing at o. Since $\langle \nu, w\rangle$ remains uniformly bounded when r is small, we have

$$\left|\int_{S_r} \langle \nu, w\rangle f d\sigma\right| \leq \text{const} \int_{S_r} |f|\, d\sigma.$$

We are left to show that there exists a sequence $\{r_k\} \to 0$ such that

$$\int_{S_r} |f|\, d\sigma \to 0,$$

because then (B.66) will follow from (B.67) by taking the limit along this sequence.

Since

$$\int_{B_1} |f|^2\, d\mu = \int_0^1 \left(\int_{S_r} |f|^2\, d\sigma\right) dr$$

and by the Cauchy-Schwarz inequality

$$F(r) := \left(\int_{S_r} |f|\, d\sigma\right)^2 \leq \sigma(S_r) \int_{S_r} |f|^2\, d\sigma = \omega_n r^{n-1} \int_{S_r} |f|^2\, d\sigma,$$

the hypothesis $f \in L^2_{loc}$ implies

$$\int_0^1 \frac{F(r)}{\omega_n r^{n-1}} dr \leq \int_{B_1} |f|^2\, d\mu < \infty. \tag{B.68}$$

We claim that

$$\liminf_{r\to 0} F(r) = 0.$$

Indeed, if this is not so then $F(r) \geq c$ for some $c > 0$ and all small enough $r < \varepsilon$, which implies

$$\int_0^1 \frac{F(r)}{r^{n-1}} dr \geq c \int_0^\varepsilon \frac{dr}{r^{n-1}} = \infty,$$

and which contradicts (B.68). Hence, there is a sequence $\{r_k\} \to 0$ such that $F(r_k) \to 0$, which was to be proved.

(b) We need to prove that, for any smooth compactly supported function φ on $\mathbb{R}^n$,

$$\int_\Omega (\Delta f)\, \varphi d\mu = \int_{\mathbb{R}^n} f \Delta\varphi d\mu.$$

If φ is supported in Ω then this is true by the Green formula. In general, we have by $f, \Delta f \in L^2_{loc} \subset L^1_{loc}$ that

$$\begin{aligned}\int_\Omega (\Delta f)\, \varphi d\mu &= \lim_{r\to 0} \int_{\mathbb{R}^n\setminus B_r} (\Delta f)\, \varphi d\mu = \lim_{r\to 0}\left(\int_{S_r}\left(\frac{\partial f}{\partial \nu}\varphi - \frac{\partial \varphi}{\partial \nu} f\right) d\sigma - \int_{\mathbb{R}^n\setminus B_r} f\Delta\varphi d\mu\right) \\ &= \lim_{r\to 0} \int_{S_r}\left(\frac{\partial f}{\partial \nu}\varphi - \frac{\partial \varphi}{\partial \nu} f\right) d\sigma - \int_{\mathbb{R}^n} f\Delta\varphi d\mu \end{aligned} \tag{B.69}$$

Since $|\varphi|$ and $\left|\frac{\partial\varphi}{\partial\nu}\right| \leq |\nabla\varphi|$ remain uniformly bounded when r is small, we have

$$\left|\int_{S_r}\left(\frac{\partial f}{\partial\nu}\varphi - \frac{\partial\varphi}{\partial\nu}f\right)d\sigma\right| \leq \text{const}\int_{S_r}(|f| + |\nabla f|)\,d\sigma.$$

Since $|f| + |\nabla f| \in L^2_{loc}$, in the same way as in part (a), we obtain sequence $\{r_k\} \to 0$ such that

$$\int_{S_{r_k}}(|f| + |\nabla f|)\,d\sigma \to 0,$$

Taking the limit in (B.69) along this sequence, we finish the proof.

(c) To show that $f \in L^2_{loc}$ it suffices to prove that

$$\int_{B_1} f^2 d\mu < \infty.$$

Indeed, we have

$$\int_{B_1} f^2 d\mu = \int_0^1\left(\int_{S_r} f^2 d\sigma\right)dr = \int_0^1 r^{-2}4\pi r^2 dr = 4\pi.$$

To verify that $\Delta f = 0$, recall the representation of Δ in the polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^2},$$

whence, using $f(r) = r^{-1}$, we obtain

$$\Delta f = f'' + \frac{2}{r}f' = 0.$$

To prove that $\Delta_{dist} f = -4\pi\delta$ we need to verify that, for any smooth compactly supported function in $\mathbb{R}^3$,

$$(f, \Delta\varphi) = -(4\pi\delta, \varphi),$$

that is,

$$\int_{\mathbb{R}^3} f\Delta\varphi d\mu = -4\pi\varphi(o).$$

As in (B.69), we have, using that $\Delta f = 0$ in $\mathbb{R}^3 \setminus \{o\}$

$$\begin{aligned}\int_{\mathbb{R}^3} f\Delta\varphi d\mu &= \lim_{r\to 0}\int_{\mathbb{R}^3\setminus B_r} f\Delta\varphi d\mu = \lim_{r\to 0}\left(\int_{S_r}\left(\frac{\partial\varphi}{\partial\nu}f - \frac{\partial f}{\partial\nu}\varphi\right)d\sigma - \int_{\mathbb{R}^3\setminus B_r}\varphi\Delta f d\mu\right)\\ &= \lim_{r\to 0}\int_{S_r}\left(\frac{\partial\varphi}{\partial\nu}f - \frac{\partial f}{\partial\nu}\varphi\right)d\sigma. \qquad\text{(B.70)}\end{aligned}$$

Since $f \in L^2_{loc}$, the term $\int_{S_r}\frac{\partial\varphi}{\partial\nu}fd\sigma$ tends to 0 along a sequence $r_k \to 0$. Let us compute the remaining term. In the polar coordinates (r, θ), we have $f(r) = r^{-1}$ and $\frac{\partial f}{\partial\nu} = -\frac{\partial f}{\partial r} = r^{-2}$ whence

$$\int_{S_r}\frac{\partial f}{\partial\nu}\varphi d\sigma = \frac{1}{r^2}\int_{S_r}\varphi d\sigma = \frac{4\pi}{\sigma(S_r)}\int_{S_r}\varphi d\sigma \longrightarrow 4\pi\varphi(o)$$

as $r \to 0$. Substituting into (B.70), we finish the proof.

4.14. (a) It is clear that $f \in C^\infty(\Omega)$ for any Ω not containing the origin o. Hence, $f \in L^2_{loc}(\mathbb{R}^n)$ is equivalent to $f \in L^2(B)$ where B is the unit ball centred at o. Integrating in polar coordinates, we obtain

$$\|f\|^2_{L^2(B)} = \int_B |x|^{2\alpha}\, d\mu = \int_0^1 r^{2\alpha}\omega_n r^{n-1} dr = \omega_n \int_0^1 r^{2\alpha+n-1} dr.$$

The latter integral is finite provided $2\alpha + n > 0$, whence the claim follows.

(b) Using $\nabla r = x/r$, we obtain

$$\nabla f = \alpha r^{\alpha-1}\nabla r = \alpha r^{\alpha-2} x$$

whence

$$|\nabla f|^2 = \alpha^2 r^{2(\alpha-2)} r^2 = \alpha^2 r^{2(\alpha-1)}.$$

Therefore,

$$\int_B |\nabla f|^2\, d\mu = \alpha^2\omega_n \int_0^1 r^{2(\alpha-1)} r^{n-1} dr = \alpha^2 \omega_n \int_0^1 r^{n+2\alpha-3} dr.$$

This integral is finite provided $n + 2\alpha - 3 > -1$, that is $\alpha > 1 - n/2$. Hence, under this condition, we have $\nabla f \in \vec{L}^2_{loc}$. The fact that $\nabla_{dist} f = \nabla f$ follows from Exercise 4.13.

(c) A computation in the polar coordinates shows that

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r}\frac{\partial f}{\partial r} = \alpha(\alpha + n - 2) r^{\alpha-2},$$

where $r = |x|$. If $\alpha - 2 > -n/2$ then by part (a) the function $|x|^{\alpha-2}$ belongs to L^2_{loc} and hence also $\Delta f \in L^2_{loc}$. Also, by parts (a) and (b), we have $f \in L^2_{loc}$ and $\nabla f \in \vec{L}^2_{loc}$. Then the fact that $\Delta_{dist} f = \Delta f$ follows from Exercise 4.13.

4.15. By hypotheses, both sequences $\{u_k\}$ and $\{\nabla u_k\}$ are bounded in L^2. By the weak compactness of balls in L^2, there is a subsequence $\{u_{k_i}\}$ that converges weakly in L^2, and also ∇u_{k_i} converges weakly in L^2, that is,

$$u_{k_i} \overset{L^2}{\rightharpoonup} u \text{ and } \nabla u_{k_i} \overset{L^2}{\rightharpoonup} w. \tag{B.71}$$

Since the weak convergence in L^2 implies the convergence in $\mathcal{D}'$, it follows that $w = \nabla u$ and, hence, $u \in W^1$. It follows from (B.71) that, for any $\varphi \in W^1$,

$$(u_{k_i}, \varphi)_{L^2} + (\nabla u_{k_i}, \nabla\varphi)_{L^2} \to (u, \varphi)_{L^2} + (\nabla u, \nabla \varphi)_{L^2}$$

whence $u_{k_i} \overset{W^1}{\rightharpoonup} u$. Hence, $\{u_{k_i}\}$ converges to u weakly in L^2 and W^1, which was to be proved.

4.16. By the principle of uniform boundedness, any weakly convergence sequence is bounded in the norm. By Exercise 4.15, there is a subsequence $\{u_{k_i}\}$ and $v \in W^1$ such that

$$u_{k_i} \overset{L^2}{\rightharpoonup} v \text{ and } \nabla u_{k_i} \overset{L^2}{\rightharpoonup} \nabla v.$$

It follows that $v = u$ and, hence,

$$u_{k_i} \overset{L^2}{\rightharpoonup} u \text{ and } \nabla u_{k_i} \overset{L^2}{\rightharpoonup} \nabla u, \tag{B.72}$$

which was to be proved.

4.17. (a) By Exercise 4.15, a subsequence of $\{u_k\}$ converges weakly in L^2 to a function from W^1, which implies that $u \in W^1$. To prove that $u_k \overset{W^1}{\rightharpoonup} u$, it suffices to verify that any subsequence of $\{u_k\}$ contains a sub-subsequence that converges to u weakly in W^1. (cf. Exercise 2.14). Renaming the subsequence back to $\{u_k\}$, it suffices to prove that $\{u_k\}$ contains a subsequence that converges to u weakly in W^1. Indeed, by Exercise 4.15 there is a subsequence $\{u_{k_i}\}$ and a function $v \in W^1$ such that

$$u_{k_i} \overset{L^2}{\rightharpoonup} v \text{ and } u_{k_i} \overset{W^1}{\rightharpoonup} v.$$

It follows that $v = u$ and, hence, $u_{k_i} \overset{W^1}{\rightharpoonup} u$, which was to be proved.

(b) By part (a), u_k converges to u weakly in W^1. Together with the convergence of the W^1-norms, this implies the strong convergence in W^1.

4.18. Since the sequence $|u - u_k|^2$ converges almost everywhere to 0 and is bounded by the integrable function u^2, it follows by the dominated convergence theorem that $\|u - u_k\|_{L^2} \to 0$, that is, $u_k \overset{L^2}{\to} u$. By hypotheses, all norms $\|u_k\|_{W^1}$ are uniformly bounded, which implies by Exercise 4.17 that $u \in W^1$ and $u_k \overset{W^1}{\rightharpoonup} u$.

Since $0 \le u_k \le u$, we have

$$\|u_k\|_{W^1}^2 \le \|u\|_{L^2}^2 + \|\nabla u_k\|_{L^2}^2 \le \|u\|_{L^2}^2 + c^2.$$

Since u_k converges to u weakly in W^1, the norm $\|u\|_{W^1}$ admits the same estimate, that is,

$$\|u\|_{W^1}^2 \le \|u\|_{L^2}^2 + c^2,$$

whence $\|\nabla u\|_{L^2} \le c$.

4.19. Left to the reader

4.20. Let U be a chart in M. Without loss of generality, we may assume that U contains a cube $Q = \left\{\left(x^1, ..., x^n\right) : \left|x^i\right| < 1\right\}$. For any (large) integer k consider a function

$$f_k(x) = \sin(kx^1)\varphi(x)$$

where $\varphi \in C_0^\infty(Q) \setminus \{0\}$. Let us show that $\|Af_k\|_{L^2}$ grows as k^2 when $k \to \infty$ whereas $\|f_k\|_{L^2}$ remains bounded as $k \to \infty$.

By the product rule,

$$-Af_k = \Delta_\mu f_k = \varphi \Delta_\mu \sin\left(kx^1\right) + 2\langle \nabla \sin\left(kx^1\right), \nabla\varphi\rangle + \sin\left(kx^1\right)\Delta_\mu\varphi. \tag{B.73}$$

We have

$$\frac{\partial}{\partial x^1}\sin\left(kx^1\right) = k\cos\left(kx^1\right) = O(k) \quad \text{as } k \to \infty.$$

Hence, all the first derivatives of $\sin\left(kx^1\right)$ grow in $L^2\left(Q\right)$ at most as k when $k \to \infty$.

For the Laplace operator, we have

$$\Delta_\mu u = g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + \left(1^{st} \text{ order derivatives}\right).$$

For function $u = \sin\left(kx^1\right)$ the only non-vanishing second order term is $\frac{\partial^2 u}{(\partial x^1)^2}$ so that

$$\Delta_\mu\left(\sin\left(kx^1\right)\right) = -g^{11}k^2 \sin\left(kx^1\right) + \left(1^{st} \text{ order derivatives}\right).$$

Hence, we see that $\Delta_\mu \sin\left(kx^1\right)$ grows in $L^2\left(Q\right)$ as k^2 when $k \to \infty$.

Since φ does not depend in k, all L^2 norms of $\varphi, \nabla\varphi, \Delta_\mu\varphi$ can be bounded by a constant. By (B.73) we have

$$\|Af_k\|_{L^2} \geq \|\varphi\Delta_\mu \sin\left(kx^1\right)\|_{L^2} - 2\|\nabla\left(\sin kx^2\right)\|_{L^2}\|\nabla\varphi\|_{L^2} - \|\sin\left(kx^1\right)\|_{L^2}\|\Delta_\mu\varphi\|_{L^2},$$

whence we see that $\|Af_k\|_{L^2}$ grows as k^2 when $k \to \infty$. The sequence $\{f_k\}$ is uniformly bounded and is supported by a relatively compact set Q. Hence, the norms $\|f_k\|_{L^2}$ are also bounded. In particular, we obtain

$$\frac{\|Af_k\|_{L^2}}{\|f_k\|_{L^2}} \to \infty \quad \text{as } k \to \infty,$$

which means that the operator A is unbounded.

4.21. By Exercise 4.2, for any $f \in C_0^\infty$ and $u \in W^1$, we have $fu \in W^1$ and

$$\|fu\|_{W^1} \leq C\|u\|_{W^1}$$

where C is a constant depending on f. If $u \in C_0^\infty$ then clearly $fu \in C_0^\infty$ and, hence, $fu \in W_0^1$. For an arbitrary $u \in W_0^1$, let $\{u_k\}$ be a sequence of functions from C_0^∞ such that

$$u_k \xrightarrow{W^1} u \text{ as } k \to \infty.$$

Then

$$\|fu_k - fu\|_{W^1} \leq C\|u_k - u\|_{W^1} \to 0$$

that is, also $fu_k \xrightarrow{W^1} fu$. Since $fu_k \in W_0^1$, it follows that also $fu \in W_0^1$.

4.22. Use the same argument as in the proof of Lemma 4.3.

4.23. For any $u \in W_0^2$, we have by Lemma 4.4

$$\int_M |\nabla u|^2 \, d\mu = -\int_M u\Delta_\mu u \, d\mu.$$

Using the inequality

$$ab \leq \frac{s}{2}a^2 + \frac{1}{2s}b^2,$$

which holds for all real a, b and $s > 0$, we obtain

$$\|\nabla u\|_{\vec{L}^2}^2 = \int_M |\nabla u|^2 \, d\mu \leq \int_M \left(\frac{s}{2}u^2 + \frac{1}{2s}(\Delta_\mu u)^2\right) d\mu = \frac{s}{2}\|u\|_{L^2}^2 + \frac{1}{2s}\|\Delta_\mu u\|_{L^2}^2.$$

Therefore,

$$\|u\|_{W^1}^2 := \|u\|_{L^2}^2 + \|\nabla u\|_{\vec{L}^2}^2 \le \left(1+\frac{s}{2}\right)\|u\|_{L^2}^2 + \frac{1}{2s}\|\Delta_\mu u\|_{L^2}^2\,,$$

and (4.31) holds with

$$c = \max\left(1+\frac{s}{2},\, \frac{1}{2s}\right).$$

The minimum value of c is attained if

$$1+\frac{s}{2} = \frac{1}{2}s^{-1},$$

which leads to $s = \sqrt{2}-1$ and $c = \frac{1+\sqrt{2}}{2}$.

4.24. (a) For any $f \in W_0^2 = \operatorname{dom}\mathcal{L}$, we have

$$\mathcal{L}f = \int_0^\infty \lambda dE_\lambda f$$

and, hence,

$$(\mathcal{L}f, f) = \int_0^\infty \lambda d\left(E_\lambda f, f\right).$$

Since

$$(E_\lambda f, f) = \left(E_\lambda^2 f, f\right) = (E_\lambda f, E_\lambda f) = \|E_\lambda f\|^2,$$

we obtain

$$(\mathcal{L}f, f) = \int_0^\infty \lambda \|dE_\lambda f\|^2.$$

On the other hand, by Lemma 4.4,

$$(\mathcal{L}f, f) = -\left(\Delta_\mu f, f\right) = \int_M |\nabla f|^2\, d\mu, \tag{B.74}$$

whence (4.32) follows.

4.25. The operator $\mathcal{L}^{1/2}$ is non-negative definite and self-adjoint. Hence, by Exercise A.13, $\operatorname{dom}\mathcal{L}^{1/2}$ is a Hilbert space with the following norm:

$$\|f\|_{\operatorname{dom}\mathcal{L}^{1/2}}^2 = \|f\|_{L^2}^2 + \|\mathcal{L}^{1/2}f\|_{L^2}^2.$$

If in addition $f \in \operatorname{dom}\mathcal{L} \subset \operatorname{dom}\mathcal{L}^{1/2}$ then, using (B.74) and

$$\left(\mathcal{L}^{1/2}f, \mathcal{L}^{1/2}f\right) = (\mathcal{L}f, f)\,,$$

we obtain

$$\|f\|_{\operatorname{dom}\mathcal{L}^{1/2}}^2 = \|f\|_{L^2}^2 + (\mathcal{L}f, f) = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 = \|f\|_{W_0^1}^2. \tag{B.75}$$

Hence, $\operatorname{dom}\mathcal{L}$ is contained in two Hilbert spaces: $\operatorname{dom}\mathcal{L}^{1/2}$ and W_0^1, and the norms of these space are identical on $\operatorname{dom}\mathcal{L}$. By definition, C_0^∞ is dense in W_0^1 and, since $C_0^\infty \subset \operatorname{dom}\mathcal{L}$, we see that $\operatorname{dom}\mathcal{L}$ is dense in W_0^1.

Let us show that $\operatorname{dom}\mathcal{L}$ is dense in $\operatorname{dom}\mathcal{L}^{1/2}$, too. Recall that, by (A.48),

$$\operatorname{dom}\mathcal{L}^{1/2} = \left\{f \in L^2 : \int_0^\infty \lambda d\|E_\lambda f\|_{L^2}^2 < \infty\right\}. \tag{B.76}$$

Consider a sequence $\{\varphi_k\}_{k=1}^{\infty}$ of continuous functions on $[0,+\infty)$ such that $0\le\varphi_k\le 1$, $\varphi_k\equiv 1$ on $[0,k]$ and $\varphi_k\equiv 0$ on $[k+1,+\infty)$. Clearly, the operator $\varphi_k(\mathcal{L})$ is bounded. Fix a function $f\in\operatorname{dom}\mathcal{L}^{1/2}$ and set $f_k=\varphi_k(\mathcal{L})f$. Let us verify that $f_k\in\operatorname{dom}\mathcal{L}$ and $f_k\to f$ in the norm of $\operatorname{dom}\mathcal{L}^{1/2}$ (which will imply that $\operatorname{dom}\mathcal{L}$ is dense in $\operatorname{dom}\mathcal{L}^{1/2}$). The first claim follows from the fact that $\lambda\varphi_k(\lambda)$ is a bounded function (cf. (A.56)). Next, we have by (A.50)

$$\|f-f_k\|_{L^2}^2=\|(\mathrm{id}-\varphi_k(\mathcal{L}))f\|_{L^2}^2=\int_0^\infty(1-\varphi_k(\lambda))^2\,d\|E_\lambda f\|_{L^2}^2,$$

and

$$\|\mathcal{L}^{1/2}f-\mathcal{L}^{1/2}f_k\|_{L^2}^2=\int_0^\infty\lambda(1-\varphi_k(\lambda))^2\,d\|E_\lambda f\|_{L^2}^2,$$

and the both integrals tend to 0 as $k\to\infty$ by the dominated convergence theorem.

Let us now show that $\operatorname{dom}\mathcal{L}^{1/2}=W_0^1$. Any function $f\in\operatorname{dom}\mathcal{L}^{1/2}$ can be approximated by a sequence $\{f_k\}\subset\operatorname{dom}\mathcal{L}$ that converges to f in the norm of $\operatorname{dom}\mathcal{L}^{1/2}$. The sequence $\{f_k\}$ is contained in W_0^1 and, thanks to (B.75), it is Cauchy in W_0^1. Let $\widetilde{f}$ be its limit in W_0^1. Then f_k converges in L^2 to both functions f and $\widetilde{f}$, which implies $\widetilde{f}=f$ and, hence, $f\in W_0^1$. The opposite inclusion is proved in the same way.

Finally, the identity (B.75) extends by continuity to all $f\in W_0^1$, which implies

$$\|\mathcal{L}^{1/2}f\|_{L^2}^2=\|\nabla f\|_{L^2}^2. \tag{B.77}$$

Using

$$\|\mathcal{L}^{1/2}f\|_{L^2}^2=\int_0^\infty\lambda d\|E_\lambda f\|_{L^2}^2, \tag{B.78}$$

we obtain (4.32).

4.26. It follows from (B.76) and

$$\operatorname{dom}(\mathcal{L}+\mathrm{id})^{1/2}=\left\{f\in L^2:\int_0^\infty(\lambda+1)\,d\|E_\lambda f\|_{L^2}^2<\infty\right\}$$

that

$$\operatorname{dom}\mathcal{L}^{1/2}=\operatorname{dom}(\mathcal{L}+\mathrm{id})^{1/2}.$$

For any $f\in W_0^1$, we obtain using (B.77) and (B.78),

$$\begin{aligned}\|(\mathcal{L}+\mathrm{id})^{1/2}f\|_{L^2}^2 &= \int_0^\infty(\lambda+1)\,d\|E_\lambda f\|_{L^2}^2\\ &= \|\mathcal{L}^{1/2}f\|_{L^2}^2+\|f\|_{L^2}^2\\ &= \|\nabla f\|_{L^2}^2+\|f\|_{L^2}^2=\|f\|_{W^1}^2,\end{aligned}$$

which was to be proved.

4.27. By (4.32), we have, for any $f\in W_0^2$,

$$\|\nabla f\|_{L^2}^2=\int_{\lambda_{\min}}^\infty\lambda\|dE_\lambda f\|_{L^2}^2\ge\lambda_{\min}\int_{\lambda_{\min}}^\infty\|dE_\lambda f\|_{L^2}^2=\lambda_{\min}\|f\|_{L^2}^2.$$

Since W_0^2 is dense in W_0^1, it follows that the inequality

$$\|\nabla f\|_{L^2}^2 \geq \lambda_{\min}\|f\|_{L^2}^2 \tag{B.79}$$

holds for all $f \in W_0^1$.

4.28. The assumption (4.35) implies that 0 is a regular value of $\mathcal{L}$ and, hence, the inverse $\mathcal{L}^{-1}$ is defined as a bounded operator in L^2. Setting $u = \mathcal{L}^{-1}f$ and noticing that $u \in \operatorname{dom}\mathcal{L} = W_0^2$, we obtain a solution to (4.36).

If u is another solution to (4.36) then $-\Delta_\mu u = f$ implies $\Delta_\mu u \in L^2$, which together with $u \in W_0^1$ yields $u \in \operatorname{dom}\mathcal{L}$. In particular, we obtain $\mathcal{L}u = f$, and, hence, $u = \mathcal{L}^{-1}f$, which proves the uniqueness of solution.

To prove (4.37), observe that

$$\mathcal{L}^{-1} = \int_{\operatorname{spec}\mathcal{L}}^{\infty} \lambda^{-1} dE_\lambda = \int_{\lambda_{\min}}^{\infty} \lambda^{-1} dE_\lambda,$$

where $\{E_\lambda\}$ is the spectral resolution of $\mathcal{L}$. Therefore, for any $f \in L^2$,

$$\|\mathcal{L}^{-1}f\|_{L^2}^2 = \int_{\lambda_{\min}}^{\infty} \lambda^{-2} d\|E_\lambda f\|_{L^2}^2 \leq \lambda_{\min}^{-2} \int_{\lambda_{\min}}^{\infty} d\|E_\lambda f\|_{L^2}^2 = \lambda_{\min}^{-2}\|f\|_{L^2}^2,$$

whence (4.37) follows.

Multiplying the equation $-\Delta_\mu u = f$ by u and using the Green formula (4.12), we obtain

$$\int_M |\nabla u|^2\, d\mu = \int_M uf\, d\mu \leq \|u\|_{L^2}\|f\|_{L^2}.$$

Estimating $\|u\|_{L^2}$ from (4.37), we obtain (4.38).

4.29. The equation

$$\Delta_\mu u + \alpha u = f$$

means that

$$(\Delta_\mu u, \varphi) + \alpha\,(u, \varphi) = (f, \varphi) \quad \text{for any } \varphi \in \mathcal{D},$$

where the brackets mean pairing of distributions with test functions. By definition of Δ_μ is distributional sense, this equation is equivalent to

$$(u, \Delta_\mu \varphi) + \alpha\,(u, \varphi) = (f, \varphi) \quad \text{for any } \varphi \in \mathcal{D}.$$

Since $u - w \in W_0^1$ and $w \in W^1$, a solution u must be in W^1. In particular, $\nabla u \in \vec{L}^2$ and, using the definition of the distributional gradient, we obtain that the above equation is equivalent to

$$-\,(\nabla u, \nabla\varphi) + \alpha\,(u, \varphi) = (f, \varphi) \quad \text{for any } \varphi \in \mathcal{D}.$$

Since $\mathcal{D}$ is dense in W_0^1, we rewrite the equation in yet another equivalent form:

$$(\nabla u, \nabla\varphi)_{L^2} - \alpha\,(u, \varphi)_{L^2} = -\,(f, \varphi)_{L^2} \quad \text{for any } \varphi \in W_0^1.$$

Setting $v = u - w$ and replacing u in the above equation by $u = v + w$, we obtain the following equation for $v \in W_0^1$:

$$(\nabla v, \nabla \varphi) - \alpha\,(v, \varphi) = -\,(\nabla w, \nabla \varphi) - (f - \alpha w, \varphi) \quad \text{for any } \varphi \in W_0^1, \tag{B.80}$$

where the brackets mean the inner product in L^2.

Let us show that the bilinear form

$$[v, \varphi]_\alpha := (\nabla v, \nabla \varphi) - \alpha\,(v, \varphi)$$

defines an inner product in W_0^1, which is equivalent to the standard inner product $[v, \varphi]_1$. If $\alpha < 0$ then this is trivial and was already used in the proof of Theorem 4.5. We need to prove the same in under the hypothesis $\alpha < \lambda_{\min}$, and we can assume in addition that $\alpha \geq 0$.

It suffices to show that

$$[\varphi, \varphi]_\alpha \geq \varepsilon\,[\varphi, \varphi]_1 \tag{B.81}$$

for some $\varepsilon \in (0,1)$ and all $\varphi \in W_0^1$, which is equivalent to

$$\|\nabla \varphi\|_{L^2}^2 \geq \frac{\varepsilon + \alpha}{1 - \varepsilon}\|\varphi\|_{L^2}^2. \tag{B.82}$$

We claim that (B.82) holds with $\varepsilon = \frac{\lambda_{\min} - \alpha}{1 + \lambda_{\min}}$. Indeed, for this value of ε we have

$$\frac{\varepsilon + \alpha}{1 - \varepsilon} = \lambda_{\min},$$

and (B.82) coincides with the estimate (B.79) of Exercise 4.27.

Hence, the Riesz representation theorem yields that (B.80) has a unique solution $v \in W_0^1$ provided the right hand side of (B.80) is a bounded linear functional of φ. The latter follows from the Cauchy-Schwarz inequality because

$$|(\nabla w, \nabla \varphi)| \leq \|\nabla w\|_{L^2}\|\nabla \varphi\|_{L^2} \leq C_1\|\varphi\|_{W^1}$$

and

$$|(f - \alpha w, \varphi)| \leq \|f - \alpha w\|_{L^2}\|\varphi\|_{L^2} \leq C_2\|\varphi\|_{W^1}$$

where $C_1 = \|\nabla w\|_{L^2}$ and $C_2 = \|f - \alpha w\|_{L^2}$.

4.30. (a) Denote by $[f, g]$ the standard inner product in W_0^1 i.e.

$$[f, g] = (\nabla f, \nabla g) + (f, g)\,.$$

Let us verify that the bilinear form $\{f, g\}$ satisfies all axioms of an inner product. Indeed, the symmetry follows from the symmetry of A, and the positiveness follows from

$$\{f, f\} \geq (\nabla f, \nabla f) + \alpha^{-1}(f, f) \geq c\,[f, f]$$

where $c = \min(1, \alpha^{-1})$.

Note also that

$$\{f, f\} \leq (\nabla f, \nabla f) + \alpha\,(f, f) \leq C\,[f, f]$$

where $C = \max(1, \alpha)$. Therefore, the norms $\{f, f\}^{\frac{1}{2}}$ and $[f, f]^{\frac{1}{2}}$ are equivalent, and hence the inner product $\{\cdot, \cdot\}$ defines a complete metric on W_0^1.

(b) We can rewrite the given equation $-\Delta_\mu u + Au = h$ in the form of an integral identity

$$\{u, v\} = (h, v), \quad \forall v \in W_0^1, \tag{B.83}$$

where a solution u is sought in W_0^1. Indeed, if $u \in W_0^1$ and u satisfies (B.83) then as follows from the definition of the distributional Laplace operator, $\Delta_\mu u = Au - h \in L^2(M)$ and hence $u \in W_0^2$.

Let us show that the right hand side of (B.83) is a bounded linear functional in $v \in W_0^1$ with respect to the norm $\{\cdot, \cdot\}^{1/2}$. Indeed, we have

$$(h, v)^2 \le (h, h)(v, v) \le (h, h)[v, v] \le c^{-1}(h, h)\{v, v\}$$

whence it follows

$$|(h, v)| \le \text{const}\, \|v\|_{W_0^1}.$$

Hence, by the Riesz representation theorem, the equation (B.83) has exactly one solution $u \in W_0^1$.

4.31. As was shown in the proof of Theorem 4.5, the fact that $u = R_\alpha f$ is equivalent to the identity

$$(\nabla u, \nabla \varphi) + \alpha (u, \varphi) = (f, \varphi)$$

for all $\varphi \in W_0^1$. Hence, for any $\varphi \in W_0^1$, we have

$$\begin{aligned} E(u + \varphi) &= \|\nabla (u + \varphi)\|^2 + \alpha \|u + \varphi - f\|^2 \\ &= \|\nabla u\|^2 + 2(\nabla u, \nabla \varphi) + \|\nabla \varphi\|^2 + \alpha \|u - f\|^2 + 2\alpha (u - f, \varphi) + \alpha \|\varphi\|^2 \\ &= E(u) + \|\nabla \varphi\|^2 + \alpha \|\varphi\|^2. \end{aligned}$$

It follows that $E(u + \varphi) > E(u)$ unless $\varphi \equiv 0$, which was to be proved.

4.32. Let $f \in L^2$ and $u = R_\alpha f$. Using $\|u\|_{L^2} \le \alpha^{-1}\|f\|_{L^2}$ and (4.21), we obtain

$$\|\nabla u\|_{L^2}^2 \le (f, u) \le \|f\|_{L^2}\|u\|_{L^2} \le \alpha^{-1}\|f\|_{L^2}^2,$$

that is,

$$\|\nabla R_\alpha f\|_{L^2} \le \alpha^{-1/2}\|f\|_{L^2},$$

whence (4.40) follows.

Let $u \in W_0^2$. Setting $\varphi = \mathcal{L}u$ in the equation

$$(\mathcal{L}u, \varphi) + \alpha (u, \varphi) = (f, \varphi)$$

and noticing that by (4.14) $(u, \varphi) \ge 0$, we obtain

$$\|\Delta_\mu u\|_{L^2}^2 \le \|f\|_{L^2}\|\Delta_\mu u\|_{L^2},$$

which implies $\|\mathcal{L} R_\alpha f\|_{L^2} \le \|f\|_{L^2}$ and, hence, (4.41).

4.33. Left to the reader

4.34. By (4.45), we have for any $f \in L^2(M)$

$$R_\alpha f = \int_0^\infty (\alpha + \lambda)^{-1}\, dE_\lambda f.$$

Hence, (4.42) follows from this and the elementary identity

$$\frac{1}{\alpha+\lambda}-\frac{1}{\beta+\lambda}=\frac{(\beta-\alpha)}{(\alpha+\lambda)(\beta+\lambda)}.$$

4.35. (a)We have

$$\varphi(t)=\|P_{t/2}f\|_{L^2}^2=\int_0^\infty e^{-\lambda t}d\|E_\lambda f\|_{L^2}^2, \tag{B.84}$$

where $\{E_\lambda\}$ is the spectral resolution of the Dirichlet Laplace operator $\mathcal{L}$. It is clear that $\varphi(t)$ is non-negative and decreasing. That it is continuous follows from (4.55). Writing for simplicity $\sigma=\|E_\lambda f\|_{L^2}^2$, we obtain by the Cauchy-Schwarz inequality

$$\varphi\left(\frac{t+s}{2}\right)=\int_0^\infty e^{-\lambda(\frac{t+s}{2})}d\sigma\le\left(\int_0^\infty e^{-\lambda t}d\sigma\right)^{1/2}\left(\int_0^\infty e^{-\lambda s}d\sigma\right)^{1/2}=\sqrt{\varphi(t)\varphi(s)},$$

which is exactly the log-convexity of φ. Alternatively, one can argue as in the proof of Lemma 2.19.

(b) Since $P_tf\in W_0^2$, we have by the Green formula (4.12) and (4.57)

$$\psi(t)=(\nabla P_tf,\nabla P_tf)=-(P_tf,\Delta_\mu(P_tf))=-\left(P_tf,\frac{d}{dt}(P_tf)\right)=-\frac{1}{2}\frac{d}{dt}\|P_tf\|_{L^2}^2.$$

By part (a), function $\|P_tf\|_{L^2}^2$ is convex whence it follows that its derivative is increasing. It follows that $\psi(t)$ is decreasing. Integrating the above identity, we obtain

$$\int_0^\infty\psi(t)\,dt=-\frac{1}{2}\left[\|P_tf\|_{L^2}^2\right]_0^\infty\le\frac{1}{2}\|f\|_{L^2}^2.$$

4.36. The function $\varphi(t)$ from (B.84) is differentiable for $t>0$ and

$$\varphi'(t)=-\int_0^\infty\lambda e^{-\lambda t}d\|E_\lambda f\|_{L^2}^2,$$

which follows from Theorem 4.9. Since $f\in W_0^1(M)$, Exercise 4.25 yields

$$\|\nabla f\|_{L^2}^2=\int_0^\infty\lambda d\|E_\lambda f\|_{L^2}^2, \tag{B.85}$$

which implies that

$$\lim_{t\to0+}\varphi'(t)=-\|\nabla f\|_{L^2}^2.$$

Hence, $\varphi'(0)$ exists and

$$\varphi'(0)=-\|\nabla f\|_{L^2}^2.$$

By the log-convexity of $\varphi(t)$ (cf. Exercise 4.35), we have

$$\log\varphi(t)\ge\log\varphi(0)+t(\log\varphi)'(0).$$

Since $\varphi(0)=\|f\|_{L^2}^2=1$, this inequality implies

$$\varphi(t)\ge\exp\left(t\varphi'(0)\right)=\exp\left(-t\|\nabla f\|_{L^2}^2\right),$$

that is,

$$\|P_{t/2}f\|_{L^2}^2 \geq \exp\left(-t\|\nabla f\|_{L^2}^2\right), \tag{B.86}$$

whence (4.63) follows by changing t to $2t$.

4.37. If $\{E_\lambda\}$ is the spectral resolution of $\mathcal{L}$ then

$$-\Delta_\mu (P_t f) = \mathcal{L}e^{-t\mathcal{L}}f = \int_0^\infty \lambda e^{-t\lambda} dE_\lambda f$$

and

$$\|\mathcal{L}e^{-t\mathcal{L}}f\|_{L^2}^2 = \int_0^\infty \lambda^2 e^{-2t\lambda} d\|E_\lambda f\|_{L^2}^2 \leq \sup_{\lambda\in[0,+\infty)} \left(\lambda^2 e^{-2t\lambda}\right) \|f\|_{L^2}^2.$$

It is easy to see that

$$\sup_{\lambda\in[0,+\infty)} \left(\lambda^2 e^{-2t\lambda}\right) = \frac{e^2}{t^2},$$

whence (4.64) follows.

The inequality (4.65) follows from (4.64) and from the inequality

$$\|\nabla u\|_{L^2}^2 = (\mathcal{L}u, u) \leq \|\mathcal{L}u\|_{L^2}\|u\|_{L^2},$$

which is true for any $u \in W_0^2$.

4.38. (*a*) Using the spectral resolution $\{E_\lambda\}$ of the Dirichlet Laplace operator $\mathcal{L}$, we obtain

$$\mathcal{E}_t(f) = \int_0^\infty \frac{1-e^{-\lambda t}}{t} d(E_\lambda f, f) = \int_0^\infty \frac{1-e^{-\lambda t}}{t} d\|E_\lambda f\|_2^2.$$

It is easy to see that the function $\frac{1-e^{-\lambda t}}{t}$ is increasing when t is decreasing and, hence, the same is true for $\mathcal{E}_t(f)$.

(*b*) Since

$$\lim_{t\to 0} \frac{1-e^{-\lambda t}}{t} = \lambda,$$

the monotone convergence theorem yields

$$\lim_{t\to 0} \mathcal{E}_t(f) = \int_0^\infty \lambda d\|E_\lambda f\|_2^2,$$

which is finite if and only if $f \in \operatorname{dom} \mathcal{L}^{1/2} = W_0^1$ (see Exercise 4.25). Using (4.32), we obtain

$$\lim_{t\to 0} \mathcal{E}_t(f) = \int_M |\nabla f|^2 d\mu,$$

which was to be proved.

(*c*) Observe that $\mathcal{E}_t$ satisfies the polarization identity

$$\mathcal{E}_t(f,g) = \frac{1}{4}\left(\mathcal{E}_t(f+g) - \mathcal{E}_t(f-g)\right), \tag{B.87}$$

which follows directly from the definition. By (b) we have, as $t \to 0$,

$$\begin{aligned}\mathcal{E}_t(f \pm g) &\to \int_M |\nabla(f \pm g)|^2\, d\mu \\ &= \int_M |\nabla f|^2\, d\mu \pm 2\int_M \langle \nabla f, \nabla g\rangle\, d\mu + \int_M |\nabla g|^2\, d\mu.\end{aligned}$$

Subtracting the these two identities and using (B.87), we obtain (4.67).

4.39. We have, for any $f \in L^2$,

$$\|P_t f - f\|_{L^2}^2 = \int_0^\infty \left(1 - e^{-\lambda t}\right)^2 d\|E_\lambda f\|_{L^2}^2. \tag{B.88}$$

Using the inequality

$$1 - e^{-\lambda t} \le \lambda t, \tag{B.89}$$

and assuming that $f \in \operatorname{dom}\mathcal{L} = W_0^2$, we obtain

$$\|P_t f - f\|_{L^2}^2 \le t^2 \int_0^\infty \lambda^2 d\|E_\lambda f\|_{L^2}^2 = t^2\|\mathcal{L}f\|_{L^2}^2 = t^2\|\Delta_\mu f\|_{L^2}^2,$$

which proves (4.68).

4.40. By Exercise 4.25, we have $W_0^1 = \operatorname{dom}\mathcal{L}^{1/2}$ and

$$\|\nabla f\|_{L^2} = \|\mathcal{L}^{1/2} f\|_{L^2}. \tag{B.90}$$

Using the inequality

$$1 - e^{-\lambda t} \le (\lambda t)^{1/2}$$

(which is a consequence of (B.89)) and assuming that $f \in \operatorname{dom}\mathcal{L}^{1/2}$, we obtain

$$\|P_t f - f\|_{L^2}^2 \le t \int_0^\infty \lambda d\|E_\lambda f\|_{L^2}^2 = t\|\mathcal{L}^{1/2} f\|_{L^2}^2 = t\|\nabla f\|_{L^2}^2,$$

which proves (4.69).

Alternatively, one can use the same approach as in Lemma 2.20, which does not require Exercise 4.25.

4.41. Similarly to (B.88), we have

$$\left\|\frac{P_t f - f}{t} - \Delta_\mu f\right\|_{L^2}^2 = \int_0^\infty \left(\frac{e^{-\lambda t} - 1}{t} + \lambda\right)^2 d\|E_\lambda f\|_{L^2}^2. \tag{B.91}$$

By (B.89), the function

$$\frac{e^{-\lambda t} - 1}{t} + \lambda \tag{B.92}$$

is non-negative and bounded by λ. By $\mathcal{L}f \in L^2$, the function λ^2 is integrable with respect to $d\|E_\lambda f\|_{L^2}^2$ and the function (B.92) tends to 0 as $t \to 0$, we conclude by the dominated convergence theorem that the right hand side of (B.91) goes to 0 as $t \to 0$, which proves (4.70).

4.42. For any $\varphi \in \mathcal{D}$, we have by Exercise 4.41

$$\frac{P_t\varphi - \varphi}{t} \overset{L^2}{\to} \Delta_\mu\varphi \text{ as } t \to 0.$$

It follows that

$$\left(\frac{P_t f - f}{t}, \varphi\right) = \left(f, \frac{P_t \varphi - \varphi}{t}\right) \to (f, \Delta_\mu \varphi) = (\Delta_\mu f, \varphi),$$

whence $\frac{P_t f - f}{t} \xrightarrow{\mathcal{D}'} \Delta_\mu f$.

4.43. Using the quadratic form $\mathcal{E}_t$ defined by (4.66), we obtain

$$\mathcal{E}_t(f) \to -(g, f)_{L^2} \text{ as } t \to 0.$$

By Exercise 4.38 $f \in W_0^1$, and by Exercise 4.42 $\Delta_\mu f = g$, whence $f \in W_0^2$.

4.44. Clearly, u is differentiable in t in the norm $L^2(\Omega)$ for all $t > 0$ and for all $t < 0$, and

$$\frac{du}{dt} = \begin{cases} \Delta_\mu u, & t > 0, \\ 0, & t < 0. \end{cases}$$

For $t < 0$ we have $\Delta_\mu u(t) = \Delta_\mu f = 0$ in Ω so that the equation $\frac{du}{dt} = \Delta_\mu u$ is satisfied in Ω both for $t > 0$ and $t < 0$. We are left to verify it for $t = 0$, which amounts to showing that $\frac{du}{dt}(0) = 0$. It is obvious that $\frac{du}{dt}(0-) = 0$. To evaluate $\frac{du}{dt}(0+)$, observe that, by Exercise 4.41,

$$\frac{P_t f - f}{t} \xrightarrow{L^2(M)} \Delta_\mu f.$$

Since $\Delta_\mu f = 0$ in Ω, we conclude that

$$\frac{du}{dt}(0+) = L^2(\Omega)\text{-}\lim_{t\to 0+} \frac{P_t f - f}{t} = 0,$$

which finishes the proof.

4.45. Using (B.90), we obtain, for any $f \in W_0^1$,

$$\|\nabla P_t f - \nabla f\|_{L^2}^2 = \|\mathcal{L}^{1/2}(P_t f - f)\|_{L^2}^2 = \int_0^\infty \lambda \left(1 - e^{-\lambda t}\right)^2 d\|E_\lambda f\|_{L^2}^2.$$

Since the function λ is integrable with respect to $d\|E_\lambda f\|_{L^2}^2$ and

$$\lambda \left(1 - e^{-\lambda t}\right)^2 \to 0 \text{ as } t \to 0,$$

we obtain by the dominated convergence theorem that

$$\|\nabla P_t f - \nabla f\|_{L^2} \to 0,$$

whence $P_t f \xrightarrow{W^1} f$.

In the case $f \in W_0^2$ it suffices to prove that $\Delta_\mu P_t f \xrightarrow{L^2} \Delta_\mu f$. Since

$$\|\Delta_\mu P_t f - \Delta_\mu f\|_{L^2}^2 = \int_0^\infty \lambda^2 \left(1 - e^{-\lambda t}\right)^2 d\|E_\lambda f\|_{L^2}^2,$$

the claim follows as above by the dominated convergence theorem.

4.46. (a) We have

$$\frac{(u(t+\varepsilon), v(t+\varepsilon)) - (u(t), v(t))}{\varepsilon} = \left(u(t+\varepsilon), \frac{v(t+\varepsilon) - v(t)}{\varepsilon}\right) + \left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}, v(t)\right).$$

When $\varepsilon \to 0$, we have $u(t+\varepsilon) \to u(t)$ and

$$\frac{v(t+\varepsilon) - v(t)}{\varepsilon} \to v'(t) \quad \text{and} \quad \frac{u(t+\varepsilon) - u(t)}{\varepsilon} \to u'(t),$$

where all the convergencies are in the norm of $\mathcal{H}$. Since the inner product is a continuous functional of the both arguments (cf. Exercise A.1), we obtain

$$\frac{(u(t+\varepsilon), v(t+\varepsilon)) - (u(t), v(t))}{\varepsilon} \to \big(u(t), v'(t)\big) + \big(u'(t), v(t)\big),$$

which was to be proved.

(b) It follows from the Hölder inequality that

$$\|uv\|_r \le \|u\|_p \|v\|_q \tag{B.93}$$

so that $w \in L^r$. We also have

$$\begin{aligned} \|w(t+\varepsilon) - w(t)\|_r &\le \|(u(t+\varepsilon) - u(t))\, v(t+\varepsilon)\|_r + \|(v(t+\varepsilon) - v(t))\, u(t)\|_r \\ &\le \|(u(t+\varepsilon) - u(t))\|_p \|v(t+\varepsilon)\|_q + \|(v(t+\varepsilon) - v(t))\|_q \|u(t)\|_p \end{aligned}$$

whence it follows that $w(t+\varepsilon) \overset{L^r}{\to} w(t)$ as $\varepsilon \to 0$.

In the same way, if $u_k \overset{L^p}{\to} u$ and $v_k \overset{L^q}{\to} v$ then $u_k v_k \overset{L^r}{\to} uv$.

(c) Write

$$\frac{w(t+\varepsilon) - w(t)}{\varepsilon} = u(t+\varepsilon)\frac{v(t+\varepsilon) - v(t)}{\varepsilon} + \frac{u(t+\varepsilon) - u(t)}{\varepsilon} v(t).$$

Since $\frac{v(t+\varepsilon)-v(t)}{\varepsilon} \overset{L^q}{\to} v'(t)$ and $u(t+\varepsilon) \overset{L^p}{\to} u(t)$, we obtain by the argument of part (b) that

$$u(t+\varepsilon)\frac{v(t+\varepsilon) - v(t)}{\varepsilon} \overset{L^r}{\to} u(t)\, v'(t).$$

Similarly, we have

$$\frac{u(t+\varepsilon) - u(t)}{\varepsilon} v(t) \overset{L^r}{\to} u'(t)\, v(t),$$

which finishes the proof.

4.47. Since the functions $u(t, \cdot)$ and $\frac{\partial u}{\partial t}(t, \cdot)$ are continuous and bounded in $\overline{\Omega}$, they both belong to $C_b(\Omega)$. By the mean value theorem, we have

$$\frac{u(t+\varepsilon, x) - u(t, x)}{\varepsilon} = \frac{\partial u}{\partial t}(t + \theta\varepsilon, x),$$

where $\theta \in (0,1)$. By the uniform continuity of the function $\frac{\partial u}{\partial t}$ in $[t/2, 3t/2] \times \overline{\Omega}$, it follows that

$$\sup_{x\in\Omega} \left| \frac{\partial u}{\partial t}(t+\theta\varepsilon, x) - \frac{\partial u}{\partial t}(t,x) \right| \to 0 \text{ as } \varepsilon \to 0,$$

that is,

$$\left\| \frac{u(t+\varepsilon,\cdot) - u(t,\cdot)}{\varepsilon} - \frac{\partial u}{\partial t}(t,\cdot) \right\|_{C_b(\Omega)} \to 0.$$

This proves that $\frac{\partial u}{\partial t}(t,\cdot)$ is the strong derivative of $u(t,\cdot)$ in $C_b(\Omega)$.

4.48. (*a*) The function $t \mapsto (u(t), x)$ is continuous as the composition of two continuous mappings $t \mapsto u(t)$ and $u \mapsto (u,x)$, and the function $t \mapsto \|u(t)\|$ is continuous as the composition of two continuous mappings $t \mapsto u(t)$ and $u \mapsto \|u\|$. By the Cauchy-Schwarz inequality, we obtain

$$\left| \int_a^b (u(t), x)\, dt \right| \le \left| \int_a^b \|u(t)\|\, \|x\|\, dt \right| = C\|x\|,$$

where $C = \int_a^b \|u(t)\| dt$. Hence, the functional

$$x \mapsto \int_a^b (u(t), x)\, dt$$

is linear and bounded, which implies by the Riesz representation theorem that it can be represented in the form (U, x) for a unique vector $U \in \mathcal{H}$.

Setting

$$\int_a^b u(t)\, dt = U,$$

we obtain

$$\left\| \int_a^b u(t)\, dt \right\| = \|U\| \le C = \int_a^b \|u(t)\| dt.$$

(*b*) For any $x \in \mathcal{H}$, we have

$$\big(u'(t), x\big) = \frac{d}{dt}(u(t), x)$$

(cf. Exercise 4.46). Therefore,

$$\int_a^b \big(u'(t), x\big)\, dt = \int_a^b \frac{d}{dt}(u(t), x)\, dt = (u(b), x) - (u(a), x) = (u(b) - u(x), x),$$

whence, by the definition of the integral,

$$\int_a^b u'(t)\, dt = u(b) - u(a).$$

4.49. It follows by a standard argument that the function u is uniformly continuous on $[a,b]$, that is, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|t-s| < \delta \implies \|u(t) - u(s)\|_{L^1(M)} < \varepsilon.$$

Consider a step function approximation u_k for u defined as follows. Fix $k \in \mathbb{N}$ and set $t_i = a + \frac{i}{k}(b-a)$ so that $\{t_i\}_{i=0}^k$ is a partition of $[a,b]$. Then define $u_k(t)$ for $t \in [a,b)$ by

$$u_k(t) = u(t_i) \ \text{ if } t \in [t_i, t_{i+1})$$

that is,

$$u_k = \sum_{i=0}^{k-1} u(t_i) \, 1_{[t_i, t_{i+1})}.$$

Function $u_k(t,x)$ is integrable on N because it is a finite sum of functions of the form $f(x) g(t)$. Let us show that the sequence $\{u_k\}_{k=1}^\infty$ is Cauchy in $L^1(N)$. By the uniform continuity of u, we have

$$\sup_{t\in[a,b]} \|u_k(t) - u(t)\|_{L^1(M)} \to 0$$

as $k \to \infty$ whence

$$\int_a^b \left(\int_M |u_k - u| \, d\mu \right) dt \to 0. \tag{B.94}$$

It follows that

$$\int_a^b \left(\int_M |u_k - u_m| \, d\mu \right) dt \to 0$$

as $k, m \to \infty$. Since $u_k - u_m \in L^1(N)$, it follows by Fubini's theorem that

$$\|u_k - u_m\|_{L^1(N)} \to 0$$

that is, $\{u_k\}$ is Cauchy in $L^1(N)$. Hence, there is $w \in L^1(N)$ such that

$$\|u_k - w\|_{L^1(N)} \to 0$$

as $k \to \infty$, whence by Fubini's theorem

$$\int_a^b \left(\int_M |u_k - w| \, d\mu \right) dt \to 0.$$

Comparing with (B.94), we obtain

$$\int_a^b \left(\int_M |u - w| \, d\mu \right) dt = 0.$$

This implies that, for any $t \in [a,b]$, $u(t)$ and $w(t,\cdot)$ coincide as functions from $L^1(M)$.

4.50. The condition (4.71) implies that $|\psi(t)| \le C|t|$ whence it follows that $\psi(u(t))$ is also in $L^2(M)$.

Fix $t \in (a,b)$. Denoting

$$r(s) := \frac{u(t+s) - u(t)}{s}$$

and $u' = \frac{du}{dt}$, we have by hypothesis

$$r(s) \xrightarrow{L^2} u'(t) \quad \text{as } s \to 0. \tag{B.95}$$

We need to prove that

$$\frac{\psi\left(u\left(t+s\right)\right)-\psi\left(u\left(t\right)\right)}{s}\overset{L^2}{\longrightarrow}\psi'\left(u\right)u' \text{ as } s\to 0. \tag{B.96}$$

It suffices to show that for any sequence $s_k \to 0$, there is a subsequence along which (B.96) holds (cf. Exercise 2.14).

By the mean value theorem, we have

$$\begin{aligned}\psi\left(u\left(t+s\right)\right)-\psi\left(u\left(t\right)\right) &= \psi\left(u\left(t\right)+sr\left(s\right)\right)-\psi\left(u\left(t\right)\right)\\ &= \psi'\left(u\left(t\right)+\xi sr\left(s\right)\right)sr\left(s\right)\end{aligned}$$

where $\xi=\xi\left(s,x\right)\in\left(0,1\right)$. Therefore,

$$\begin{aligned}\frac{\psi\left(u\left(t+s\right)\right)-\psi\left(u\left(t\right)\right)}{s}-\psi'\left(u\right)u' &= \left[\psi'\left(u\left(t\right)+\xi sr\left(s\right)\right)-\psi'\left(u\left(t\right)\right)\right]u'\left(t\right)\\ &\quad+\psi'\left(u\left(t\right)+\xi sr\left(s\right)\right)\left[r\left(s\right)-u'\left(t\right)\right]\end{aligned}$$

and, hence,

$$\begin{aligned}&\left\|\frac{\psi\left(u\left(t+s\right)\right)-\psi\left(u\left(t\right)\right)}{s}-\psi'\left(u\right)u'\right\|_{L^2}\\ \leq &\left(\int_M\left|\psi'\left(u\left(t\right)+\xi sr\left(s\right)\right)-\psi'\left(u\left(t\right)\right)\right|^2\left|u'\left(t\right)\right|^2d\mu\right)^{1/2}+\sup\left|\psi'\right|\left\|r\left(s\right)-u'\left(t\right)\right\|_{L^2}\end{aligned} \tag{B.97}$$

When $s\to 0$, the second term in (B.97) tends to 0 by (B.95). Let us show that, for any sequence $s_k\to 0$, there is a subsequence along which the first term in (B.97) tends to 0. The sequence of functions $s_kr\left(s_k\right)$ tends to 0 in L^2 because the norms $\|r\left(s\right)\|_{L^2}$ remain bounded as $s\to 0$. Therefore, there is a subsequence s_{k_i}, which will be renumbered by $\{s_k\}$, along which $s_kr\left(s_k,\cdot\right)\to 0$ a.e. Since $\xi_k:=\xi\left(s_k\right)$ is bounded, we also have $\xi_ks_kr\left(s_k\right)\to 0$ a.e., and by the continuity of ψ',

$$\psi'\left(u\left(t\right)+\xi_ks_kr\left(s_k\right)\right)\to\psi'\left(u\left(t\right)\right) \text{ a.e.}$$

Hence, the function under the integral sign in (B.97) tends to 0 almost everywhere. Since this function is bounded for all s by the integrable function $4C^2\left|u'\right|^2$, we conclude by the dominated convergence theorem that the integral in (B.97) tends to 0, which finishes the proof.

4.51. By (4.72), the function $\Phi\left(\lambda\right)e^{-t\lambda}$ is bounded for any $t>0$, which implies that the right hand side of (4.73) is defined for all $f\in\mathcal{H}$ and determines $v\left(t\right)$ as an element of $\mathcal{H}$. The first equality in (4.74) is proved exactly in the same way as the existence of the strong derivative in Theorem 4.9. Since the function $\lambda\Phi\left(\lambda\right)e^{-\lambda t}$ is bounded for any $t>0$, we obtain from the functional calculus that, for any $t>0$,

$$\mathcal{L}v\left(t\right)=\mathcal{L}\Phi\left(\mathcal{L}\right)e^{-t\mathcal{L}}v\left(t\right)=\int_0^\infty\lambda\Phi\left(\lambda\right)e^{-t\lambda}dE_\lambda f, \tag{B.98}$$

which gives the second equality in (4.74).

The existence of $\frac{d^k v}{dt^k}$ for any k and the identity (4.75) are proved by induction, using the fact that the function $\lambda^k \Phi(\lambda)$ satisfies the condition (4.72) for any k.

4.52. (a) This follows from the functional calculus since functions $\cos\left(t\sqrt{\lambda}\right)$ and $\sin\left(t\sqrt{\lambda}\right)$ are real valued and bounded in $\lambda \in [0, +\infty)$.

(b) We have

$$\begin{aligned}\frac{d}{dt}C_t f &= \lim_{s\to 0}\frac{C_{t+s}f - C_t f}{s}\\ &= \lim_{s\to 0}\int_0^\infty \frac{\cos\left((t+s)\lambda^{1/2}\right) - \cos\left(t\lambda^{1/2}\right)}{s} dE_\lambda f. \end{aligned} \tag{B.99}$$

Note that

$$\lim_{s\to 0}\frac{\cos\left((t+s)\lambda^{1/2}\right) - \cos\left(t\lambda^{1/2}\right)}{s} = -\lambda^{1/2}\sin\left(t\lambda^{1/2}\right).$$

By Exercise A.30, in order to interchange the limit and the integral in (B.99) it suffices to prove that

$$\left|\frac{\cos\left((t+s)\lambda^{1/2}\right) - \cos\left(t\lambda^{1/2}\right)}{s}\right| \leq \Phi(\lambda),$$

where Φ is a function such that

$$\int_0^\infty \Phi^2(\lambda)\, d\|E_\lambda f\|^2 < \infty.$$

By the mean value theorem, we have

$$\frac{\cos\left((t+s)\lambda^{1/2}\right) - \cos\left(t\lambda^{1/2}\right)}{s} = -\lambda^{1/2}\sin\left(\xi\lambda^{1/2}\right),$$

where $\xi \in (t, t+s)$. It follows that

$$\left|\frac{\cos\left((t+s)\lambda^{1/2}\right) - \cos\left(t\lambda^{1/2}\right)}{s}\right| \leq \lambda^{1/2} =: \Phi(\lambda)$$

If $f \in \operatorname{dom}\mathcal{L}^{1/2}$ then

$$\int_0^\infty \Phi^2(\lambda)\, d\|E_\lambda f\|^2 = \int_0^\infty \lambda d\|E_\lambda f\|^2 < \infty.$$

Hence, for such f,

$$\begin{aligned}\frac{d}{dt}C_t f &= \int_0^\infty \lim_{s\to 0}\frac{\cos\left((t+s)\lambda^{1/2}\right) - \cos\left(t\lambda^{1/2}\right)}{s} dE_\lambda f\\ &= -\int_0^\infty \lambda^{1/2}\sin\left(t\lambda^{1/2}\right) dE_\lambda f.\end{aligned}$$

Evaluating at $t = 0$ we obtain

$$\frac{d}{dt} C_t f \Big|_{t=0} = 0.$$

Obviously, we have $C_0 f = f$.

Similarly, we have $S_0 g = 0$ and

$$\frac{d}{dt} S_t g = \int_0^\infty \lambda^{1/2} \cos\left(t\lambda^{1/2}\right) dE_\lambda g,$$

whence it follows that

$$\frac{d}{dt} S_t g \Big|_{t=0} = \int_0^\infty \lambda^{1/2} dE_\lambda g = \mathcal{L}^{1/2} g.$$

(c) If $f \in \operatorname{dom} \mathcal{L}$ then $f \in \operatorname{dom} \mathcal{L}^{1/2}$ and, by the previous argument, we have

$$\frac{d}{dt} C_t f = -\int_0^\infty \lambda^{1/2} \sin\left(t\lambda^{1/2}\right) dE_\lambda f.$$

Applying similar argument again and using $f \in \operatorname{dom} \mathcal{L}$, we obtain

$$\frac{d^2}{dt^2} C_t f = -\int_0^\infty \lambda \cos\left(t\lambda^{1/2}\right) dE_\lambda f.$$

On the other hand,

$$\mathcal{L} C_t f = \int_0^\infty \lambda \cos\left(t\lambda^{1/2}\right) dE_\lambda f,$$

whence

$$\frac{d^2}{dt^2} C_t f = -\mathcal{L}\left(C_t f\right).$$

In the same way, one handles $S_t f$.

(d) By Exercise 1.3, we have, for any $\lambda \in \mathbb{R}$,

$$e^{-t\lambda^2} = \int_{-\infty}^{+\infty} e^{-is\lambda} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{s^2}{4t}\right) ds,$$

which implies, by taking the real part and using the symmetry of the integral,

$$e^{-t\lambda^2} = \int_0^{+\infty} \cos(s\lambda) \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) ds.$$

As in Lemma 5.10, replacing λ by $\mathcal{L}^{1/2}$, we obtain from this functional identity the operator identity

$$\left(e^{-t\mathcal{L}} f, g\right) = \int_0^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) \left(C_s f, g\right) ds, \tag{B.100}$$

for all $f, g \in \mathcal{H}$, which is equivalent to (4.76).

4.53. It suffices to prove that $\varphi(\alpha) \geq \varphi(\beta)$ for all $a < \alpha < \beta < b$. Assume first that $\varphi'(t) < 0$ for all $t \in [\alpha, \beta]$. Let ξ be a point where φ attains its minimum value on $[\alpha, \beta]$. If $\xi = \beta$ then there is nothing to prove.

If $\xi < \beta$ then, at the minimum point ξ, the right derivative $\varphi'(\xi)$ must be non-negative, which contradicts $\varphi'(\xi) < 0$.

Consider now the general case with a non-strict inequality $\varphi'(t) \leq 0$. Fix $\varepsilon > 0$ and consider the function $\varphi(t) - \varepsilon t$ whose right derivative is obviously strictly negative. By the previous argument, we have

$$\varphi(\alpha) - \varepsilon\alpha \geq \varphi(\beta) - \varepsilon\beta,$$

whence letting $\varepsilon \to 0$ we obtain $\varphi(\alpha) \geq \varphi(\beta)$.

4.54. The proof goes the same way as that of Theorem 4.10. It suffices to show that any path $u(t)$, that solves the right Cauchy problem with the initial function 0, is identical 0. We have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = \left(\frac{du}{dt}, u\right) = (\mathcal{L}u, u) \leq 0.$$

Since the function $\varphi(t) = \|u(t)\|^2$ is continuous in $t > 0$ and its right derivative is non-positive, we conclude by Exercise 4.53 that $\varphi(t)$ is decreasing in t. Since $\varphi(t) \to 0$ as $t \to 0$, it follows that $\varphi(t) \equiv 0$.

Solutions to Chapter 5

5.1. Left to the reader

5.2. Let η be a mollifier in $\mathbb{R}$, and set

$$\psi_k = \psi * \eta_{1/k}.$$

Then, by Lemma 2.1, $\psi_k \in C^\infty(\mathbb{R})$ and

$$\psi_k' = \psi' * \eta_{1/k}.$$

Hence, we obtain, by Exercise 2.3,

$$\sup|\psi_k'| \leq \sup|\psi'|$$

and, by Lemma 2.4,

$$\psi_k(t) \to \psi(t) \quad \text{and} \quad \psi_k'(t) \to \psi'(t),$$

where the convergence is locally uniform in t.

We have satisfied all the conditions except for $\psi_k(0) = 0$. To satisfy it, just replace the function $\psi_k(t)$ by

$$\widetilde{\psi}_k(t) = \psi_k(t) - \psi_k(0).$$

Since $\psi_k(0) \to \psi(0) = 0$, we obtain

$$\widetilde{\psi}_k(t) \to \psi(t),$$

and all other requirements are trivially satisfied.

5.3. We have

$$\begin{aligned}\max(u, v) &= v + (u - v)_+ \\ \min(u, v) &= u - (u - v)_+\end{aligned}$$

so that both functions $\max(u,v)$ and $\min(u,v)$ are in W_0^1 by Lemma 5.2 (see also Example 5.3).

5.4. Since any function $f \in W^1(M)$ has compact support, we have $W^1(M) = W_c^1(M)$, whence it follows from Lemma 5.5 that $W^1(M) \subset W_0^1(M)$, which was to be proved.

5.5. It suffices to prove that $\nabla u = 0$ in any relatively compact open set $\Omega \subset M$. Since $c \in W^1(\Omega)$ and, hence, $u - c \in W^1(\Omega)$, we obtain by Theorem 5.7 (see also Example 5.8) that $(u-c)_+ \in W^1(\Omega)$ and $\nabla(u-c)_+ = 0$ on the set $\{u - c = 0\} \cap \Omega$.The same holds also for $\nabla(u-c)_-$, whence we obtain that, on the set $\{u = c\} \cap \Omega$,

$$\nabla u = \nabla(u-c) = \nabla(u-c)_+ - \nabla(u-c)_- = 0.$$

5.6. For any $c > 0$, we have $(u-c)_+ = (u_+ - c)_+$ so that we can rename u_+ by u and assume $u \geq 0$. By the dominated convergence theorem, we obtain $(u-c)_+ \overset{L^2}{\longrightarrow} u$ as $c \to 0+$. By (5.12), we have

$$\nabla(u-c)_+ = \begin{cases} \nabla u, & u > c, \\ 0, & u \leq c, \end{cases}$$

which implies that

$$\|\nabla u - \nabla(u-c)_+\|_{L^2}^2 = \int_{\{u \leq c\}} |\nabla u|^2 \, d\mu \to \int_{\{u=0\}} |\nabla u|^2 \, d\mu$$

as $c \to 0+$. Since $\nabla u = 0$ on the set $\{u = 0\}$, we conclude that

$$\|\nabla u - \nabla(u-c)_+\|_{L^2} \to 0,$$

which finishes the proof.

5.7. Fix some $c > 0$. By Theorem 5.7, we have $(f-c)_+ \in W^1(M)$. By hypothesis, the set $\{f \geq c\}$ is relatively compact, which implies that $\operatorname{supp}(f-c)_+$ is compact. Hence, $(f-c)_+ \in W_c^1(M)$, which implies by Lemma 5.5 that $(f-c)_+ \in W_0^1(M)$. By Theorem 5.7 (see also Exercise 5.6),

$$(f-c)_+ \overset{W^1}{\to} f_+ \quad \text{as } c \to 0,$$

whence it follows that $f_+ \in W_0^1(M)$. In the same way, $f_- \in W_0^1(M)$ and, hence, $f \in W_0^1(M)$.

5.8. Let Ω be any relatively compact open subset of M. Then $u \in W_{loc}^1(M)$ implies $u \in W^1(\Omega)$. By Theorem 5.7, we obtain that $\psi(u) \in W^1(\Omega)$ and

$$\nabla \psi(u) = \varphi(u) \nabla u.$$

It follows that $\psi(u) \in W_{loc}^1(M)$, which was to be proved.

5.9. Let Ω be any relatively compact open set containing $\operatorname{supp} u$. Then by Lemma 5.5 $u \in W_0^1(\Omega)$, while it follows from the definition of W_{loc}^2 that

$v \in W^2(\Omega)$. Hence, the Green formula (4.12) of Lemma 4.4 can be applied in Ω, which finishes the proof.

5.10. (*a*) Let us first show that, for any $\lambda \geq 0$ and $t > 0$,

$$\lim_{\alpha \to +\infty} e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\alpha^{2k} t^k}{(\alpha+\lambda)^k k!} = e^{-\lambda t}. \tag{B.101}$$

Indeed, by changing αt to α and λt to λ, it suffices to consider the case $t = 1$. Then we have

$$\begin{aligned} e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{(\alpha+\lambda)^k k!} &= e^{-\alpha} \sum_{k=0}^{\infty} \frac{\left(\frac{\alpha^2}{\alpha+\lambda}\right)^k}{k!} \\ &= \exp\left(-\alpha + \frac{\alpha^2}{\alpha+\lambda}\right) = \exp\left(-\frac{\alpha\lambda}{\alpha+\lambda}\right), \end{aligned} \tag{B.102}$$

whence (B.101) follows.

Since

$$R_\alpha^k f = \int_0^\infty \frac{1}{(\alpha+\lambda)^k} dE_\lambda f, \tag{B.103}$$

the right hand side of (5.33) can be transformed as follows:

$$e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\alpha^{2k} t^k}{k!} R_\alpha^k f = \int_0^\infty e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\alpha^{2k} t^k}{k! (\alpha+\lambda)^k} dE_\lambda f \longrightarrow \int_0^\infty e^{-\lambda t} dE_\lambda f = P_t f,$$

where we have passed to the limit as $\alpha \to +\infty$ using the dominated convergence theorem (indeed, as follows from (B.102), the integrand remains uniformly bounded by 1 for all $\alpha > 0$).

(*b*) If $f \leq 1$ then $R_\alpha f \leq \alpha^{-1}$ and, by induction, $R_\alpha^k f \leq \alpha^{-k}$. Substituting into (5.33), we obtain

$$P_t f \leq \lim_{\alpha \to +\infty} e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\alpha^k t^k}{k!} = \lim_{\alpha \to +\infty} e^{-\alpha t} e^{\alpha t} = 1.$$

5.11. (*a*) By the definition of the gamma function,

$$\Gamma(k) = \int_0^\infty \tau^{k-1} e^{-\tau} d\tau,$$

which implies, for any $s > 0$,

$$\int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-st} dt = s^{-k}.$$

Similarly to the proof of Lemma 5.10, we obtain, for all $f, g \in L^2$,

$$
\begin{aligned}
\int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-\alpha t} (P_t f, g)_{L^2}\, dt &= \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-\alpha t} \left(\int_0^\infty e^{-\lambda t} d(E_\lambda f, g)_{L^2} \right) dt \\
&= \int_0^\infty \left(\int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-(\alpha+\lambda)t} dt \right) d(E_\lambda f, g)_{L^2} \\
&= \int_0^\infty (\alpha+\lambda)^{-k}\, d(E_\lambda f, g)_{L^2} \\
&= \left(R_\alpha^k f, g \right)_{L^2}, \qquad \text{(B.104)}
\end{aligned}
$$

which was to be proved.

(b) The identity $R^k R^l = R^{k+l}$ follows from the functional calculus because R is bounded self-adjoint operator in L^2. If $f \geq 0$ then by Theorem 5.11, $P_t f \geq 0$ which implies for any non-negative $g \in C_0^\infty(M)$ that $(P_t f, g) \geq 0$. From the identity (B.104) we conclude $\left(R^k f, g\right) \geq 0$ whence $R^k f \geq 0$. If $f \leq 1$ then by Theorem 5.11 $P_t f \leq 1$, and (B.104) implies, for any g as above,

$$
\int_M \left(R^k f\right) g d\mu \leq \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-t} dt \int_M g d\mu = \int_M g d\mu,
$$

where in the last identity we have used the definition of gamma function. This obviously yields $R^k f \leq 1$.

(c) Since $\operatorname{spec} \mathcal{L} \subset [0, +\infty)$, the function $\log(1+\lambda)$ is defined (and even non-negative) on $\operatorname{spec} \mathcal{L}$, which implies that $L = \log(\mathrm{id} + \mathcal{L})$ is defined by the functional calculus as a self-adjoint operator. Since for any $\lambda \geq 0$ and $k > 0$

$$
(1+\lambda)^{-k} = \exp(-k \log(1+\lambda)),
$$

we obtain from the functional calculus a similar operator identity

$$
(\mathrm{id} + \mathcal{L})^{-k} = \exp(-k \log(\mathrm{id} + \mathcal{L})),
$$

whence the claim follows.

5.12. For any non-negative $g \in L^2(M)$, we obtain using (5.29),

$$
\begin{aligned}
(P_t R_\alpha f, g) &= (R_\alpha f, P_t g) = \left(\int_0^\infty e^{-\alpha s} P_s f\, ds, P_t g \right) = \int_0^\infty e^{-\alpha s} (P_s f, P_t g)\, ds \\
&= \int_0^\infty e^{-\alpha s} (P_{t+s} f, g)\, ds = \int_t^\infty e^{-\alpha(\tau - t)} (P_\tau f, g)\, d\tau \\
&= e^{\alpha t} \int_t^\infty e^{-\alpha \tau} (P_\tau f, g)\, d\tau \leq \left(e^{\alpha t} R_\alpha f, g \right),
\end{aligned}
$$

whence the claim follows.

5.13. (a) Function $u = R_\lambda f$ must satisfy the equation $\mathcal{L} u + \lambda u = f$, that is,

$$
-u'' + \lambda u = f.
$$

Moreover, u is the unique solution of this equation in $L^2\left(\mathbb{R}^1\right)$. Solving this ODE by the method of variation of constants and selecting a solution from L^2, we obtain (5.35).

(b) Using the explicit formula for $P_t f$, we obtain

$$R_\lambda f=\int_0^\infty\int_{-\infty}^{+\infty}\frac{1}{(4\pi t)^{1/2}}\exp\left(-\frac{|x-y|^2}{4t}\right)e^{-\lambda t}f(y)\,dydt$$

Comparing with (5.35) and setting $r=|x-y|$, we obtain

$$\frac{1}{2\sqrt{\lambda}}e^{-\sqrt{\lambda}r}=\int_0^\infty\frac{1}{(4\pi t)^{1/2}}\exp\left(-\frac{r^2}{4t}\right)e^{-t\lambda}dt.$$

Differentiating this identity in r, we obtain

$$e^{-\sqrt{\lambda}r}=\int_0^\infty\frac{r}{\sqrt{4\pi t^3}}\exp\left(-\frac{r^2}{4s}\right)e^{-t\lambda}dt,$$

whence (5.36) follows by renaming the variables. The limiting case $\lambda=0$ follows by passing to the limit as $\lambda\to0+$.

5.14. (a) The identity (5.37) follows from (5.36) by substituting $\mathcal{L}$ in place of λ.

(b) If $f\geq0$ then $P_tf\geq0$, whence $Q_tf\geq0$ just by the positivity of the integral kernel in (5.37). If $f\leq1$ then $P_tf\leq1$, whence $Q_tf\leq1$ follows from the identity

$$\int_0^\infty\frac{t}{\sqrt{4\pi s^3}}\exp\left(-\frac{t^2}{4s}\right)ds=1,$$

which is a particular case of (5.36) for $\lambda=0$.

(c) It follows from (5.37) that the integral kernel $q_t(x)$ of Q_t is related to the integral kernel $p_t(x)$ of P_t by the identity

$$q_t(x)=\int_0^\infty\frac{t}{\sqrt{4\pi s^3}}\exp\left(-\frac{t^2}{4s}\right)p_s(x)\,ds.$$

Substituting $p_t(x)$ from (2.50), we obtain

$$\begin{aligned}q_t(x)&=\int_0^\infty\frac{t}{\sqrt{4\pi s^3}}\exp\left(-\frac{t^2}{4s}\right)\frac{1}{(4\pi s)^{n/2}}\exp\left(-\frac{|x|^2}{4s}\right)ds\\&=\frac{t}{(4\pi)^{\frac{n+1}{2}}}\int_0^\infty s^{-(n+3)/2}\exp\left(-\frac{t^2+|x|^2}{4s}\right)ds.\qquad\text{(B.105)}\end{aligned}$$

Applying (A.60) in order to evaluate the integral (B.105), we obtain

$$q_t(x)=\Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}\frac{t}{\left(t^2+|x|^2\right)^{n+1}}.$$

Using the value of ω_{n+1} from (3.94), we obtain (5.38).

5.15. (a) For example, for $\Psi(s) = s^2/2$ the claim amounts to the fact that $\|P_t f\|_{L^2}$ is decreasing in t, which follows from Theorem 4.9 or Theorem 4.10.

The function F is non-negative and finite because $0 \le \Psi(s) \le \frac{1}{2}|s|^2$ and $P_t f \in L^2$. Observe that for all real a, b

$$|\Psi(a) - \Psi(b)| \le (|a| + |b|)\,|a - b|, \tag{B.106}$$

since for some $\xi \in (a, b)$

$$|\Psi(a) - \Psi(b)| = \left|\Psi'(\xi)\right| |a - b| \le |\xi|\,|a - b| \le (|a| + |b|)\,|a - b|.$$

Set $u_t = P_t f$ and recall that by Theorem 4.9 $\|u_t\|_{L^2} \le \|f\|_{L^2}$. Using (B.106) we obtain for all $t, \tau \in [0, +\infty)$

$$\begin{aligned} |F(\tau) - F(t)| &\le \int_M |\Psi(u_\tau) - \Psi(u_t)|\, d\mu \\ &\le \int_M (|u_\tau| + |u_t|)\ |u_\tau - u_t|\, d\mu \\ &\le 2\|f\|_{L^2}\|\,u_\tau - u_t\|_{L^2}. \end{aligned}$$

Since by Theorem 4.9 the mapping $t \mapsto u_t$ is continuous in L^2 we conclude that the function $F(t)$ is continuous.

Next, let us prove that $F(t)$ is differentiable for $t > 0$ and

$$F'(t) = \int_M \Psi'(u_t)\,\frac{du_t}{dt}\,d\mu. \tag{B.107}$$

This formula allows to finish the proof as follows. Since by Theorem 4.9 $du_t/dt = \Delta_\mu u_t$ and, by the previous claim, $\Psi'(u_t) \in W_0^1$, we obtain by Lemma 4.4

$$F'(t) = \int_M \Psi'(u_t)\,\Delta_\mu u_t d\mu = -\int_M \langle \nabla \Psi'(u_t), \nabla u_t\rangle d\mu = -\int_M \Psi''(u_t)\,|\nabla u_t|^2\, d\mu \le 0,$$

which implies that $F(t)$ is decreasing.

To prove (B.107) let us observe that for all real a, b

$$\left|\Psi(a) - \Psi(b) - \Psi'(b)(a - b)\right| \le \frac{1}{2}(a - b)^2,$$

because there exists $\xi \in (a, b)$ such that

$$\left|\Psi(a) - \Psi(b) - \Psi'(b)(a - b)\right| = \frac{1}{2}\left|\Psi''(\xi)\right|(a - b)^2 \le \frac{1}{2}(a - b)^2.$$

Therefore, for all $\tau, t > 0$,

$$\int_M \left|\Psi(u_\tau) - \Psi(u_t) - \Psi'(u_t)(u_\tau - u_t)\right| d\mu \le \frac{1}{2}\|u_\tau - u_t\|_{L^2}^2,$$

whence dividing by $\tau - t$ and passing to the limit as $\tau \to t$, we obtain

$$\lim_{\tau \to t} \int_M \left| \frac{\Psi(u_\tau) - \Psi(u_t)}{\tau - t} - \Psi'(u_t)\,\frac{u_\tau - u_t}{\tau - t} \right| d\mu = 0.$$

Finally, since

$$\frac{u_\tau - u_t}{\tau - t} \xrightarrow{L^2} \frac{du_t}{dt}$$

and $\Psi'(u_t) \in L^2$, we obtain

$$\int_M \Psi'(u_t)\frac{u_\tau - u_t}{\tau - t} d\mu \longrightarrow \int_M \Psi'(u_t)\frac{du_t}{dt} d\mu$$

and

$$F'(t) = \lim_{\tau \to t} \int \frac{\Psi(u_\tau) - \Psi(u_t)}{\tau - t} d\mu = \int_M \Psi'(u_t)\frac{du_t}{dt} d\mu.$$

(b) There exists a C^∞-function Ψ on $\mathbb{R}$ satisfying the conditions of the previous claim and in addition $\Psi(s) = 0$ for $s \le 1$ and $\Psi(s) > 0$ for $s > 1$ (see Fig. B.1).

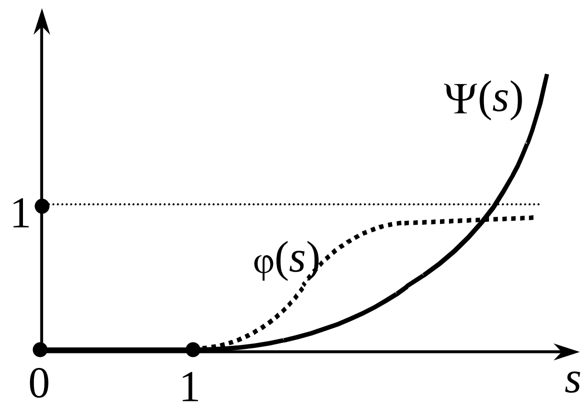

FIGURE B.1. Functions Ψ and $\varphi = \Psi''$

Indeed, one can find Ψ from the equation $\Psi'' = \varphi$ where φ is a smooth function such that $0 \le \varphi \le 1$, $\varphi(s) = 0$ for $s \le 1$ and $\varphi(s) > 0$ for $s > 1$ (think of $\varphi(s)$ as a smooth approximation to a step function $1_{\{s>1\}}$ and of $\Psi(s)$ as a smooth approximation to $\frac{1}{2}(s-1)_+^2$). Hence, the function $F(t)$ defined by (5.39) is decreasing in t. The condition $f \le 1$ implies $F(0) = 0$. Since $F(t)$ is non-negative and decreasing, we conclude that $F(t) = 0$ for all $t > 0$. This implies that $\Psi(P_t f) \equiv 0$ which is only possible if $P_t f \le 1$ almost everywhere.

5.16. Let M be the interval $(0,4)$ and let $u \in C^\infty(M)$ be a function such that

$$\begin{aligned} u(x) &= x + x\sin\frac{1}{x} \text{ for } x \in (0,1), \\ u(x) &\in [0,2x] \text{ for } x \in (1,2) \\ u(x) &= 0 \text{ for } x \in (2,4). \end{aligned}$$

Clearly, $u \geq 0$ and $u \in W^1_{loc}(M)$. Consider also the function

$$v(x) = \begin{cases} 2x, & x \in (0,2], \\ 8 - 2x, & x \in (2,4). \end{cases}$$

Obviously, $u \leq v$ and $v \in W^1_0(M)$, whence

$$u \leq 0 \mod W^1_0.$$

On the other hand, it is easy to verify that

$$\int_0^1 (u')^2 \, dx = \infty$$

so that $u \notin W^1(M)$.

5.17. Set $u = w - v$ so that u satisfies the following conditions:

$$\begin{array}{ll} \frac{du}{dt} = \Delta_\mu u, & \text{for } t \in (0,T), \\ u(t,\cdot) = w(t,\cdot) \mod W^1_0(M) & \text{for } t \in (0,T), \\ u(t,\cdot) \xrightarrow{L^2} 0 & \text{as } t \to 0. \end{array}$$

By Theorem 5.16, $w \geq 0$ implies $u \geq 0$ and, hence $w \geq v$.

5.18. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are trivial.

Proof of $(iii) \Rightarrow (i)$. Assume from the contrary that $v_\alpha(x) \not\rightrightarrows c$ as $x \to \infty$, that is, there is $\varepsilon > 0$ such that, for any compact set $K \subset M$,

$$\sup_{\alpha \in A} \sup_{x \in M \setminus K} |v_\alpha(x) - c| \geq \varepsilon.$$

Take any compact exhaustion sequence $\{\Omega_l\}$ in M and choose a point $x_l \in M \setminus \overline{\Omega_l}$ so that

$$\sup_{\alpha \in A} |v_\alpha(x_l) - c| \geq \varepsilon/2.$$

Then the sequence $\{x_l\}$ leaves any compact in M, but there is no subsequence $\{x_{k_i}\}$ such that $v(x_{k_i}) \rightrightarrows c$.

Proof of $(i) \Leftrightarrow (iv)$. If (i) holds then, using the set K_ε from (5.59) we obtain that $V_\varepsilon \subset K_\varepsilon$ which implies that V_ε is relatively compact. If (iv) holds then setting $K_\varepsilon = \overline{V}_\varepsilon$ we obtain (5.59) although with a non-strict inequality.

The case $c = \pm\infty$ is treated similarly or can be reduced to the case $c = 0$ by switching to the function $\frac{1}{v_\alpha(x)}$.

5.19. Observe that $(v) \Rightarrow (ii)$, because every sequence $\{x_k\}$ as in (v) leaves any compact in Ω.

Let us show that $(iii) \Rightarrow (v)$. Indeed, if $\{x_k\} \subset \Omega$ is a sequence leaving any compact in Ω, then it is either bounded and, hence, contains a convergent subsequence $\{x_{k_i}\}$ whose limit x must be then at the boundary $\partial\Omega$, or $\{x_k\}$ is unbounded and, hence, contains a subsequence $\{x_{k_i}\}$ such that $|x_{k_i}| \to \infty$. In the both cases, we have by (v) that $v_\alpha(x_{k_i}) \rightrightarrows c$.

5.20. The hypothesis $v_+(x) \to 0$ as $x \to \infty$ means that, for any $\varepsilon > 0$, there is a compact set $K \subset M$ such that

$$\sup_{M\setminus K} v \le \varepsilon. \tag{B.108}$$

Let Ω be a relatively compact open subset of M containing K. We claim that the function $u = v - \varepsilon$ satisfies the hypotheses of Theorem 5.13 in Ω. Indeed, obviously $u \in W^1(\Omega)$,

$$-\Delta_\mu u + \alpha u = -\Delta_\mu v + \alpha(v - \varepsilon) \le 0,$$

and

$$u_+ = (v - \varepsilon)_+ \in W_0^1(\Omega),$$

which is equivalent to

$$u \le 0 \mod W_0^1(\Omega).$$

By Theorem 5.13, we obtain $u \le 0$ in Ω, that is, $v \le \varepsilon$ in Ω. Letting $\varepsilon \to 0$ and expanding Ω, we obtain $v \le 0$ in M.

*Second solution.*Let $\varphi(s)$ be a C^∞ function on $\mathbb{R}$ such that $\varphi(s) = 0$ for $s \le \varepsilon$, $\varphi(s) > 0$ for $s > \varepsilon$, and $\varphi'(s) \ge 0$. Then $\varphi(v)$ is a C^2 function on M that vanishes outside K (where K is defined by (B.108)), whence $\varphi(v) \in C_0^2(M)$.

Multiplying the inequality $-\Delta_\mu v + \alpha v \le 0$ by $\varphi(v)$ and integrating over M, we obtain

$$\int_M |\nabla v|^2 \varphi'(v)\, d\mu + \alpha \int_M v\varphi(v)\, d\mu \le 0.$$

Since $\varphi' \ge 0$ and $\alpha > 0$, this implies

$$\int_M v\varphi(v)\, d\mu = 0$$

whence $v\varphi(v) = 0$ and $v \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude $v \le 0$.

5.21. By the hypothesis $u_+(x,t) \rightrightarrows 0$ as $x \to \infty$, for any $\varepsilon > 0$ there is a compact set $K \subset M$ such that

$$\sup_{t\in I} \sup_{x\in M\setminus K} u(t,x) < \varepsilon.$$

Choose a function $\varphi(s) \in C^\infty(-\infty, +\infty)$ such that $\varphi(s) \equiv 0$ for $s \le \varepsilon$, $\varphi(s) \equiv 1$ for $s \ge 2\varepsilon$, and $\varphi'(s) > 0$ for all $s \in (\varepsilon, 2\varepsilon)$. Clearly, for any $t \in I$, the function $\varphi(u(t,\cdot))$ is of the class $C^2(M)$ and is supported in the set K.

Multiplying the inequality $\frac{\partial u}{\partial t} \le \Delta_\mu u$ by $\varphi(u)$, integrating over M and using the Green formula, we obtain, for any $t \in I$,

$$\int_M \varphi(u) \frac{\partial u}{\partial t} d\mu \le \int_M \varphi(u)\, \Delta_\mu u\; d\mu = -\int_M |\nabla u|^2 \varphi'(u)\, d\mu \le 0. \tag{B.109}$$

Next, set

$$\Phi(s) = \int_0^s \varphi(\xi)\, d\xi,$$

and observe $\Phi(s) = 0$ for $s \leq \varepsilon$ so that the function $\Phi(u(t,\cdot))$ is also supported in K for any $t \in I$. Using (B.109), we obtain

$$\frac{d}{dt}\int_M \Phi(u)\,d\mu = \int_M \Phi'(u)\frac{\partial u}{\partial t}d\mu = \int_M \varphi(u)\frac{\partial u}{\partial t}d\mu \leq 0.$$

Hence, the function

$$t \mapsto \int_M \Phi(u(t,\cdot))\,d\mu \tag{B.110}$$

is decreasing in t.

Since $\Phi(s) \leq s_+$ and $u_+(t,\cdot) \overset{L^1(K)}{\longrightarrow} 0$ as $t \to 0$, we obtain

$$\int_M \Phi(u(t,\cdot))\,d\mu = \int_K \Phi(u(t,\cdot))\,d\mu \leq \int_K u_+(t,\cdot)\,d\mu \to 0 \text{ as } t \to 0.$$

Since the function (B.110) is monotone decreasing and non-negative, we conclude that

$$\int_M \Phi(u(t,\cdot))\,d\mu \equiv 0,$$

which implies $u(t,\cdot) \leq \varepsilon$. Letting $\varepsilon \to 0$, we obtain $u(t,\cdot) \leq 0$, which was to be proved.

5.22. Since $u_+ \in C(M) \cap W_0^1(M)$ and $u = u_+$ on U_a, we rename u_+ by u and assume in the sequel that $u \geq 0$.

Assume first in addition that the support of u is compact. Then also $\operatorname{supp}(u-c)_+$ is compact for any $c > 0$. For any $c > a$, we have

$$\operatorname{supp}(u-c)_+ = \{x \in M : u(x) \geq c\} \subset U_a,$$

whence it follows that $(u-c)_+ \in W_c^1(U_a)$ and, by Lemma 5.5,

$$(u-c)_+ \in W_0^1(U_a).$$

By Exercise 5.6, we have

$$(u-c)_+ \overset{W^1}{\longrightarrow} (u-a)_+ \text{ as } c \to a+$$

whence $(u-a)_+ \in W_0^1(U_a)$.

For a general non-negative function $u \in C(M) \cap W_0^1(M)$, there is a sequence $\{\varphi_k\} \in C_0^\infty(M)$ such that $\varphi_k \overset{W^1}{\to} u$. Consider the functions

$$u_k := \min\left((\varphi_k)_+, u\right) = u - \left(u - (\varphi_k)_+\right)_+ \tag{B.111}$$

Clearly, $u_k \in C \cap W_0^1(M)$ and $u_k \overset{W^1}{\to} u$ as $k \to \infty$ (cf. Theorem 5.7 and Example 5.8). Since $\operatorname{supp} u_k \subset \operatorname{supp} \varphi_k$, the argument in the first part of the proof applies to u_k, whence we obtain

$$(u_k - a)_+ \in W_0^1(U_a^{(k)})$$

where

$$U_a^{(k)} := \{x \in M : u_k(x) > a\}.$$

Since $0 \le u_k \le u$, we have $U_a^{(k)} \subset U_a$ and, hence,

$$(u_k - a)_+ \in W_0^1(U_a).$$

Letting $k \to \infty$, we finish the proof.

5.23. It suffices to assume that Ω is relatively compact, because otherwise we can exhaust Ω by a sequence $\{\Omega_k\}$ of relatively compact open subsets and then let $k \to \infty$ (cf. Theorems 5.22 and 5.23).

The inequality (5.72) is trivial in $M \setminus \Omega$ so it suffices to verify it in Ω. Set $C = \operatorname{esup}_{M\setminus K} R_\alpha f$ and consider the function

$$u = R_\alpha f - R_\alpha^\Omega f - C$$

that belongs to $W^1(\Omega)$ and satisfies in Ω the relation

$$-\Delta_\mu u + \alpha u = -\alpha C \le 0.$$

By Theorem 5.7, $u_+ \in W^1(\Omega)$. By the choice of C, we have $u \le 0$ in $\Omega \setminus K$ so that $\operatorname{supp} u_+ \subset K$, which implies by Lemma 5.5 that $u_+ \in W_0^1(\Omega)$. We conclude by Theorem 5.13 that $u \le 0$ in Ω, which is equivalent to (5.72).

5.24. Fix some $T > 0$ and set

$$C = \sup_{s\in[0,T]} \operatorname*{esup}_{M\setminus K} P_s^\Omega f.$$

It suffices to prove that $P_t f - P_t^\Omega f \le C$ in $[0,T] \times \Omega$. Consider the function

$$u(t,\cdot) = P_t f - P_t^\Omega f - C$$

that belongs to $W^1(\Omega)$ for any $t > 0$. Clearly, it satisfies the heat equation in $\mathbb{R}_+ \times \Omega$ in the sense of Theorem 5.13. By the choice of C, we have $u(t,\cdot) \le 0$ in $M \setminus K$ for all $t \in (0,T)$, which implies that $u_+(t,\cdot)$ is supported in Ω and, hence, $u_+(t,\cdot) \in W_0^1(\Omega)$. Finally, we have $u(t,\cdot) \overset{L^2(\Omega)}{\to} -C$ as $t \to 0$, whence it follows that $u_+(t,\cdot) \overset{L^2(\Omega)}{\to} 0$. By Theorem 5.16, we conclude that $u(t,\cdot) \le 0$ in $[0,T] \times \Omega$, which was to be proved.

5.25. Splitting f into the positive and negative parts, we can assume that $f \ge 0$. For all $s > 0$ and $t \ge 0$, we have

$$\left\| P_{t+s}^{\Omega_i} f - P_t^{\Omega_i} f \right\| = \left\| P_t^{\Omega_i}\left(P_s^{\Omega_i} f - f\right) \right\| \le \left\| P_s^{\Omega_i} f - f \right\|.$$

Since $P_s^{\Omega_1} f \le P_s^{\Omega_i} f \le P_t^\Omega f$, it follows that

$$\left\| P_{t+s}^{\Omega_i} f - P_t^{\Omega_i} f \right\| \le \left\| P_s^{\Omega_1} f - f \right\| + \left\| P_s^{\Omega} f - f \right\|,$$

whence

$$\sup_{i\in\mathbb{N}} \left\| P_{t+s}^{\Omega_i} f - P_t^{\Omega_i} f \right\| \le \left\| P_s^{\Omega_1} f - f \right\| + \left\| P_s^{\Omega} f - f \right\| \to 0 \text{ as } s \to 0,$$

which means the right equicontinuity. If $t > s > 0$ then we have

$$\left\| P_{t-s}^{\Omega_i} f - P_t^{\Omega_i} f \right\| = \left\| P_{t-s}^{\Omega_i}\left(f - P_s^{\Omega_i} f\right) \right\| \le \left\| P_s^{\Omega_i} f - f \right\|,$$

which similarly implies the left equicontinuity.

5.26. (*a*) The boundedness of A follows from

$$\|Af\|_{L^2}^2 = \int_M (af)^2\, d\mu \leq \sup |a|^2 \|f\|_{L^2}^2,$$

the non-negative definiteness from

$$(Af, f) = \int_M af^2 d\mu \geq 0,$$

and the self-adjointness from

$$(Af, g) = \int_M afg d\mu = (f, Ag)\,.$$

Since the operator A is bounded, we have

$$e^{-tA} = \sum_{k=0}^{\infty} \frac{(-tA)^k}{k!}$$

whence

$$e^{-tA} f = \sum_{k=0}^{\infty} \frac{(-ta)^k}{k!} f = e^{-ta} f,$$

which obviously implies (5.74).

(*b*) Let us show by induction in n that, for any $s \geq 0$,

$$0 \leq \left(e^{-s\mathcal{L}} e^{-sA}\right)^n f \leq e^{-ns\mathcal{L}} f. \tag{B.112}$$

Set $g = \left(e^{-s\mathcal{L}} e^{-sA}\right) f$. The inductive basis for $n = 1$ means that $0 \leq g \leq e^{-s\mathcal{L}} f$. Indeed, we have

$$g = e^{-s\mathcal{L}} \left(e^{-sA} f\right),$$

which implies by (5.74) and Theorem 5.11

$$0 \leq g \leq e^{-s\mathcal{L}} f.$$

To prove the inductive step from $n - 1$ to n, observe that

$$\left(e^{-s\mathcal{L}} e^{-sA}\right)^n f = \left(e^{-s\mathcal{L}} e^{-sA}\right)^{n-1} g,$$

which implies by the inductive hypothesis that $\left(e^{-s\mathcal{L}} e^{-sA}\right)^n f \geq 0$ and

$$\left(e^{-s\mathcal{L}} e^{-sA}\right)^n f \leq e^{-(n-1)s\mathcal{L}} g \leq e^{-(n-1)s\mathcal{L}} e^{-s\mathcal{L}} f = e^{-ns\mathcal{L}} f.$$

Finally, it follows from (B.112) that

$$0 \leq \left(e^{-\frac{t}{n}\mathcal{L}} e^{-\frac{t}{n}A}\right)^n f \leq e^{-t\mathcal{L}} f,$$

which together with (5.76) yields (5.75).

(*c*) Consider the sequence $\{A_k\}_{k=1}^{\infty}$ of operators in $L^2(M)$ where A_k is the multiplication operator by $k 1_{M \setminus \Omega}$. By part (*b*), we have

$$e^{-t(\mathcal{L}+A_k)} f \leq e^{-t\mathcal{L}} f.$$

This implies the claim because

$$e^{-t\mathcal{L}^{\Omega}} f = \lim_{k\to\infty} e^{-t(\mathcal{L}+A_k)} f.$$

See also [**96**, Theorem 2.1.6].

Solutions to Chapter 6

6.1. Use the same argument as in Example 6.2.

6.2. (*a*) If $f \in C_0^\infty(\Omega)$ then (6.24) is just the standard integration-by-parts formula. Then (6.24) extends by continuity to all $f \in W_0^1(\Omega)$ because the both sides of (6.24) are bounded functional with respect to $\|f\|_{W^1}$.

(*b*) Using (6.24), we obtain

$$\begin{aligned}\left(f, e^{i\xi x}\right)_{W^1} &= \left(f, e^{i\xi x}\right)_{L^2} + \sum\nolimits_j \left(\partial_j f, \partial_j e^{i\xi x}\right)_{L^2} \\ &= \int_\Omega f e^{-i\xi x} dx - \sum\nolimits_j \int_\Omega f\, \partial_j^2 e^{-i\xi x} dx \\ &= \int_{\mathbb{R}^n} f e^{-i\xi x} dx + \sum\nolimits_j \xi_j^2 \int_{\mathbb{R}^n} f e^{-i\xi x} dx,\end{aligned}$$

whence (6.25) follows.

(*c*) The weak convergence in $W^1(\mathbb{R}^n)$ implies that, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$(f_k, \varphi)_{W^1(\mathbb{R}^n)} \to (f, \varphi)_{W^1(\mathbb{R}^n)}. \tag{B.113}$$

Choose a function $\psi \in C_0^\infty(\mathbb{R}^n)$ so that $\psi|_{\overline{\Omega}} \equiv 1$. Then, applying (B.113) to $\varphi(x) = \psi(x) e^{i\xi x}$ and using (6.25), we obtain

$$\begin{aligned}\left(1 + |\xi|^2\right) \widehat{f_k}(\xi) &= \left(f_k, e^{i\xi x}\right)_{W^1(\Omega)} \\ &= \left(f_k, \psi e^{i\xi x}\right)_{W^1(\mathbb{R}^n)} \longrightarrow \left(f, \psi e^{i\xi x}\right)_{W^1(\mathbb{R}^n)} = \left(1 + |\xi|^2\right) \widehat{f}(\xi),\end{aligned}$$

whence the first claim follows.

By the Cauchy-Schwarz inequality,

$$|\widehat{f_k}(\xi)| = \left| \int_\Omega e^{-ix\xi} f_k(x)\, dx \right| \le \mu(\Omega)^{1/2} \|f_k\|_{L^2},$$

whence it follows that the sequence $\{\widehat{f_k}(\xi)\}$ is bounded uniformly for all $\xi \in \mathbb{R}^n$. By the dominated convergence theorem, the uniform boundedness of $\{\widehat{f_k}(\xi)\}$ and the pointwise convergence imply that $\widehat{f_k} \to \widehat{f}$ in $L^2_{loc}(\mathbb{R}^n)$.

(*d*) Since the sequence $\{f_k\}$ is bounded in $W^1(\mathbb{R}^n)$ and $W^1(\mathbb{R}^n)$ is a Hilbert space (see Exercise 2.28), by the weak compactness of balls in Hilbert spaces, there is a subsequence of $\{f_k\}$ that converges weakly in $W^1(\mathbb{R}^n)$. Let now $\{f_k\}$ denote this subsequence, and let $f \in W^1(\mathbb{R}^n)$ be its weak limit in $W^1(\mathbb{R}^n)$. Since $W_0^1(\Omega)$ is a closed subspace of $W^1(\mathbb{R}^n)$, we have $f \in W_0^1(\Omega)$. The proof will be concluded if we show that $f_k \to f$ in $L^2(\mathbb{R}^n)$. Due to the Plancherel identity, it suffices to prove that $\widehat{f_k} \to \widehat{f}$ in $L^2(\mathbb{R}^n)$.

By Exercise 2.34, we have

$$\|f_k\|_{W^1(\mathbb{R}^n)} \simeq \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right) |\widehat{f_k}(\xi)|^2 d\xi.$$

Therefore, there exists a constant C such that, for all k,

$$\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)|\widehat{f}_k(\xi)|^2 d\xi \le C.$$

In particular, for any ball $B_r = \{|\xi| < r\}$, we obtain

$$\int_{B_r^c}|\widehat{f}_k(\xi)|^2 d\xi \le \frac{1}{1+r^2}\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)|\widehat{f}_k(\xi)|^2 d\xi \le \frac{C}{1+r^2},$$

and the same inequality holds for f, too. Hence, for any $r > 0$,

$$\begin{aligned}\|\widehat{f}_k-\widehat{f}\|_{L^2}^2 &= \int_{B_r}|\widehat{f}_k-\widehat{f}|^2 d\xi+\int_{B_r^c}|\widehat{f}_k-\widehat{f}|^2 d\xi \\ &\le \int_{B_r}|\widehat{f}_k-\widehat{f}|^2 d\xi+\frac{C'}{1+r^2}.\end{aligned}$$

Passing to the limit as $k\to\infty$ and using the fact that, by part (c), $\widehat{f}_k\to\widehat{f}$ in $L^2(B_r)$, we obtain

$$\limsup_{k\to\infty}\|\widehat{f}_k-\widehat{f}\|_{L^2}^2 \le \frac{C'}{1+r^2}.$$

Letting $r\to\infty$, we finish the proof.

6.3. The inductive basis for $m=-1$ is covered by Lemma 6.7. Assuming $m \ge 0$, let us prove the inductive step from $m-1$ to m. By the inductive hypothesis, we have

$$\|u\|_{W^{m+1}} \le C\|Lu\|_{W^{m-1}} \le C\|Lu\|_{W^m}. \tag{B.114}$$

We are left to show that any partial derivative $\partial_l u$ admits the estimate

$$\|\partial_l u\|_{W^{m+1}} \le C\|Lu\|_{W^m}. \tag{B.115}$$

By (6.43), we have

$$L(\partial_l u) = \partial_l(Lu) - \partial_i\left[\left(\partial_l a^{ij}\right)\partial_j u\right],$$

whence it follows that

$$\|L(\partial_l u)\|_{W^{m-1}} \le \|Lu\|_{W^m} + C\|u\|_{W^{m+1}}.$$

Combining with (B.114), we obtain

$$\|L(\partial_l u)\|_{W^{m-1}} \le C\|Lu\|_{W^m}. \tag{B.116}$$

Applying the inductive hypothesis to the function $\partial_l u$, we obtain

$$\|\partial_l u\|_{W^{m+1}} \le C\|L(\partial_l u)\|_{W^{m-1}},$$

which together with (B.116) gives (B.115).

6.4. Set

$$L_0 = \partial_i\left(a^{ij}(x)\partial_j\right)$$

so that

$$L_0 u = Lu - b^j\partial_j u - cu. \tag{B.117}$$

Fix an open set $U \Subset \Omega$ and notice that $u \in W^k_{loc}(U)$ for some integer k. Let k be the maximal integer $\leq m+2$ with this property. If $k \leq m+1$ then the hypothesis $Lu \in W^m_{loc}(\Omega)$ implies $Lu \in W^{k-1}_{loc}(U)$. It follows from (B.117) that $L_0 u \in W^{k-1}_{loc}(U)$, whence, by Lemma 6.14 (or by Theorem 6.15), $u \in W^{k+1}_{loc}(U)$. Hence, we conclude that $k = m+2$, which was to be proved.

6.5. (a) The inductive basis for $m=-1$ and $m=0$ is covered by Lemmas 6.16 and 6.17, respectively. Assuming $m \geq 1$, let us prove the inductive step from $m-2$ and $m-1$ to m. By the inductive hypothesis, we have

$$\|u\|_{V^{m+1}} \leq C\|\mathcal{P}u\|_{V^{m-1}} \leq C\|\mathcal{P}u\|_{V^m}. \tag{B.118}$$

We need to show that any partial derivative $\partial_t u$, $\partial_j u$, $\partial_i\partial_j u$ has also the V^m-norm bounded by $C\|\mathcal{P}u\|_{V^m}$.

Applying the inductive hypothesis to $\partial_t u$, we obtain

$$\|\partial_t u\|_{V^m} \leq C\|\mathcal{P}(\partial_t u)\|_{V^{m-2}} = C\|\partial_t \mathcal{P}u\|_{V^{m-2}} \leq C\|\mathcal{P}u\|_{V^m}.$$

It follows from (6.82) that

$$\|\mathcal{P}(\partial_l u)\|_{V^{m-1}} \leq \|\mathcal{P}u\|_{V^m} + C\|u\|_{V^{m+1}}.$$

Combining with (B.118), we obtain

$$\|\mathcal{P}(\partial_l u)\|_{V^{m-1}} \leq C\|\mathcal{P}u\|_{V^m}.$$

Applying the inductive hypothesis to $\partial_l u$, we obtain

$$\|\partial_l u\|_{V^{m+1}} \leq C\|\mathcal{P}(\partial_l u)\|_{V^{m-1}} \leq C'\|\mathcal{P}u\|_{V^m}.$$

Therefore, the V^m-norms of the second derivatives $\partial_i\partial_j u$ are also bounded by $C'\|\mathcal{P}u\|_{V^m}$, which finishes the proof.

(b) Let us first prove a weaker inequality

$$\|u\|_{V^{m+2}(\Omega')} \leq C\left(\|u\|_{V^{m+1}(\Omega)} + \|\mathcal{P}u\|_{W^m(\Omega)}\right). \tag{B.119}$$

Let $\psi \in \mathcal{D}(\Omega)$ be such that $\psi \equiv 1$ on Ω'. Then $\psi u \in \mathcal{D}(\Omega'')$ where Ω'' is a small neighborhood of $\operatorname{supp}\psi$, and by part (a) we have

$$\|u\|_{V^{m+2}(\Omega')} \leq \|\psi u\|_{V^{m+2}(\Omega)} \leq C\|\mathcal{P}(\psi u)\|_{V^m}.$$

Next, by (6.82),

$$\|\mathcal{P}(\psi u)\|_{V^m} \leq C\left(\|u\|_{V^{m+1}} + \|\mathcal{P}u\|_{V^m}\right),$$

which together with the previous line implies (B.119).

Finally, (B.119) implies (6.88) by induction in $m \geq -1$.

6.6. Solution is similar to Exercise 6.4.

Solutions to Chapter 7

7.1. Since the operator $(\mathcal{L}+\mathrm{id})^{-s/2}$ is bounded and, hence, its domain in $L^2(M)$, the range of $(\mathcal{L}+\mathrm{id})^{s/2}$ is L^2. Hence, the mapping

$$f \mapsto (\mathcal{L}+\mathrm{id})^{s/2} f$$

is a bijection between $\mathcal{W}^s$ and L^2; it is obviously linear and norm preserving, which implies that $\mathcal{W}^s$ is isometric to L^2 as a normed linear space, whence it follows that $\mathcal{W}^s$ is Hilbert space.

7.2. By Exercise 4.25, we have $\operatorname{dom}(\mathcal{L}+\mathrm{id})^{1/2} = W_0^1$ and, for any $f \in W_0^1$,

$$\|f\|_{W^1} = \|(\mathcal{L}+\mathrm{id})^{1/2} f\|_{L^2},$$

which implies

$$\|f\|_{W^1} = \|f\|_{\mathcal{W}_0^1}.$$

Hence, the spaces W_0^1 and $\mathcal{W}_0^1$ are identical including the identity of the norms.

Since $W_0^2 = \operatorname{dom}\mathcal{L}$ and $\mathcal{W}_0^2 = \operatorname{dom}(\mathcal{L}+\mathrm{id})$, we obviously have $W_0^2 = \mathcal{W}_0^2$. For the norms, we have

$$\|f\|_{W_0^2}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 + \|\Delta_\mu f\|_{L^2}^2$$

and

$$\begin{aligned}\|f\|_{\mathcal{W}_0^2}^2 &= \|(\mathcal{L}+\mathrm{id}) f\|_{L^2}^2 = (\mathcal{L}f+f, \mathcal{L}f+f)_{L^2} \\ &= (f,f)_{L^2} + 2(\mathcal{L}f,f)_{L^2} + (\mathcal{L}f,\mathcal{L}f)_{L^2} \\ &= \|f\|_{L^2}^2 + 2\|\nabla f\|_{L^2}^2 + \|\Delta_\mu f\|_{L^2}^2.\end{aligned}$$

Obviously, the two norms are equivalent, although not equal.

7.3. If $\{E_\lambda\}$ is the spectral resolution of $\mathcal{L}$ then, for any $\alpha > 0$,

$$\operatorname{dom}\mathcal{L}^\alpha = \left\{ f \in L^2 : \int_0^\infty \lambda^{2\alpha} d\|E_\lambda f\|_{L^2}^2 < \infty \right\}$$

and

$$\operatorname{dom}(\mathcal{L}+\mathrm{id})^\alpha = \left\{ f \in L^2 : \int_0^\infty (1+\lambda)^{2\alpha} d\|E_\lambda f\|_{L^2}^2 < \infty \right\}.$$

Since

$$\int_0^\infty d\|E_\lambda f\|_{L^2}^2 = \|f\|_{L^2} < \infty,$$

it follows that

$$\operatorname{dom}\mathcal{L}^\alpha = \operatorname{dom}(\mathcal{L}+\mathrm{id})^\alpha$$

and that this domain shrinks when α is increasing. Hence, if $f \in \mathcal{W}_0^{2k}$ then $f \in \operatorname{dom}\mathcal{L}^k$ which implies that, for any $l = 0, ..., k-1$, $\mathcal{L}^l f \in \operatorname{dom}\mathcal{L}$ and, hence, $\Delta_\mu^l f \in W_0^1$ and $\Delta_\mu^k f \in L^2$. Conversely, assuming that (7.12) holds, we obtain $\Delta_\mu^l f \in \operatorname{dom}\mathcal{L}$ for any $l = 0, ..., k-1$, which implies by induction in l that $\mathcal{L}^l f \in \operatorname{dom}\mathcal{L}$ and $\mathcal{L}^l f = \Delta_\mu^l f$. Applying this for $l = k-1$ yields $f \in \operatorname{dom}\mathcal{L}^k = \mathcal{W}_0^{2k}$.

7.4. It follows from Exercise 7.3 that $f \in \mathcal{W}_0^{2k}$ implies $\mathcal{L}^l f \in L^2(M)$ for all $l = 0, 1, ..., k$, that is, $f \in \mathcal{W}^{2k}$, which proves that $\mathcal{W}_0^{2k} \subset \mathcal{W}^{2k}$. If $f \in \mathcal{W}_0^{2k}$ then

$$\|f\|^2_{\mathcal{W}_0^{2k}} = \left((\mathcal{L}+\mathrm{id})^k f, (\mathcal{L}+\mathrm{id})^k f\right)_{L^2} = \sum_{i,j=0}^{k} \binom{k}{i}\binom{k}{j} \left(\mathcal{L}^i f, \mathcal{L}^j f\right)_{L^2}.$$

Using the fact that $\mathcal{L}$ is positive definite and symmetric, we obtain

$$0 \le \left(\mathcal{L}^i f, \mathcal{L}^j f\right)_{L^2} \le \|\mathcal{L}^i f\|^2_{L^2} + \|\mathcal{L}^j f\|^2_{L^2},$$

which implies

$$\|f\|^2_{\mathcal{W}_0^{2k}} \simeq \sum_{i=0}^{k} \|\mathcal{L}^i f\|^2_{L^2} = \|f\|^2_{\mathcal{W}^{2k}}.$$

7.5. We have, for any $i = 1, ..., k-1$

$$\|\mathcal{L}^i f\|^2_{L^2} = \left(\mathcal{L}^i f, \mathcal{L}^i f\right)_{L^2} = \left(\mathcal{L}^{i-1} f, \mathcal{L}^{i+1} f\right)_{L^2} \le \|\mathcal{L}^{i-1} f\|_{L^2}\|\mathcal{L}^{i+1} f\|_{L^2}.$$

Hence, the sequence $a_i = \log\|\mathcal{L}^i f\|_{L^2}$ is convex, which implies

$$a_l \le \frac{(k-l)\,a_0 + l a_k}{k}$$

and, hence, (7.13).

7.6. If $f \in C^\infty(M)$ and $\nabla f = 0$ then f is a constant in any connected chart whence it follows by the connectedness of M, that f is a constant in M.

Let us show that the present hypotheses actually imply that $f \in C^\infty(M)$. For that, let us verify that $\Delta_\mu f = 0$ in the distributional sense. Indeed, for any $\varphi \in \mathcal{D}(M)$, we have

$$(\Delta_\mu f, \varphi) = (f, \Delta_\mu \varphi) = -(\nabla f, \nabla \varphi) = 0.$$

The conditions $f \in L^2_{loc}(M)$ and $\Delta_\mu f = 0$ imply by Corollary 7.3 that $f \in C^\infty(M)$.

7.7. Assume that $f := 1_\Omega \in W^1(M)$. We have $\nabla f = 0$ in Ω, but also $\nabla f = 0$ a.e. on $M \setminus \Omega$ by Exercise 5.5. Hence, $\nabla f = 0$ a.e. in M. By Exercise 7.6, we conclude that $f = \mathrm{const}$ on M which contradicts the definition of f. Hence, $f \notin W^1(M)$.

If $f \in W_0^1(\Omega)$ then by a Claim in Section 5.5, $f \in W_0^1(M)$, which is not the case by the above argument. Hence, $f \notin W_0^1(\Omega)$.

7.8. Set $a = \sup_{\partial\Omega} u$ and prove that $u \le a$ in Ω. If $a = +\infty$ then there is nothing to prove. Assuming $a < +\infty$, consider the open set

$$U = \{x \in M : u(x) > a\}.$$

By Exercise 5.22, we have $(u-a)_+ \in W_0^1(U)$. Since U does not intersect $\partial\Omega$, U is a disjoint union of the open sets $V = U \cap \Omega$ and $U \setminus \overline{\Omega}$.

Assume that V is non-empty. Clearly, the function $v = (u-a)|_V$ belongs to $W_0^1(V)$. Since $\Delta_u v = 0$ in V, we obtain that $v \in W_0^2(V)$. By the Green formula (4.12), we obtain

$$\int_V |\nabla v|^2 \, d\mu = -\int_V v \Delta_\mu v \, d\mu = 0,$$

whence $\nabla v = 0$ in V. Extending v by 0 outside V so that $v \in W_0^1(M)$, we obtain $\nabla v = 0$ in $M \setminus V$ (cf. Exercise 5.5). By Exercise 7.6, we conclude that $v = \text{const}$ on M, which contradicts to the fact that $v > 0$ in V and $v = 0$ in $M \setminus \overline{\Omega}$. This contradiction shows that V must be empty, whence $u \leq a$ in Ω.

To prove the second claim, set $K = M \setminus \Omega$. By hypothesis, K is compact. Let K_ε be the closed ε-neighborhood of K. If $\varepsilon > 0$ is small enough then K_ε is compact. Set $\Omega_\varepsilon = M \setminus K_\varepsilon$ and prove that

$$\sup_{\Omega_\varepsilon} u = \sup_{\partial\Omega_\varepsilon} u. \tag{B.120}$$

Let φ be a cutoff function of K in K_ε. Then the function

$$w := (1-\varphi)\, u = u - \varphi u$$

is continuous in $\overline{\Omega}$ and vanishes in K (where $\varphi = 1$), whence $w \in C(M)$. It is clear that also $w \in W_0^1(M)$ and $w = u$ in Ω_ε (where $\varphi = 0$). In particular, w is harmonic in Ω_ε. It follows from the first part that

$$\sup_{\Omega_\varepsilon} w = \sup_{\partial\Omega_\varepsilon} w,$$

which is equivalent to (B.120). Letting $\varepsilon \to 0$ we finish the proof.

7.9. Consider first the case when the closure $\overline{\Omega}''$ is contained in a chart U. Consider the following operator in the chart U

$$L = \rho^{-1} \partial_i \left(\rho g^{ij} \partial_j \right),$$

where $\rho = \frac{d\mu}{d\lambda}$ and λ is the Lebesgue measure in U. As was shown in the proof of Theorem 7.1, if $u \in L^2_{loc}$ and $\Delta_\mu u \in L^2_{loc}$ then $Lu = \Delta_\mu u$ in U. Hence, $Lu \in L^2_{loc}(U)$ and, by Corollary 6.11, we conclude that $u \in W^2_{loc}(U)$. In particular, the partial derivatives $\partial_i u$ exist in $L^2_{loc}(U)$ and satisfy the estimate

$$\int_{\Omega'} \sum_{i=1}^n (\partial_i u)^2 \, d\lambda \leq C \int_{\Omega''} u^2 d\lambda + C \int_{\Omega''} (Lu)^2 \, d\lambda \tag{B.121}$$

(cf. 6.47). Note that the measures λ and μ are comparable in Ω'' so that λ in (B.121) can be replaced by μ. By Exercise 4.11, the distributional gradient ∇u in U as a part of M is given by

$$(\nabla u)^i = g^{ik} \partial_k u,$$

whence

$$|\nabla u|_{\mathbf{g}}^2 = g_{ij} (\nabla u)^i (\nabla u)^j = g^{ij}\partial_i u \partial_j u \leq C \sum_{i=1}^n (\partial_i u)^2 .$$

Combining with (B.121), we obtain

$$\int_{\Omega'} |\nabla u|_{\mathbf{g}}^2 \, d\mu \leq C \int_{\Omega''} u^2 d\mu + C \int_{\Omega''} (\Delta_\mu u)^2 \, d\mu. \tag{B.122}$$

In the general case when $\overline{\Omega}''$ is not contained in a chart, the same estimate follows by covering $\overline{\Omega}''$ by a finite number of charts.

Finally, since $u \in W^1(\Omega')$ for any $\Omega' \Subset M$, it follows that $u \in W^1_{loc}(M)$.

7.10. Let U be a chart on M with the coordinates $x^1, ..., x^n$. Let $d\mu = \rho(x)\, d\lambda$ where λ is the Lebesgue measure, and hence

$$\Delta_\mu = \rho^{-1}\partial_i \left(\rho g^{ij} \partial_j\right).$$

For any $u \in \mathcal{D}'(M)$ and $\varphi \in \mathcal{D}(U)$, we obtain

$$\begin{aligned}(\Delta_\mu u, \varphi) &= (u, \Delta_\mu \varphi) = \left(u, \rho^{-1}\partial_i \left(\rho g^{ij}\partial_j \varphi\right)\right) = \left(\rho^{-1}u, \partial_i \left(\rho g^{ij}\partial_j\varphi\right)\right) \\ &= -\left(\partial_i \left(\rho^{-1}u\right), \rho g^{ij}\partial_j \varphi\right) = \left(\partial_j \left(\rho g^{ij}\partial_i \left(\rho^{-1}u\right)\right), \varphi\right).\end{aligned}$$

Hence, considering u as a distribution in U, we obtain that

$$\Delta_\mu u = Lv$$

where $v = \rho^{-1}u$ and

$$L = \partial_j \left(\rho g^{ij} \partial_i\right).$$

The hypothesis $\Delta_\mu u \in C^\infty(M)$ implies $Lv \in C^\infty(U)$ whence, by Theorem 6.15, $v \in C^\infty(U)$ and $u \in C^\infty(U)$.

7.11. The fact that u_k is harmonic implies that $(\Delta u_k, \varphi) = 0$ for any $\varphi \in C_0^\infty(M)$, whence $(u_k, \Delta_\mu \varphi) = 0$. Since

$$(u_k, \Delta_\mu \varphi) = \int_M u_k \Delta_\mu \varphi d\mu$$

and $u_k \to u$ in L^2_{loc}, we obtain that also $(u, \Delta_\mu \varphi) = 0$. Hence, $\Delta_\mu u = 0$ in the distributional sense and, by Corollary 7.3, we conclude that $u \in C^\infty$ and, hence, u is harmonic (cf. Exercise 7.10).

By Corollary 7.2, in order to prove that

$$u_k \overset{C^\infty}{\longrightarrow} u$$

it suffices to show that

$$u_k \overset{\mathcal{W}^\infty_{loc}}{\longrightarrow} u.$$

The latter means that

$$u_k \overset{L^2_{loc}}{\longrightarrow} u \tag{B.123}$$

and, for all positive integers l,

$$\Delta_\mu^l u_k \overset{L^2_{loc}}{\longrightarrow} \Delta_\mu^l u. \tag{B.124}$$

However, (B.123) is given by hypothesis, and (B.124) is trivial because

$$\Delta_\mu u_k = \Delta_\mu u = 0.$$

7.12. The equation (7.15) means that, for any $\varphi \in \mathcal{D}(M)$,

$$-(u_k, \Delta_\mu \varphi) + \alpha_k (u_k, \varphi) = (f_k, \varphi).$$

Passing to the limit as $k \to \infty$ we obtain (7.16).

To prove the convergence

$$u_k \overset{\mathcal{W}^{2m+2}_{loc}}{\longrightarrow} u,$$

set $v_k = u - u_k$ and observe that

$$-\Delta_\mu v_k + \alpha v_k = h_k,$$

where

$$h_k := f - f_k \overset{\mathcal{W}^{2m}_{loc}}{\longrightarrow} 0. \tag{B.125}$$

The identity

$$\Delta_\mu v_k = \alpha v_k - h_k$$

implies by induction that, for any positive integer l,

$$\Delta_\mu^l v_k = \alpha^l v_k - \alpha^{l-1} h_k - \alpha^{l-2} \Delta_\mu h_k - ... - \Delta_\mu^{l-1} h_k,$$

whence, for any open set $\Omega \Subset M$,

$$\|v_k\|_{\mathcal{W}^{2m+2}(\Omega)} \le C \|h_k\|_{\mathcal{W}^{2m}(\Omega)}$$

where C depends on m and α. Finally, using (B.125), we conclude that

$$v_k \overset{\mathcal{W}^{2m+2}_{loc}}{\longrightarrow} 0,$$

which was to be proved.

If $f_k \overset{C^\infty}{\longrightarrow} f$ then, by Corollary 7.2, $f_k \overset{\mathcal{W}^\infty_{loc}}{\longrightarrow} f$ and, by the previous part of the proof, $u_k \overset{\mathcal{W}^\infty_{loc}}{\longrightarrow} u$. Applying again Corollary 7.2, we obtain $u_k \overset{C^\infty}{\longrightarrow} u$.

7.13. Since $\{u_k(x)\}$ is increasing and converging to u pointwise, it follows that $u_k \overset{L^2_{loc}}{\longrightarrow} u$. By Exercise 7.12, there is a version $\widetilde{u}$ of u that is C^∞ smooth and that $u_k \overset{C^\infty}{\longrightarrow} \widetilde{u}$. In particular, $u_k(x) \to \widetilde{u}(x)$ for any $x \in M$. Since also $u_k(x) \to u(x)$, it follows that $u(x) = \widetilde{u}(x)$ for all $x \in M$, which finishes the proof.

7.14. Since $\Delta_\mu^l u = \alpha^l u$, we have

$$\|u\|^2_{\mathcal{W}^{2k}(\Omega)} = \sum_{l=0}^{k} \|\Delta_\mu^l u\|^2_{L^2(\Omega)} = \left(1 + \alpha^2 + ... + \alpha^{2k}\right) \|u\|^2_{L^2(\Omega)}.$$

Hence, the result follows immediately from estimate (7.2) of Theorem 7.1.

7.15. By Theorem 4.5, for any $f \in L^2(M)$, the function $u := R_\alpha f$ also belongs to $L^2(M)$ (in fact, even to $W_0^2(M)$) and satisfies in M the equation

$$-\Delta_\mu u + \alpha u = f. \tag{B.126}$$

By Corollary 7.3, $f \in C^\infty$ implies $u \in C^\infty$, whence the claim follows.

7.16. Set $u = R_\alpha f$, $u_i = R_\alpha^{\Omega_i} f$ and note u satisfies the equation (B.126) in M, and u_i satisfies the same equation

$$-\Delta_\mu u_i + \alpha u_i = f$$

in Ω_i. Fix an open set $\Omega \Subset M$. For large enough i, Ω_i contains Ω and, by Corollary 7.3, $u_i \in C^\infty(\Omega)$. Obviously, both u_i and u satisfy the same equation (B.126) in Ω. By Theorem 5.22, we have $u_i \overset{L^2(\Omega)}{\longrightarrow} u$, and we conclude by Exercise 7.12 (with $f_i = f$) that $u_i \overset{C^\infty(\Omega)}{\longrightarrow} u$, which finishes the proof.

7.17. Arguing similarly to the proof of Theorem 7.6, we obtain

$$\sup_K \left|\Delta_\mu^m (P_t f)\right| \le C \|\Delta_\mu^m (P_t f)\|_{\mathcal{W}^{2\sigma}(M)},$$

whereas

$$\begin{aligned} \|\Delta_\mu^m (P_t f)\|_{\mathcal{W}^{2\sigma}} &= \sum_{k=0}^{\sigma} \|\Delta_\mu^{m+k} P_t f\|_{L^2} \\ &\le C \left(\sum_{k=0}^{\sigma} \left(\frac{m+k}{t}\right)^{m+k} e^{-(m+k)} \right) \|f\|_{L^2} \\ &\le C' t^{-m} \left(1 + t^{-\sigma}\right) \|f\|_{L^2}. \end{aligned}$$

Combining these two estimates, we obtain (7.46).

7.18. Set $u(t,x) = P_t f(x)$ and $u_i(t,x) = P_t^{\Omega_i} f(x)$. By Theorem 7.10, u is a smooth function in $\mathbb{R}_+ \times M$ and satisfies in $\mathbb{R}_+ \times M$ the heat equation. The same applies to the function u_i in $\mathbb{R}_+ \times \Omega_i$.

It was shown in the proof of Theorem 5.23, that, for any $t > 0$,

$$u_i(t,\cdot) \overset{\text{a.e.}}{\longrightarrow} u(t,\cdot).$$

That is, the set of points (t,x) where $u_i(t,x) \not\to u(t,x)$, has measure 0 on M for every fixed t and, hence, it has measure 0 on $\mathbb{R}_+ \times M$. By Theorem 5.23, we also have $0 \le u_i \le u$. Hence, the sequence $\{u_i\}$ increases and converges a.e. on $\mathbb{R}_+ \times M$ to the function $u \in L^2_{loc}(\mathbb{R}_+ \times M)$, which implies by the dominated convergence theorem that

$$u_i \overset{L^2_{loc}(\mathbb{R}_+ \times M)}{\longrightarrow} u. \tag{B.127}$$

Fix an open set $\Omega \Subset M$. For large enough i, Ω_i contains Ω. Hence, all functions u_i with large enough i satisfy the heat equation in Ω. By Theorem 7.4, (B.127) implies

$$u_i \overset{C^\infty(\mathbb{R}_+ \times \Omega)}{\longrightarrow} u,$$

whence the claim follows.

7.19. By Corollary 7.2, it suffices to prove that

$$P_t f \overset{\mathcal{W}^\infty_{loc}}{\longrightarrow} f \text{ as } t \to 0,$$

which it turn will be the case provided we show that, for any $k \in \mathbb{N}$,

$$\mathcal{L}^k (P_t f) \overset{L^2}{\longrightarrow} \mathcal{L}^k f \ \text{ as } t \to 0. \tag{B.128}$$

Using the spectral resolution of $\mathcal{L}$ and arguing as in the proof of Lemma 4.8, we obtain

$$\left\|\mathcal{L}^k (P_t f) - \mathcal{L}^k f\right\|^2 = \int_0^\infty \left|\lambda^k \left(e^{-t\lambda} - 1\right)\right|^2 d\left\|E_\lambda f\right\|^2.$$

The integrand is bounded by λ^{2k}, which is an integrable function with respect to $d\left\|E_\lambda f\right\|^2$ because $\mathcal{L}^k f \in L^2$.By the dominated convergence theorem, we can pass to the limit as $t \to \infty$ under the sign of the integral and obtain (B.128).

7.20. Observe that $f \in \operatorname{dom}\left(\mathcal{L}^k\right)$ for any $k \in \mathbb{N}$ and use the approach of the third proof of Theorem 7.10 to prove that $u \in C^\infty$. That u satisfies the wave equation follows then from Exercise 4.52.

7.21. By (7.48) and the Cauchy-Schwarz inequality,

$$p_t (x, y) = \left(p_{t/2,x}, p_{t/2,y}\right)_{L^2} \le \|p_{t,x}\|_{L^2} \|p_{t,y}\|_{L^2}.$$

Using

$$p_t (x, x) = \left(p_{t/2,x}, p_{t/2,x}\right)_{L^2} = \|p_{t,x}\|^2_{L^2}$$

and a similar identity for $p_t (y, y)$, we obtain (7.61).

7.22. Since

$$p_t (x, x) = \left(p_{t/2,x}, p_{t/2,x}\right) = \|p_{t/2,x}\|^2_{L^2}, \tag{B.129}$$

it suffices to prove that $\|p_{t,x}\|_{L^2}$ is non-increasing in t. For any $0 < s < t$, we have by (7.56)

$$p_t (x, y) = (p_{t-s,y}, p_{s,x}) = P_{t-s} p_{s,x} (y),$$

for all $y \in M$. Since $p_{t,x} = p_t (x, \cdot)$ a.e., we obtain

$$p_{t,x} = P_{t-s} p_{s,x} \text{ a.e.}$$

Since $\|P_{t-s}\| = \|e^{-(t-s)\mathcal{L}}\| \le 1$, we conclude that

$$\|p_{t,x}\|_{L^2} \le \|p_{s,x}\|_{L^2},$$

which was to be proved.

7.23. (*a*) It follows from (7.61) and (B.129) that

$$S (t) = \sup_{x \in K} p_t (x, x) = \sup_{x \in K} \|p_{t/2,x}\|^2_{L^2}. \tag{B.130}$$

Hence, $S (t)$ is non-increasing by Exercise 7.22.

(*b*) By Theorem 7.7, we have

$$\sup_{x \in K} \|p_{t,x}\|_{L^2} \le C \left(1 + t^{-\sigma}\right),$$

which together with (B.130) settles the claim.

7.24. The pullback operator J_* on functions, defined by

$$J_* f = f \circ J,$$

obviously maps $\mathcal{D}(M)$ onto $\mathcal{D}(M)$ and, by Lemma 3.27, J_* commutes with Δ_μ on $\mathcal{D}(M)$. In the same way, J_* commutes with gradient ∇.

It follows from Lemma 3.27 that, for all $f, g \in L^2(M)$,

$$(J_* f, J_* g)_{L^2} = (f, g)_{L^2}, \tag{B.131}$$

which implies that J_* is an isometry in $L^2(M)$. In the same way, J_* is an isometry in the spaces $W^1(M)$ and $W_0^1(M)$.

Extend J_* to $\mathcal{D}'(M)$ by the identity

$$(J_* u, J_* \varphi) = (u, \varphi),$$

for all $u \in \mathcal{D}'(M)$ and $\varphi \in \mathcal{D}(M)$. Then Δ_μ commutes with J_* in $\mathcal{D}'(M)$, because

$$\begin{aligned}(J_* \Delta_\mu u, J_* \varphi) &= (\Delta_\mu u, \varphi) = (u, \Delta_\mu \varphi) = (J_* u, J_* \Delta_\mu \varphi) \\ &= (J_* u, \Delta_\mu J_* \varphi) = (\Delta_\mu J_* u, J_* \varphi).\end{aligned}$$

Hence, J_* is an isometry of $W_0^2(M)$, and the Dirichlet Laplace operator $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ commutes with J_*.

By the spectral theory, also the heat semigroup operator $P_t = e^{-t\mathcal{L}}$ commutes with J_*, that is, for any $f \in L^2(M)$,

$$P_t(f \circ J) = (P_t f) \circ J.$$

In the terms of the heat kernel, this means that the following identity holds

$$\int_M p_t(x, y) f(Jy) \, d\mu(y) = \int_M p_t(Jx, y) f(y) \, d\mu(y).$$

By (B.131), we have

$$\int_M p_t(Jx, y) f(y) \, d\mu(y) = \int_M p_t(Jx, Jy) f(Jy) \, d\mu(y),$$

and the comparison with the previous line yields $p_t(x, y) = p_t(Jx, Jy)$.

7.25. We have by the Cauchy-Schwarz inequality

$$\begin{aligned}P_t(fg)(x) &= \int_M p_t(x, \cdot) fg d\mu = \int_M \sqrt{p_t(x, \cdot)} f \sqrt{p_t(x, \cdot)} g d\mu \\ &\leq \left(\int_M p_t(x, \cdot) f^2 d\mu\right)^{1/2} \left(\int_M p_t(x, \cdot) g^2 d\mu\right)^{1/2} \\ &= \left(P_t f^2\right)^{1/2} \left(P_t g^2\right)^{1/2},\end{aligned}$$

which proves the first claim. The second claim follows from the first one by setting $g = 1$ and using $P_t 1 \leq 1$.

7.26. Assume that the first alternative fails, that is, there is $\tau > 0$ such that $a := \sup P_\tau 1 < 1$. Then

$$P_{2\tau} 1 = P_\tau (P_\tau 1) \le P_\tau a = a P_\tau 1 \le a^2.$$

By induction, we obtain $P_{n\tau} 1 \le a^n$ for all $n \in \mathbb{N}$. If $t \in (n\tau, (n+1)\tau)$ then

$$P_t 1 = P_{t-n\tau} P_{n\tau} 1 \le P_{t-n\tau} a^n \le a^n \le a^{t/\tau - 1} = a^{-1} \exp\left(-\frac{t}{\tau} \ln \frac{1}{a}\right),$$

so that the second alternative holds.

7.27. Denote for simplicity $q(y) = p_t(x, y)$ and assume that q is unbounded. Consider the following sets

$$\Omega_k = \{x \in M : k < q(x) < k+1\}.$$

Since q is a continuous function, Ω_k is an open set, and $\sup q = \infty$ implies that Ω_k is non-empty for all large enough k. Choose a compact subset $E_k \subset \Omega_k$ of positive measure and consider the function

$$f = \sum_k c_k 1_{E_k},$$

where c_k are positive constant to be specified. For this function, we have

$$\int_M f \, d\mu = \sum_k c_k \mu(E_k)$$

and

$$\int_M f \, q \, d\mu \ge \sum_k k c_k \mu(E_k).$$

Choosing c_k from the condition $c_k \mu(E_k) = 1/k^2$, we obtain that $f \in L^1(M)$ but

$$P_t f(x) = \int_M f \, q \, d\mu = \infty,$$

which contradicts Theorem 7.19.

7.28. (a) Assume first that $f \in \mathcal{F}$ is non-negative. Then, using (7.62), (7.51), and Fubini's theorem for non-negative functions, we obtain

$$\begin{aligned} P_t(P_s f)(x) &= \int_M p_t(x,y) P_s f(y) \, d\mu(y) \\ &= \int_M p_t(x,y) \left(\int_M p_s(y,z) f(z) \, d\mu(z) \right) d\mu(y) \\ &= \int_M \left(\int_M p_t(x,y) p_s(y,z) \, d\mu(y) \right) f(z) \, d\mu(z) \\ &= \int_M p_{t+s}(x,z) f(z) \, d\mu(z) \\ &= P_{t+s} f(x). \end{aligned} \tag{B.132}$$

If f is a signed function from $\mathcal{F}$ then applying the above argument to $|f|$, we obtain that the function

$$(y,z) \mapsto p_t(x,y)\, p_s(y,z)\, f(z)$$

is integrable on $M \times M$. Hence, we can repeat the above computation for f, using Fubini's theorem for integrable functions.

(b) Set $\varepsilon = t - s$. By the semigroup identity, we have

$$P_t f - P_s f = P_s (P_\varepsilon f - f) \quad \text{if } \varepsilon > 0,$$

and

$$P_t f - P_s f = P_t \left(f - P_{|\varepsilon|} f\right) \quad \text{if } \varepsilon < 0.$$

In the both cases, we obtain using $\|P_\cdot f\|_{\mathcal{F}} \le \|f\|_{\mathcal{F}}$,

$$\|P_t f - P_s f\|_{\mathcal{F}} \le \|P_{|\varepsilon|} f - f\|_{\mathcal{F}} \to 0 \text{ as } \varepsilon \to 0,$$

which was to be proved.

7.29. For any relatively compact open set $\Omega \subset M$, we have $f \in W^1(\Omega)$. Consider the constant path $u(t,\cdot) = f$ in $W^1(\Omega)$. Then we obviously have

$$\frac{du}{dt} - \Delta_\mu u \ge 0,$$

which implies by Corollary 5.17 that $u \ge P_t^\Omega f$. Exhausting M by sets like Ω, we conclude by Theorem 5.23 that $u \ge P_t f$, that is, $f \ge P_t f$.

7.30. (a) Applying the operator P_s to the inequality $P_t f \le f$, where the both sides are non-negative, and using (B.132), we obtain

$$P_{t+s} f(x) \le P_s f(x).$$

This means that $P_t f(x)$ is decreasing in t.

(b) The inequality $P_t f \le f$ implies that $P_t f \in L^1_{loc}(\mathbb{R}_+ \times M)$ and, by Theorem 7.15, $P_t f$ is smooth in $\mathbb{R}_+ \times M$ and satisfies the heat equation.

(c) By part (a) and by $P_t f(x) \le f(x)$ we conclude that the limit

$$h(x) := \lim_{t \to 0} P_t f(x) \tag{B.133}$$

exists for all x and $h(x) \le f(x)$. Let us show that $h(x) = f(x)$ μ-a.e. Indeed, $P_t f \le h$ implies that, for all $t, s > 0$,

$$P_{t+s} f = P_s (P_t f) \le P_s h.$$

Letting $t \to 0$, we obtain $P_s f \le P_s h$. On the other hand, $h \le f$ implies $P_s h \le P_s f$, whence it follows that $P_s f \equiv P_s h$.

Hence, the function $v = f - h$ is non-negative and $P_s v \equiv 0$. Let us show that this implies $v = 0$ μ-a.e.. If $v \in L^1(M)$ then this follows from Theorem 7.19. In general, we have $v \in L^1_{loc}(M)$ and, hence, $v \in L^1(\Omega)$ for any relatively compact open set Ω. Then $0 \le P_s^\Omega v \le P_s v$ implies that $P_s^\Omega v \equiv 0$, and by the above argument $v = 0$ μ-a.e.in Ω. Exhausting M by such sets Ω, we prove that $v = 0$ μ-a.e. in M.

Hence, $f = h$ μ-a.e., and it follows from (B.133) that $P_t f(x)$ increases as $t \downarrow 0$ and converges to $f(x)$ μ-a.e.. By the dominated (or monotone) convergence theorem, we conclude that $P_t f \xrightarrow{L^1_{loc}} f$.

(*d*) By parts (*a*) and (*b*), the function $u(t,x) = P_t f(x)$ is a smooth solution to the heat equation in $\mathbb{R}_+ \times M$ and $u(t,x)$ is decreasing in t for any $x \in M$. Therefore,

$$\Delta_\mu u = \frac{\partial u}{\partial t} \le 0,$$

that is, $\Delta_\mu P_t f \le 0$. By part (*c*), we have $P_t f \xrightarrow{\mathcal{D}'} f$, which implies $\Delta_\mu P_t f \xrightarrow{\mathcal{D}'} \Delta_\mu f$ and, hence, $\Delta_\mu f \le 0$.

7.31. Since $0 \le u(t,x) \le f(x)$, the function $u(t,x)$ belongs to $L^1_{loc}(\mathbb{R} \times M)$. Let us show that u satisfies the heat equation in $N = \mathbb{R} \times U$ in the distributional sense. Then by Theorem 7.4, we can conclude that $u \in C^\infty(N)$, which will settle the claim.

It remains to prove that, for any function $\varphi \in \mathcal{D}(N)$,

$$\int_N u\left(\frac{\partial \varphi}{\partial t} + \Delta_\mu \varphi\right) dt d\mu = 0. \tag{B.134}$$

For any $\varepsilon > 0$, set

$$N_\varepsilon = (\varepsilon, +\infty) \times U \quad \text{and} \quad N_{-\varepsilon} = (-\infty, -\varepsilon) \times U.$$

Note that the function u is C^∞ smooth separately in N_ε and $N_{-\varepsilon}$, and satisfies the heat equation in each of these domain (here we use the fact that $\Delta_\mu f = 0$ in U). For simplicity of notation, set $u_t = u(t,\cdot)$ and $\varphi_t = \varphi(t,\cdot)$. Using the integration by parts in t and the Green formula, we obtain

$$\begin{aligned}\int_{N_\varepsilon} u\left(\frac{\partial \varphi}{\partial t} + \Delta_\mu \varphi\right) dt d\mu &= -\int_U u_\varepsilon \varphi_\varepsilon \, d\mu - \int_{N_\varepsilon} \varphi\left(\frac{\partial u}{\partial t} - \Delta_\mu u\right) dt d\mu \\ &= -\int_U u_\varepsilon \varphi_\varepsilon \, d\mu. \end{aligned} \tag{B.135}$$

Similarly,

$$\int_{N_{-\varepsilon}} u\left(\frac{\partial \varphi}{\partial t} + \Delta_\mu \varphi\right) dt d\mu = \int_U u_{-\varepsilon} \varphi_{-\varepsilon} \, d\mu = \int_U f \varphi_{-\varepsilon} d\mu. \tag{B.136}$$

As $\varepsilon \to 0$, we have by Exercise 7.30 that $u(\varepsilon,\cdot) \xrightarrow{L^1_{loc}} f$. Using also that $0 \le u_\varepsilon \le f$, we obtain

$$\begin{aligned}\left|\int_U u_\varepsilon \varphi_\varepsilon \, d\mu - \int_U f \varphi_0 \, d\mu\right| &\le \left|\int_U u_\varepsilon (\varphi_\varepsilon - \varphi_0) \, d\mu\right| + \left|\int_U (u_\varepsilon - f) \varphi_0 \, d\mu\right| \\ &\le \int_U f \left|\varphi_\varepsilon - \varphi_0\right| \, d\mu + \left|\int_U (u_\varepsilon - f) \varphi_0 \, d\mu\right|,\end{aligned}$$

which obviously goes to 0 as $\varepsilon \to 0$. It follows that

$$\int_U u_\varepsilon \varphi_\varepsilon \, d\mu \to \int_U f \varphi_0 \, d\mu \text{ as } \varepsilon \to 0.$$

Adding up (B.135) and (B.136) and letting $\varepsilon \to 0$, we obtain (B.134).

7.32. (a) Applying the operator P_s to the inequality $P_t f \geq f$ and using (B.132), we obtain

$$P_{t+s} f(x) \geq P_s f(x),$$

which implies that $P_t f(x)$ is increasing in t.

(b) The function $P_t f(x)$ is non-negative and measurable on $\mathbb{R}_+ \times M$. For any compact set $K \subset M$ and any interval $[a,b] \subset I := (0,T)$, we have

$$\int_{[a,b]\times K} P_t f(x)\, d\mu(x)\, dt \leq (b-a) \int_K P_b(x)\, d\mu(x) < \infty,$$

whence $P_t f \in L^1_{loc}(I \times M)$. By Theorem 7.15, $P_t f$ is smooth in $I \times M$ and satisfies the heat equation.

(c) By part (a) and by $P_t f(x) \geq f(x)$, the limit

$$h(x) := \lim_{t\to 0} P_t f(x) \tag{B.137}$$

exists for all x and $h(x) \geq f(x)$. Let us show that $h(x) = f(x)$ μ-a.e. Indeed, $P_t f \geq h$ implies that, for all $t, s > 0$,

$$P_{t+s} f = P_s(P_t f) \geq P_s h.$$

Letting $t \to 0$, we obtain $P_s f \geq P_s h$. On the other hand, $h \geq f$ implies $P_s h \geq P_s f$, whence it follows that $P_s f \equiv P_s h$. Arguing as in the solution to Exercise 7.30(c), we conclude that $f = h$ μ-a.e.. It follows from (B.137) that $P_t f(x)$ decreases as $t \downarrow 0$ and converges to $f(x)$ μ-a.e.. By the dominated convergence theorem, we obtain $P_t f \xrightarrow{L^1_{loc}} f$.

(d) By parts (a) and (b), the function $u(t,x) = P_t f(x)$ is a smooth solution to the heat equation in $\mathbb{R}_+ \times M$ and $u(t,x)$ is increasing in $t \in (0,T)$ for any $x \in M$. Therefore,

$$\Delta_\mu u = \frac{\partial u}{\partial t} \geq 0,$$

that is, $\Delta_\mu P_t f \geq 0$. By part (c), we have $P_t f \xrightarrow{\mathcal{D}'} f$, which implies $\Delta_\mu P_t f \xrightarrow{\mathcal{D}'} \Delta_\mu f$ and, hence, $\Delta_\mu f \geq 0$.

(e) The straightforward computation yields, for $t < \alpha$,

$$
\begin{aligned}
P_t f(x) &= \int_{\mathbb{R}^n} p_t(x,y) f(y)\, dy \\
&= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t} + \frac{|y|^2}{4\alpha}\right) dy \\
&= \frac{1}{(4\pi t)^{n/2}} \exp\left(\frac{x^2}{4(\alpha - t)}\right) \int_{\mathbb{R}^n} \exp\left(-\left(\frac{1}{4t} - \frac{1}{4\alpha}\right)\left|y - \frac{x}{1 - t/\alpha}\right|^2\right) dy \\
&= \frac{1}{(4\pi t)^{n/2}} \left(\frac{\pi}{\left(\frac{1}{4t} - \frac{1}{4\alpha}\right)}\right)^{n/2} \exp\left(\frac{x^2}{4(\alpha - t)}\right) \\
&= \frac{1}{1 - t/\alpha} \exp\left(\frac{x^2}{4(\alpha - t)}\right).
\end{aligned}
$$

It is clear that $P_t f(x)$ increasing in $t \in (0, \alpha)$ whence the claim follows. It is also obvious that $P_t f \equiv \infty$ for $t \geq \alpha$.

7.33. The proof follows verbatim the first part of the proof of Theorem 7.16 since the continuity of f in that theorem was used only for the proof of the initial condition.

7.34. Let us prove that $P_t f \to f$ as $t \to 0$ uniformly on any compact set $K \subset M \setminus \partial\Omega$. It suffices to consider separately the following two cases.

Case $K \subset \Omega$. There exists a function $\varphi \in C_0(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on K. Clearly, $\varphi \leq f \leq 1$ on M whence

$$P_t \varphi \leq P_t f \leq 1.$$

By Theorem 7.16, function $P_t \varphi$ converges to φ uniformly on K as $t \to 0$. Since $\varphi \equiv 1 \equiv f$ on K, it follows that $P_t f$ converges to f uniformly on K.

Case $K \subset M \setminus \overline{\Omega}$. There exists a function $\varphi \in C_0\left(M \setminus \overline{\Omega}\right)$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on K. Set $\psi = 1 - \varphi$ so that $\psi = 1$ on Ω and $\psi = 0$ on K. Obviously, we have $0 \leq f \leq \psi$ on M, whence

$$0 \leq P_t f \leq P_t \psi.$$

By Theorem 7.16, $P_t \psi$ converges to ψ uniformly on K as $t \to 0$. Since $\psi \equiv 0 \equiv f$ on K, it follows that $P_t f$ converges to f uniformly on K.

7.35. By Theorem 7.16, the convergence (7.73) holds for any bounded continuous function, in particular, for a constant function. Hence, by adding to f a constant and renormalizing it, we can assume that $0 < f < 1$. Set $a = f(x)$ and let U_ε be an open neighborhood of x where $|f - a| < \varepsilon$, which by hypothesis exists for any $\varepsilon > 0$. Consider the function $\varphi = (a - \varepsilon) 1_{U_\varepsilon}$. By Exercise 7.34 we have

$$P_t \varphi(x) \to \varphi(x) \quad \text{as } t \to 0.$$

Since $f \geq \varphi$ on M and, hence, $P_t f \geq P_t \varphi$, we obtain

$$\liminf_{t\to 0} P_t f(x) \geq \lim_{t\to 0} P_t \varphi(x) = \varphi(x) = f(x) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{t\to 0} P_t f(x) \geq f(x).$$

Applying the same argument to the function $1 - f$, we obtain

$$\liminf_{t\to 0} P_t (1-f)(x) \geq 1 - f(x).$$

Since $P_t 1(x) \to 1$, it follows that

$$\limsup_{t\to 0} P_t f(x) \leq f(x),$$

which finishes the proof.

7.36. (a) This statement is a particular case Exercise 2.20. Nevertheless, let us give an independent proof. The case $r = 1$ is covered by Theorem 7.19, and the case $r = \infty$ is covered by Exercise 7.33 (see also Theorem 7.16). So, assume in the sequel $1 < r < \infty$.

By Theorem 7.15, the function $P_t f(x)$ is measurable. To estimate $\|P_t f\|_{L^r}$, let us first estimate $P_t f(x)$ using the Hölder inequality and (7.50):

$$\begin{aligned} |P_t f(x)| &= \left| \int_M p_t(x,\cdot) f d\mu \right| \leq \int_M p_t^{1-1/r}(x,\cdot) \left(p_t^{1/r}(x,\cdot) |f| \right) d\mu \\ &\leq \left(\int_M p_t(x,\cdot) d\mu \right)^{1-1/r} \left(\int_M p_t(x,\cdot) |f|^r d\mu \right)^{1/r} \\ &\leq \left(\int_M p_t(x,\cdot) |f|^r d\mu \right)^{1/r}. \end{aligned} \tag{B.138}$$

Next, applying Fubini's theorem and (7.50), we obtain

$$\begin{aligned} \|P_t f\|_{L^r}^r &= \int_M |P_t f(x)|^r d\mu(x) \\ &\leq \int_M \left(\int_M p_t(x,y) |f|^r(y) d\mu(y) \right) d\mu(x) \\ &= \int_M \left(\int_M p_t(x,y) d\mu(x) \right) |f|^r(y) d\mu(y) \\ &\leq \int_M |f|^r(y) d\mu(y), \end{aligned}$$

whence (7.74) follows.

(b) Integrating (7.74) in dt, we obtain $P_t f \in L^r_{loc}(\mathbb{R}_+ \times M)$, whence the claim follows from Theorem 7.15.

7.37. Prove first the L^r analogue of Lemma 7.18, and then use the same argument as in the proof of Theorem 7.19. The only place that requires an explanation is why

$$P_t^{\Omega_k} f_k \overset{L^2(\Omega_k)}{\longrightarrow} f_k \text{ as } t \to 0$$

implies

$$P_t^{\Omega_k} f_k \overset{L^r(\Omega_k)}{\longrightarrow} f_k \text{ as } t \to 0, \tag{B.139}$$

using the notation of the proof of Theorem 7.19. If $r \le 2$ then this is true by the embedding $L^2(\Omega_k) \hookrightarrow L^r(\Omega_k)$. If $r > 2$ then we use the obvious interpolation inequality

$$\|h\|_r^r \le \|h\|_\infty^{r-2} \|h\|_2^2,$$

which is true for any measurable function h. Indeed, since function f_k is bounded by k, function $P_t^{\Omega_k} f_k$ is also bounded by k. Therefore, we obtain

$$\|P_t^{\Omega_k} f_k - f_k\|_r^r \le (2k)^{r-2} \|P_t^{\Omega_k} f_k - f_k\|_2^2,$$

whence (B.139) follows.

7.38. This is a particular case of Exercise 2.21 with $K = 1$ and $C = F(t)$.

7.39. (a) For any $t > 0$, the function $p_t(x,y) f(y)$ is measurable in x, y. Hence, the measurability of

$$P_t f(x) = \int_M p_t(x,y) f(y) \, d\mu(y) \tag{B.140}$$

follows from Fubini's theorem.

(b) If f is signed then the convergence of the integral (B.140) means that $P_t f_+$ and $P_t f_-$ are finite almost everywhere. It follows from $P_t f = P_t f_+ - P_t f_-$ and part (a) that $P_t f$ is measurable.

(c) See Exercise 7.28(a).

7.40. (a) By Theorem 5.23, we have $P_t^\Omega f \le P_t f$ for any non-negative $f \in L^2(\Omega)$. In terms of the heat kernels this means that, for all $x \in \Omega$ and $t > 0$,

$$\int_\Omega p_t^\Omega(x,y) f(y) \, d\mu(y) \le \int_\Omega p_t(x,y) f(y) \, d\mu(y),$$

whence $p_t^\Omega(x,y) \le p_t(x,y)$ follows.

(b) For simplicity of notation, define $p_t^\Omega(x,y)$ for all $x, y \in M$ by setting $p_t(x,y) = 0$ if x or y is outside Ω.

By part (a), the sequence $\left\{p_t^{\Omega_i}(x,y)\right\}$ is increasing for all $t > 0$ and $x, y \in M$ and, hence, has a pointwise limit

$$q_t(x,y) := \lim_{i\to\infty} p_t^{\Omega_i}(x,y) \le p_t(x,y). \tag{B.141}$$

The function $p_t(x,y)$ is smooth in t, x, y and, hence,

$$p_t(x,y) \in L^2_{loc}(\mathbb{R}_+ \times M \times M).$$

By the dominated convergence theorem, we obtain from (B.141)

$$p_t^{\Omega_i}(x,y) \overset{L^2_{loc}(\mathbb{R}_+\times M\times M)}{\longrightarrow} q_t(x,y). \tag{B.142}$$

By Theorem 5.23, for any non-negative $f \in L^2(M)$, we have

$$P_t^{\Omega_i} f(x) \to P_t f(x), \tag{B.143}$$

for all $t > 0$ and for almost all[1] $x \in M$, that is,

$$\int_{\Omega_i} p_t^{\Omega_i}(x,y) f(y)\, d\mu(y) \to \int_M p_t(x,y) f(y)\, d\mu(y).$$

On the other hand, by (B.141) and the monotone convergence theorem, we have also

$$\int_{\Omega_i} p_t^{\Omega_i}(x,y) f(y)\, d\mu(y) \to \int_M q_t(x,y) f(y)\, d\mu(y),$$

for all $t > 0$ and $x \in M$, whence it follows that

$$q_t(x,y) = p_t(x,y),$$

for all $t > 0$ and almost all $x, y \in M$.

It follows from (B.142) that

$$p_t^{\Omega_i}(x,y) \overset{L^2_{loc}(\mathbb{R}_+ \times M \times M)}{\longrightarrow} p_t(x,y). \tag{B.144}$$

Fix an open set $\Omega \Subset M$. For large enough i, Ω_i contains Ω and, hence, both functions $p_t^{\Omega_i}(x,y)$ and $p_t(x,y)$ satisfy in $\mathbb{R}_+ \times \Omega \times \Omega$ the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}(\Delta_x + \Delta_y) u,$$

where $\Delta_x + \Delta_y$ is the Laplace operator on the manifold $M \times M$ (see the proof of Theorem 7.20). By Theorem 7.4, applied to the manifold $\Omega \times \Omega$, the convergence (B.144) implies that

$$p_t^{\Omega_i}(x,y) \overset{C^\infty(\mathbb{R}_+ \times \Omega \times \Omega)}{\longrightarrow} p_t(x,y),$$

which was to be proved.

(c) The claim follows from parts (a), (b) and from the monotone convergence theorem:

$$P_t^{\Omega_i} f(x) = \int_M p_t^{\Omega_i}(x,\cdot) f d\mu \to \int_M p_t(x,\cdot) f d\mu = P_t f(x). \tag{B.145}$$

(d) Splitting $f = f_+ - f_-$, it suffices to consider the case $f \geq 0$. By part (c), we have, for any $t > 0$,

$$P_t^{\Omega_i} f(x) \uparrow P_t f(x) \tag{B.146}$$

pointwise in x. Since $P_t f(x)$ is a locally bounded function on $\mathbb{R}_+ \times M$, (B.146) implies by the dominated convergence theorem that the convergence in (B.146) is also in $L^2_{loc}(\mathbb{R}_+ \times M)$. By Theorem 7.16, both functions $P_t^{\Omega_i} f(x)$ and $P_t f(x)$ solve the heat equation. Hence, we conclude by Theorem 7.4 that the convergence in (B.146) is also in $C^\infty(\mathbb{R}_+ \times M)$.

[1] In fact, by Exercise 7.18, the convergence (B.143) is in $C^\infty(\mathbb{R}_+ \times M)$ but we will not use this.

7.41. Let P_t be the heat semigroup on M. Consider the integral operator Q_t on functions on M defined by

$$Q_tf(x,y)=\int_M p_t^X(x,x')\,p_t^Y(y,y')\,f(x',y')\,d(\mu\times\mu)(x',y').$$

We will show that $Q_tf=P_tf$ for all $f\in L^2(M)$, which will imply (7.92). We will use the fact that, for any $f\in L^2(M)$, the path $t\mapsto P_tf$ is a unique solution to the L^2-Cauchy problem on M with the initial function f (see Corollary 4.11).

Let function f be of the form

$$f(x,y)=g(x)\,h(y), \tag{B.147}$$

where $g\in L^2(X)$ and $h\in L^2(Y)$. Then we obviously have

$$Q_tf(x,y)=P_t^Xg(x)\,P_t^Yh(y),$$

where P_t^X and P_t^Y are the heat semigroups on X and Y, respectively. The paths $t\mapsto P_t^Xg$ and $t\mapsto P_t^Yh$ solve the L^2-Cauchy problems on X and Y, respectively, with the initial functions g and h.

Let us show that $u(t,\cdot)=Q_tf$ solves the L^2-Cauchy problem on M with the initial function f. Since $P_t^Xg\in W_0^1(X)$ and $P_t^Yh\in W_0^1(Y)$, we easily obtain that $u(t,\cdot)\in W_0^1(M)$.

Next, we have

$$\begin{aligned}\frac{du}{dt}&=\frac{d}{dt}\left(P_t^Xg\right)\,P_t^Yh+P_t^Xg\,\frac{d}{dt}\left(P_t^Yh\right)\\&=\Delta_X\left(P_t^Xg\right)\,P_t^Yh+P_t^Xg\,\Delta_Y\left(P_t^Yh\right),\end{aligned}\tag{B.148}$$

where Δ_X and Δ_Y are (distributional) Laplace operators on X and Y. On smooth functions, we have

$$\Delta_X+\Delta_Y=\Delta_\mu$$

(cf. Section 3.8), whence it follows that the same identity holds on distributions. Since the right hand side in (B.148) belongs to $L^2(M)$, we conclude that the strong derivative $\frac{du}{dt}$ exists in $L^2(M)$ and is equal to

$$\frac{du}{dt}=\Delta_Xu+\Delta_Yu=\Delta_\mu u.$$

In particular, $\Delta_\mu u\in L^2(M)$ and, hence, $u(t,\cdot)\in W_0^2(M)$. Finally, $u(t,\cdot)\overset{L^2(M)}{\longrightarrow}f$ as $t\to0$ because

$$P_t^Xg\overset{L^2(X)}{\longrightarrow}g\ \text{ and }\ P_t^Yh\overset{L^2(Y)}{\longrightarrow}h.$$

By Corollary 4.11, we conclude that $u=P_tf$, that is,

$$Q_tf=P_tf, \tag{B.149}$$

for all f of the form (B.147). Since Q_t and P_t are linear bounded operator in $L^2(M)$ (cf. Exercise 2.20) and the functions of the form (B.147) span all $L^2(M)$, we conclude that (B.149) holds for all $f \in L^2(M)$, whence the claim follows.

7.42. It follows from (7.93) that

$$\begin{aligned} f(x) - P_t f(x) &= \int_M f(x)\, p_t(x,y)\, d\mu(y) - \int_M f(y)\, p_t(x,y)\, d\mu(y) \\ &= \int_M (f(x) - f(y))\, p_t(x,y)\, d\mu(y), \end{aligned}$$

whence

$$(f - P_t f, f) = \int_M \int_M (f(x) - f(y)) f(x)\, p_t(x,y) d\mu(y) d\mu(x).$$

Switching x and y in the integral and using the symmetry of the heat kernel, we obtain

$$(f - P_t f, f) = \int_M \int_M (f(y) - f(y)) f(y)\, p_t(x,y) d\mu(x) d\mu(y).$$

Adding up the above two lines, we obtain

$$(f - P_t f, f) = \frac{1}{2} \int_M \int_M (f(x) - f(y))^2 p_t(x,y) d\mu(y) d\mu(x),$$

whence (7.94) follows.

7.43. Note that the operators $\mathcal{L}$ and $R := (\mathcal{L} + \mathrm{id})^{-1}$ are self-adjoint with the spectra in $[0, +\infty)$ so that $\mathcal{L}^k$ and R^k are defined by the functional calculus for all real $k > 0$. Since R is bounded, R^k is also bounded and, hence, its domain is L^2.

Using Exercise 5.11 and (7.49), we obtain, for all $f, g \in L^2$,

$$\begin{aligned} \left(R^k f, g\right) &= \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-t} (P_t f, g)\, dt \\ &= \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-t} \left(\int_M P_t f(x)\, g(x)\, d\mu(x) \right) dt \\ &= \int_M \left(\int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-t} P_t f(x)\, dt \right) g(x)\, d\mu(x), \end{aligned}$$

whence, it follows

$$R^k f(x) = \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-t} P_t f(x)\, dt, \tag{B.150}$$

for almost all $x \in M$.

7.44. (*a*) Using the Cauchy-Schwarz inequality, (7.51), and (7.96), we obtain, for all $x \in M$ and $t > 0$,

$$\begin{aligned} |P_t f(x)| &= \left| \int_M p_t(x,y) f(y) \, d\mu(y) \right| \\ &\le \left(\int_M p_t^2(x,y) \, d\mu(y) \right)^{1/2} \|f\|_2 \\ &= p_{2t}(x,x)^{1/2} \|f\|_2. \end{aligned}$$

By hypothesis, $p_t(x,x) \le ct^{-\gamma}$ for $t < 1$. By Exercise 7.22, $p_{2t}(x,x)$ is a non-increasing function of t and, hence, $p_t(x,x) \le c$ for $t \ge 1$. Combining these estimates together, we obtain that, for all $x \in M$ and $t > 0$,

$$|P_t f(x)| \le c \left(1 + t^{-\gamma/2}\right) \|f\|_2. \tag{B.151}$$

Therefore, (B.150) yields, for almost all $x \in M$,

$$\left| R^k f(x) \right| \le c \|f\|_2 \int_0^\infty \left(1 + t^{-\gamma/2}\right) \frac{t^{k-1}}{\Gamma(k)} e^{-t} dt.$$

We are left to observe that if $k > \gamma/2$ then this integral converges, which implies

$$\left| R^k f(x) \right| \le C \|f\|_2, \tag{B.152}$$

for almost all x.

Let us prove that the function $R^k f(x)$ has a continuous version. In fact, the latter is given by the right hand side of (B.150). Denoting it by $\widetilde{R^k f}$, we have, for all $x, y \in M$,

$$\widetilde{R^k f}(y) - \widetilde{R^k f}(x) = \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-t} \left(P_t f(y) - P_t f(x)\right) dt.$$

Since the function $x \mapsto P_t f(x)$ is continuous, $P_t f(y) - P_t f(x) \to 0$ as $y \to x$. By (B.151), the function under integration in the previous line is uniformly bounded by an integrable function, and the dominated convergence theorem implies that the integral converges to 0 as $y \to x$, which finishes the proof.

(*b*) Let $\{E_\lambda\}$ be the spectral resolution of $\mathcal{L}$. Then

$$\operatorname{dom} \mathcal{L}^k = \left\{ u \in L^2 : \int_0^\infty \lambda^{2k} d\|E_\lambda u\|_2^2 \right\}$$

and

$$\operatorname{dom} (\mathcal{L} + \mathrm{id})^k = \left\{ u \in L^2 : \int_0^\infty (1+\lambda)^{2k} \, d\|E_\lambda u\|_2^2 \right\},$$

whence it follows that these two domains are identical. Applying part (*b*) to $f = (\mathcal{L} + \mathrm{id})^k u$, we obtain that $u = R^k f$ and, hence, u is continuous and

$$\sup_M |u| \le C \| (\mathcal{L} + \mathrm{id})^k u \|_2. \tag{B.153}$$

Finally, the inequality

$$(1+\lambda)^{2k} \le 2^{2k}\left(1+\lambda^{2k}\right)$$

implies that

$$\|\left(\mathcal{L}+\mathrm{id}\right)^k u\|_2 \le 2^k\left(\|u\|_2+\|\mathcal{L}^k u\|_2\right),$$

which leads to

$$\sup|u| \le C\left(\|u\|_2+\|\mathcal{L}^k u\|_2\right).$$

7.45. Fix $f\in L^2$, $t>0$ and apply (7.98) to the function $u=P_t f$. Using also

$$\|u\|_2=\|P_t f\|_2\le\|f\|_2$$

and

$$\|\mathcal{L}^k u\|_2^2=\int_0^\infty \lambda^{2k}e^{-\lambda t}d\|E_\lambda f\|_2^2\le \sup_{\lambda\in(0,+\infty)}\left(\lambda^{2k}e^{-\lambda t}\right)\|f\|_2^2=\frac{c_k^2}{t^{2k}}\|f\|_2^2,$$

where $c_k=(2k/e)^k$, we obtain

$$\sup_M|P_t f|\le C\left(\|u\|_2+\|\mathcal{L}^k u\|_2\right)\le C\left(1+\frac{c_k}{t^k}\right)\|f\|_2.$$

Arguing further as in the proof of Theorem 7.7, we obtain

$$\sup_{x\in M}\|p_{t,x}\|_2\le C\left(1+\frac{c_k}{t^k}\right)$$

whence

$$p_t(x,x)=\|p_{t/2,x}\|_2^2\le C\left(1+\frac{c_k}{t^k}\right)^2,$$

which finishes the proof.

7.46. (a) The identity $W^1(\mathbb{R}^n)=W_0^1(\mathbb{R}^n)$ (cf. Exercise 2.30) implies that $\operatorname{dom}\mathcal{L}=W^2(\mathbb{R}^n)$. Let us show that $u\in W^k(\mathbb{R}^n)$ implies $u\in\operatorname{dom}\mathcal{L}^{k/2}$. If k is even then this easily follows by induction because $u\in W^k$ implies $\mathcal{L}u=-\Delta u\in W^{k-2}$. Also, expanding $\left(\mathcal{L}+\mathrm{id}\right)^{k/2}u$ by the binomial formula, we obtain that $\left(\mathcal{L}+\mathrm{id}\right)^{k/2}u$ is a combination of the (weak) derivatives of u up to the order k, which yields

$$\|\left(\mathcal{L}+\mathrm{id}\right)^{k/2}u\|_{L^2}\le C\|u\|_{W^k}.$$

If k is odd, then write $k=l+1$ and notice that $u\in W^k$ implies $\mathcal{L}^{l/2}u\in W^1=W_0^1$. Since by Exercise 4.25 $W_0^1=\operatorname{dom}\mathcal{L}^{1/2}$, we obtain that $\mathcal{L}^{l/2}u\in\operatorname{dom}\mathcal{L}^{1/2}$, whence it follows that $u\in\operatorname{dom}\mathcal{L}^{k/2}$. Since for any $f\in W_0^1$,

$$\|\left(\mathcal{L}+\mathrm{id}\right)^{1/2}f\|_{L^2}=\|f\|_{W^1},$$

(see Exercise 4.25), we also obtain

$$\begin{aligned}\|\left(\mathcal{L}+\mathrm{id}\right)^{k/2}u\|_{L^2} &= \|\left(\mathcal{L}+\mathrm{id}\right)^{1/2}\left(\mathcal{L}+\mathrm{id}\right)^{l/2}u\|_{L^2}\\ &= \|\left(\mathcal{L}+\mathrm{id}\right)^{l/2}u\|_{W^1}\le C\|u\|_{W^{l+1}}=C\|u\|_{W^k}.\end{aligned}$$

(*b*) Since the heat kernel in $\mathbb{R}^n$ is given by

$$p_t(x,x) = (4\pi t)^{-n/2}$$

and, hence, satisfies the hypothesis of Exercise 7.44 with $\gamma = n/2$, we obtain that if $k > n/2$ then every function $u \in \operatorname{dom} \mathcal{L}^{k/2}$ is continuous and satisfies the estimate

$$\sup_{\mathbb{R}^n} |u| \le C \| (\mathcal{L} + \mathrm{id})^{k/2} u \|_{L^2}$$

(cf. (B.153)). By part (*a*) we conclude that every function $u \in W^k$ with $k > n/2$ is continuous and

$$\sup_{\mathbb{R}^n} |u| \le C \|u\|_{W^k}.$$

(*c*) If α is a multiindex such that $|\alpha| \le m$ then $u \in W^k(\mathbb{R}^n)$ implies $\partial^\alpha u \in W^{k-m}$ and, by part (*b*), $\partial^\alpha u$ is continuous and

$$\sup_{\mathbb{R}^n} |\partial^\alpha u| \le C \|\partial^\alpha u\|_{W^{k-m}} \le C \|u\|_{W^k}.$$

Therefore, $u \in C^m(\mathbb{R}^n)$ and

$$\|u\|_{C^m} \le C \|u\|_{W^k}.$$

(*d*) For any $\psi \in C_0^\infty(\Omega)$, we have $\psi u \in W^k(\mathbb{R}^n)$ and, by part (*c*) of the proof, $\psi u \in C^m(\mathbb{R}^n)$ and

$$\|\psi u\|_{C^m(\mathbb{R}^n)} \le C \|\psi u\|_{W^k(\mathbb{R}^n)}.$$

If Ω' and Ω'' are two open sets such that $\Omega' \Subset \Omega'' \Subset \Omega$ then ψ can be chosen so that $\psi \equiv 1$ on Ω' and $\operatorname{supp} \psi \subset \Omega''$. Since $\psi u = u$ on Ω', it follows from the above that $u \in C^m(\Omega')$ and

$$\|u\|_{C^m(\Omega')} \le C' \|u\|_{W^k(\Omega'')},$$

which finishes the proof.

7.47. (*a*) We need to prove that any bounded sequence $\{f_k\}$ in W_0^1 has a convergent subsequence in L^2. Since $\{f_k\}$ is bounded in L^2, there exists a subsequence, denoted again by $\{f_k\}$, which converges weakly in L^2 to a function $f \in L^2$. Let us show that, in fact, $\{f_k\}$ converges to f in L^2-norm.

For any $t > 0$, we have by the triangle inequality

$$\|f_k - f\|_2 \le \|f_k - P_t f_k\|_2 + \|P_t f_k - P_t f\|_2 + \|P_t f - f\|_2. \tag{B.154}$$

Since $f_k \in W_0^1$, we have, by the inequality (4.69) of Exercise 4.40,

$$\|f_k - P_t f_k\|_2 \le \sqrt{t}\, \|\nabla f_k\|_2.$$

By the hypothesis, the norms $\|\nabla f_k\|_2$ are uniformly bounded so that we can write

$$\|f_k - P_t f_k\|_2 \le C\sqrt{t}, \tag{B.155}$$

for all $t > 0$ and k. Since $\{f_k\}$ converges to f weakly in L^2, we obtain that, for almost all $x \in M$,

$$P_t f_k(x) = (p_{t,x}, f_k) \to (p_{t,x}, f) = P_t f(x) \text{ as } k \to \infty.$$

On the other hand, we have by the inequality (7.75) of Exercise 7.36

$$\|P_t f_k\|_\infty \le \left(\sup_{x\in M} p_t(x,x)\right)^{1/2} \|f_k\|_2.$$

Applying the hypothesis (7.99) and using the fact that all the norms $\|f_k\|_2$ are uniformly bounded, we obtain, that

$$\|P_t f_k\|_\infty \le S(t),$$

where $S(t)$ is a finite function of t. Hence, for any fixed $t>0$, the sequence $\{P_t f_k\}$ is bounded and converges to $P_t f$ almost everywhere. Since $\mu(M) < \infty$, the dominated convergence theorem yields

$$\|P_t f_k - P_t f\|_2 \to 0 \ \text{ as } k\to\infty. \tag{B.156}$$

Hence, we obtain from (B.154), (B.155), and (B.156) that, for any $t>0$,

$$\limsup_{k\to\infty} \|f_k - f\|_2 \le C\sqrt{t} + \|P_t f - f\|_2.$$

Since by Theorem 4.9 $\|P_t f - f\|_2 \to 0$ as $t\to 0$, we finish the proof by letting $t\to 0$.

(*b*) Let us apply part (*a*) to the weighted manifold (Ω,μ). Since Ω is relatively compact subset of M, we have $\mu(\Omega) < \infty$. We are left to verify the condition

$$\sup_{x\in\Omega} p_t^\Omega(x,x) < \infty,$$

where p_t^Ω is the heat kernel of Ω. It follows from Theorem 5.23 that $p_t^\Omega \le p_t$. By Theorem 7.7, there exist a finite function $F_\Omega(t)$ such that

$$\sup_{x\in\Omega} \|p_{t,x}\|_2 \le F_\Omega(t).$$

Recalling that $p_t(x,x) = \|p_{t/2,x}\|_2^2$, we compete the proof.

7.48. It suffices to prove that, for any $t\in I$ and any sequence of reals $\varepsilon_k \to 0$,

$$\|h(t+\varepsilon_k) - h(t)\| \to 0 \text{ as } k\to\infty. \tag{B.157}$$

Since the sequence of vectors

$$\left\{\frac{h(t+\varepsilon_k)-h(t)}{\varepsilon_k}\right\}_{k=1}^\infty$$

is weakly convergent, it is weakly bounded and, hence, strongly bounded, by the principle of uniform boundedness. Therefore, there is a constant C such that

$$\left\|\frac{h(t+\varepsilon_k)-h(t)}{\varepsilon_k}\right\| \le C$$

for all k, whence (B.157) follows.

Solutions to Chapter 8

8.1. Observe that the function $v(t,x) = e^{\alpha t} h(x)$ solves the heat equation because by hypothesis $\Delta_\mu h = \alpha h$ and

$$\frac{\partial v}{\partial t} = \alpha v = \Delta_\mu v.$$

Hence, $v(t,x)$ is a non-negative solution to the heat equation with the initial function h. By Theorem 8.1, we conclude

$$v(t,x) \geq P_t h(x),$$

which was to be proved.

8.2. By Theorem 8.4, we have $0 \leq R_\alpha f \leq u$ and $R_\alpha f$ solves (8.11). Setting $v = u - R_\alpha f$, we obtain that $v \in L^2_{loc}(M)$, v solves the equation $-\Delta_\mu v + \alpha v = 0$, and $v(x) \to 0$ as $x \to \infty$. By Corollary 7.3, $v \in C^\infty(M)$, and by Exercise 5.20 we conclude that $v = 0$.

Note that if $f \in L^2(M)$ and $u \in W^1(M)$ then one can use Corollary 5.15 instead of Theorem 8.4.

8.3. By Corollary 7.3, $u \in C^\infty(M)$. Consider the open set

$$\Omega = \{x \in M : u(x) > 0\}$$

and notice that

$$u(x) \to 0 \text{ as } x \to \infty \text{ in } \Omega. \tag{B.158}$$

Indeed, $x \to \infty$ in Ω means a sequence $\{x_k\}$ such that either $x_k \to \infty$ in M or $x_k \to \partial\Omega$.

Function u_+ satisfies in Ω the equation $\Delta_\mu u_+ + \lambda u_+ = 0$. Choose some $\alpha > |\lambda|$ and rewrite this equation in the form

$$-\Delta_\mu u_+ + \alpha u_+ = f$$

where $f = (\alpha + \lambda) u_+$. Using (B.158), $f \in L^2(\Omega)$ and $f \geq 0$, we obtain by Exercise 8.2 that $u_+ = R^\Omega_\alpha f$. It follows that $u_+ \in \operatorname{dom}(\mathcal{L}^\Omega)$ and, hence, $u_+ \in W^1_0(\Omega)$. Since $u_+ \equiv 0$ outside Ω, it follows that $u_+ \in W^1_0(M)$. In the same way, $u_- \in W^1_0(M)$, which implies $u \in W^1_0(M)$.

8.4. Left to the reader

8.5. Let $\{\Omega_k\}$ be a compact exhaustion sequence in M. By Corollary 8.16, we have

$$\inf_{\Omega_k} u = \inf_{\partial\Omega_k} u.$$

Let x_k be a point on $\partial\Omega_k$ such that

$$\inf_{\Omega_k} u \geq u(x_k) - \frac{1}{k}.$$

Since $x_k \to \infty$ as $k \to \infty$, we obtain by hypothesis that

$$\limsup_{k\to\infty} u(x_k) \geq 0.$$

Passing to a subsequence, we can assume that in fact

$$\liminf_{k\to\infty} u(x_k) \geq 0,$$

whence

$$\inf_M u = \lim_{k\to\infty} \inf_{\Omega_k} u \geq \liminf_{k\to\infty} \left(u(x_k) - \frac{1}{k} \right) \geq 0.$$

Hence, $u \geq 0$ in M, which was to be proved.

8.6. Let $\{\Omega_k\}$ be a compact exhaustion sequence in M, and let $\{\tau_k\}$ and $\{T_k\}$ be two sequences of reals such that

$$0 < \tau_k < T_k < T$$

and $\tau_k \to 0$, $T_k \to T$ as $k \to \infty$. Applying the minimum principle of Theorem 8.10 to function u in the cylinder $\mathcal{C}_k = (\tau_k, T_k) \times \Omega_k$, we obtain

$$\inf_{\mathcal{C}_k} u = \inf_{\partial_p \mathcal{C}_k} u.$$

Choose a point $(t_k, x_k) \in \partial_p \mathcal{C}_k$ such that

$$\inf_{\mathcal{C}_k} u \geq u(t_k, x_k) - \frac{1}{k}.$$

Note that $(t_k, x_k) \in \partial_p \mathcal{C}_k$ means that either $t_k = \tau_k$ or $x_k \in \partial\Omega_k$. We claim that the sequence $\{(t_k, x_k)\}$ contains a subsequence that escapes from $N = (0, T) \times M$. First of all, pass to a subsequence such that $t_k \to t \in [0, T]$ and $x_k \to \infty$ or $x_k \to x \in M$ (the former case occurs when $\{x_k\}$ leaves any compact in M while the latter case occurs when infinitely many terms x_k stay in the same compact subset of M). If $x_k \to x \in M$ then x_k cannot be on the boundary $\partial\Omega_k$ for large k. Hence, in this case we must have $t_k = \tau_k$, which implies $t_k \to 0$. Hence, by definition, $\{(t_k, x_k)\}$ escapes from N.

By hypothesis, we obtain

$$\limsup_{k\to\infty} \inf_{\mathcal{C}_k} u \geq \limsup_{k\to\infty} \left(u(t_k, x_k) - \frac{1}{k} \right) \geq 0.$$

Passing to a subsequence of $\mathcal{C}_k$, we obtain

$$\lim_{k\to\infty} \inf_{\mathcal{C}_k} u \geq 0.$$

Since the union of all cylinders $\mathcal{C}_k$ is N, it follows that $\inf_N u \geq 0$, which was to be proved.

Second solution. The conclusion follows also from Corollary 5.20 if we show that

(*i*) $u_-(t, x) \rightrightarrows 0$ as $x \to \infty$ in M, where the convergence is uniform in $t \in (0, T)$;

(*ii*) $u_-(t, x) \to 0$ as $t \to 0$ locally uniformly in x.

The hypothesis (8.24) is equivalent to

$$\lim_{k\to\infty} u_-(t_k, x_k) = 0, \tag{B.159}$$

for any sequence $\{(t_k, x_k)\}$ that escapes from N.

If (i) fails then there is $\varepsilon > 0$ such that, for any compact exhaustion sequence $\{\Omega_l\}$ in M,

$$\sup_{t\in(0,T)} \sup_{M\setminus\Omega_l} u_-(t,x) \geq \varepsilon.$$

Choose $t_l \in (0,T)$ and $x_l \in M \setminus \Omega_l$ so that

$$u_-(t_l, x_l) \geq \varepsilon/2.$$

Passing to a subsequence, we can assume that the sequence $\{t_l\}$ converges in $[0,T]$. Since $x_l \to \infty$ in M, we see that (t_l, x_l) escapes from N, which contradicts (B.159).

In the same way one proves that $u_-(t,\cdot) \rightrightarrows 0$ as $t \to 0$ where the convergence is locally uniform in $x \in M$, which implies (ii).

8.7. If u is a bounded solution to the equation $-\Delta_\mu u + \alpha u = 0$ on a compact manifold M then $u \in C_0^\infty(M)$ which implies by the Green formula that

$$(\nabla u, \nabla u)_{L^2} + \alpha (u,u)_{L^2} = 0.$$

Since $\alpha > 0$, this is possible only if $u \equiv 0$. By Theorem 8.18, we conclude that M is stochastically complete.

8.8. By Theorem 1.7, the bounded Cauchy problem in $\mathbb{R}^n$ has a unique solution, which implies that $\mathbb{R}^n$ is stochastically complete by Theorem 8.18.

8.9. For all $s \in (0,t)$, we have by the semigroup identity and $P_s 1 \leq 1$ that

$$P_t 1 = P_{t-s} P_s 1 \leq P_{t-s} 1 \leq 1. \tag{B.160}$$

If $P_t 1(x) = 1$ holds for some $x \in M$, we obtain that, for this x, all the inequalities in (B.160) become equalities. In particular, we have

$$P_{t-s}(P_s 1)(x) = 1,$$

which is only possible if

$$P_s 1 \equiv 1. \tag{B.161}$$

We are left to extend (B.161) to $s \geq t$. Assume first $s < 2t$. Then $s/2 < t$ and we obtain

$$P_s 1 = P_{s/2}\left(P_{s/2} 1\right) = P_{s/2} 1 = 1,$$

that is, (B.161) holds also for $s \in (0,2t)$. By induction, we prove (B.161) for $s \in (0, 2^k t)$, whence it follows for all $s > 0$.

8.10. If M is stochastically complete then

$$R_\alpha 1 = \int_0^\infty e^{-\alpha t} (P_t 1)\, dt = \int_0^\infty e^{-\alpha t} dt = \alpha^{-1}.$$

Conversely, if $P_t 1(x) < 1$ for some x and t then the above identity shows that $R_\alpha 1(x) < \alpha^{-1}$.

8.11. (a) The function $u(x) = |x|^2$ satisfies in $\mathbb{R}^n$ the equation $\Delta u = 2n$, which implies that $\Delta u \leq u$ for $|x| \geq C$ where C is large enough. Hence, u is

1-superharmonic, and since $u(x) \to \infty$ as $x \to \infty$ then $\mathbb{R}^n$ is stochastically complete by Theorem 8.20.

(b) Let us construct 1-superharmonic function in $\mathbb{R}^n \setminus \{0\}$ outside the compact set $K = \{1 \le |x| \le C\}$. Indeed, in domain $\{|x| > C\}$ the function from part (a) will do. In domain $\{0 < |x| < 1\}$ set

$$u(x) = \begin{cases} |x|^{2-n}, & n > 2, \\ \log \frac{1}{|x|}, & n = 2, \end{cases}$$

so that $u(x)$ harmonic and positive in this domain (cf. Exercise 3.24), which implies that u is 1-superharmonic. Obviously, $u(x) \to \infty$ as $x \to \infty$ (in this context, "$x \to \infty$" means leaving any compact, which is equivalent to $|x| \to \infty$ or $|x| \to 0$ – cf. Exercise 5.19). Hence, $\mathbb{R}^n \setminus \{0\}$ is stochastically complete by Theorem 8.20.

In $\mathbb{R}^1 \setminus \{0\}$, function $e^{-|x|}$ is a bounded solution to the equation $\Delta u = u$, which implies by Theorem 8.18 that this manifold is stochastically incomplete.

(c) Without loss of generality, we can assume that $0 \notin \overline{\Omega}$, which implies that, for some $\varepsilon > 0$, a ball B_ε is disjoint with $\overline{\Omega}$. Consider in $\mathbb{R}^n \setminus \{0\}$ the function $u(x) = e^{-\alpha|x|}$ where $\alpha > 0$ is to be chosen. Writing $u = e^{-\alpha r}$ where $r = |x|$ and computing Δu in the polar coordinates, we obtain

$$\Delta u = u'' + \frac{n-1}{r} u' = \left(\alpha^2 - \frac{n-1}{r} \alpha \right) u.$$

In $\overline{\Omega}$ we have $r \ge \varepsilon$ which implies

$$\Delta u \ge \left(\alpha^2 - \frac{n-1}{\varepsilon} \alpha \right) u = \alpha' u,$$

where $\alpha' > 0$ provided α is large enough. Hence, for such α, u is a bounded positive α'-subharmonic function in Ω, which implies that Ω is stochastically incomplete by Theorem 8.23.

8.12. Let μ be the Lebesgue measure in Ω and define measure $\widetilde{\mu}$ by $d\widetilde{\mu} = h^2 d\mu$. Then the Laplace operator $\Delta_{\widetilde{\mu}}$ satisfies the identity

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ \Delta \circ h$$

and the corresponding heat semigroup $\widetilde{P}_t^\Omega$ satisfies a similar identity

$$\widetilde{P}_t^\Omega = \frac{1}{h} \circ P_t^\Omega \circ h$$

(see Theorem 9.15). Therefore,

$$P_t^\Omega h = h \widetilde{P}_t^\Omega 1,$$

and the required identity $P_t^\Omega h = h$ is equivalent to $\widetilde{P}_t^\Omega 1 = 1$, that is, to the stochastic completeness of $(\Omega, \widetilde{\mu})$.

To prove the latter, let us use Theorem 8.20 which says that it suffices to construct an α-superharmonic function $v(x)$ in the exterior of a compact

K in $(\Omega, \widetilde{\mu})$, such that $v(x) \to +\infty$ as $x \to \infty$ in Ω. Take K to be any closed ball, say $\overline{B}(0, \varepsilon)$ assuming that the origin 0 is contained in Ω and $\varepsilon > 0$ is small enough, and consider the function $u(x) = u(r) = e^{cr}$ where $r = |x|$ and $c > 0$ is a constant to be chosen. Then we have

$$\Delta u - \alpha u = u'' + \frac{n-1}{r} u' - \alpha u = \left(c^2 + \frac{n-1}{r} c - \alpha\right) e^{cr}.$$

Given $c > 0$, choose α so big that

$$c^2 + \frac{n-1}{\varepsilon} c - \alpha < 0,$$

which yields that $\Delta u - \alpha u < 0$ in $\Omega \setminus K$. Therefore, setting $v = \frac{u}{h}$, we obtain

$$\Delta_{\widetilde{\mu}} v - \alpha v = \frac{1}{h} (\Delta (vh) - \alpha (vh)) = \frac{1}{h} (\Delta u - \alpha u) < 0$$

so that v is α-superharmonic outside K in $(\Omega, \widetilde{\mu})$.

We are left to ensure that $v(x) \to +\infty$ as $x \to \infty$ in Ω. The latter means that, for any sequence $\{x_k\} \subset \Omega$ leaving any compact in Ω, $v(x_k) \to +\infty$ or, equivalently, for any such sequence $\{x_k\}$ leaving any compact in Ω, there is a subsequence $\{x_{k_i}\}$ such that $v(x_{k_i}) \to +\infty$ (see Exercise 5.18). If $\{x_k\}$ leaves any compact in Ω then it has a subsequence $\{x_{k_i}\}$ that converges to either a point on $\partial\Omega$ or to ∞ in $\mathbb{R}^n$. In the former case, we have $h(x_{k_i}) \to 0$ whence

$$v(x_{k_i}) = \frac{u(x_{k_i})}{h(x_{k_i})} \geq \frac{1}{h(x_{k_i})} \to +\infty. \tag{B.162}$$

In the latter case, we have $|x_{k_i}| \to \infty$. Using the hypothesis $h(x) = e^{O(|x|)}$, that is, $h(x) \leq e^{C|x|}$ as $|x| \to \infty$, we obtain

$$v(x) = \frac{u(x)}{h(x)} \geq \frac{e^{c|x|}}{e^{C|x|}} \to +\infty \text{ as } |x| \to \infty,$$

provided c is chosen to be larger than C.

8.13. (*a*) Set $u(t, \cdot) = P_t f$ so that the family $u(t, x)$ increases as $t \downarrow 0$ and converges to $v(x)$ pointwise. Fix $s > 0$. By the monotone convergence theorem we obtain that

$$\lim_{t \to \infty} P_s u(t, \cdot) = P_s v,$$

where the convergence is pointwise. Since

$$P_s u(t, \cdot) = P_s P_t f = u(t + s, \cdot)$$

and, hence,

$$\lim_{t \to \infty} P_s u(t, \cdot) = v(x),$$

we obtain that $P_s v = v$. By Theorem 7.15, the function $v(x)$ must satisfy the heat equation in variables s, x which yields $\Delta_\mu v = 0$.

(*b*) If $h \leq f$ then $P_t h \leq P_t f$ for any $t > 0$. Both functions $P_t h(x)$ and $h(x)$ as functions of t and x are bounded solutions to the Cauchy problem with the initial function h. Hence, by Theorem 8.18, they coincide, that is, $P_t h \equiv h$. This implies that $h \leq P_t f$ for any $t > 0$ whence $h \leq v$.

Note that without the hypothesis of the stochastic completeness this statement is false. Indeed, consider $f(x) \equiv 1$. Then f is harmonic so that the largest harmonic minorant of f is f. On the other hand, if M is stochastically incomplete then $P_t f(x) < 1$ for some x and t, which implies $v(x) < f(x)$.

8.14. Set $a = \sup v$ and note that $a \leq 1$. Using Exercise 8.13, we obtain

$$v = P_t v \leq a P_t 1.$$

Letting $t \to \infty$ and using that $P_t 1 \to v$, we obtain $v \leq av$. If $a < 1$ then this is only possible if $v \equiv 0$. Otherwise, we have $a = 1$ and, hence, $\sup v = 1$.

To prove the second claim, set $b = \inf v$ and assume that $b > 0$. By Exercise 8.13, we have

$$P_t v = v$$

for all $t > 0$. On the other hand, $v - b$ is a non-negative harmonic function, which implies by Exercise 7.29 that

$$P_t(v - b) \leq v - b.$$

Comparing the above two lines, we obtain that $P_t b \geq b$, which is only possible if $P_t 1 \equiv 1$ and, hence, $v \equiv 1$.

8.15. Let B be the closed unit ball centered at the origin in $\mathbb{R}^n$ and $\Omega = B^c$. By Exercise 8.13(a), the function $v(x) = \lim_{t\to\infty} P_t^\Omega 1(x)$ is a harmonic function on Ω. Clearly, $0 \leq v \leq 1$ and $v \not\equiv 1$, the latter because Ω is not stochastically complete by Exercise 8.11.

By the symmetry argument, $v(x)$ must depend only on the polar radius r. By Exercise 3.24, we obtain

$$v(x) = \begin{cases} a|x|^{2-n} + b, & n \geq 3, \\ a \ln \frac{1}{|x|} + b, & n = 2, \\ ax + b, & n = 1, \end{cases} \tag{B.163}$$

where a, b are real constants. In the case $n = 1, 2$, the boundedness of v implies $a = 0$, whence $v = \text{const}$. Since $v \neq 1$, it follows from Exercise 8.13(c) that $v \equiv 0$.

In the case $n \geq 3$, consider the function

$$h = 1 - |x|^{2-n},$$

which is a harmonic function in Ω that vanishes on ∂B and $h(x) \to 1$ as $|x| \to \infty$. By Exercise 8.12 we have $P_t^\Omega h = h$. Hence,

$$P_t^\Omega 1 \geq P_t^\Omega h = h.$$

Passing to the limit as $t \to \infty$, we obtain $v \geq h$. Since also $v \leq 1$, we see that $v(x) \to 1$ as $|x| \to \infty$, whence it follows that $v(x)$ must have the form

$$v(x) = a|x|^{2-n} + 1.$$

Since $v \neq \mathrm{const}$, it follows from Exercise 8.13(c) that $\inf v = 0$.On the other hand, we have $\inf v = a + 1$, whence $a = -1$ and $v(x) = 1 - |x|^{2-n}$.

8.16. Use the same argument as in the proof of Theorem 8.24.

Solutions to Chapter 9

9.1. We have $d\mu = h^2(x)\,dx$ where the function $h(x) = \exp(c \cdot x)$ satisfies the equation

$$\Delta h - |c|^2 h = 0.$$

By Theorem 9.15, the heat kernel of $(\mathbb{R}^n, \mathbf{g}_{\mathbb{R}^n}, \mu)$ is obtained from the Gauss-Weierstrass heat kernel by (9.24) with $\alpha = -|c|^2$, whence the claim follows.

9.2. Let $p_t^{\mathbb{R}^n}(x,y)$ be the heat kernel in $\mathbb{R}^n$ so that the formula (9.41) can be rewritten in the form

$$p_t(x,y) = p_t^{\mathbb{R}^n}(x,y) - p_t^{\mathbb{R}^n}(x,\overline{y}).$$

Obviously, the function $u(t,x) = p_t(x,y)$ solves the heat equation in $\mathbb{R}_+ \times \mathbb{R}^n$ and

$$u(t,x) \xrightarrow{\mathcal{D}'} \delta_y - \delta_{\overline{y}} \text{ as } t \to 0.$$

Hence, if $y \in M$, then $\delta_{\overline{y}} = 0$ in M, whence it follows that $u(t,x)$ is a fundamental solution of M at y. Since $|x-y| > |x - \overline{y}|$ for any $x \in M$, it follows that $u(t,x) > 0$.

Let us show that $u(t,x) \rightrightarrows 0$ as $x \to \infty$ on M (where the convergence is uniform in $t \in (0,+\infty)$). Indeed, if $\{x_k\}$ is a sequence leaving any compact in M then, passing to a subsequence, we can assume that either $|x_k| \to \infty$ or $x_k \to x \in \partial M$ (cf. Exercise 5.19). In the first case, both $p_t^{\mathbb{R}^n}(x_k,y)$ and $p_t^{\mathbb{R}^n}(x_k,\overline{y})$ go to 0 as $k \to \infty$ uniformly in t (cf. Exercise 1.5), so that $u(t,x_k) \rightrightarrows 0$. In the second case, $|x-y| = |x-\overline{y}|$ whence it follows that $u(t,x) = 0$. By the uniform continuity of the heat kernel (cf. Exercise 1.5), we obtain $u(t,x_k) \rightrightarrows 0$. Hence, $u(t,x_k) \rightrightarrows 0$ as $x \to \infty$ in M. We conclude by Theorem 9.7 that $u(t,x)$ is the heat kernel at y, which was to be proved.

9.3. The formula (9.42) makes sense for all $x, y \in \mathbb{R}^n$. If $x \in \partial M$ then $x^i = x^{i+1}$ for some index i. It follows that the two rows of the determinant (9.42) are the same, whence $p_t(x,y) = 0$.

In order to investigate further properties of $p_t(x,y)$, let us use the full expansion of the determinant, which gives

$$p_t(x,y) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} \prod_{i=1}^n p_t^{\mathbb{R}^1}\left(x^i, y^{\sigma(i)}\right) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} p_t^{\mathbb{R}^n}(x, y^\sigma), \tag{B.164}$$

where S_n is the group of permutations of $\{1,\dots,n\}$, $p_t^{\mathbb{R}^n}$ is the heat kernel in $\mathbb{R}^n$, and

$$y^\sigma = \left(y^{\sigma(1)}, \dots, y^{\sigma(n)}\right).$$

It follows that $p_t(x,y)$ solves the heat equation in $\mathbb{R}_+ \times \mathbb{R}^n$ and

$$p_t(x,y) \to \sum_{\sigma \in S_n} (-1)^\sigma \delta_{y^\sigma} \text{ as } t \to 0.$$

If $y \in M$ then $y^\sigma \notin M$ for all $\sigma \neq \mathrm{id}$, which implies $\delta_{y^\delta} = 0$ in M for all $\sigma \neq \mathrm{id}$ and, hence, $p_t(x,y) \to \delta_y$ as $t \to 0$. Therefore, $p_t(x,y)$ is a fundamental solution of M at y.

By Theorem 9.7 and Remark 9.8, in order to show that $p_t(x,y)$ is the heat kernel, it suffices to verify, for any fixed $y \in M$, the following two conditions:

(i) For any sequence $\{(t_k, x_k)\}$ such that $t_k \to 0$ and $x_k \to x \in M$,

$$\limsup_{k\to\infty} p_{t_k}(x_k, y) \geq 0. \tag{B.165}$$

(ii) For any sequence $\{x_k\}$, such that $x_k \to \partial M$ or $|x_k| \to \infty$,

$$p_t(x_k, y) \rightrightarrows 0 \text{ as } k \to \infty, \tag{B.166}$$

whence the convergence is uniform in $t \in (0, +\infty)$.

Proof of (i). Choose $\varepsilon > 0$ so that $B_\varepsilon(x) \subset M$; we can assume that $x_k \in B_{\varepsilon/2}(x)$ for all k. If $\sigma \neq \mathrm{id}$ then $y^\sigma \notin M$ whence it follows that $|x_k - y^\sigma| \geq \varepsilon/2$ and

$$p_{t_k}^{\mathbb{R}^n}(x_k, y^\sigma) \leq \frac{1}{(4\pi t_k)^{n/2}} \exp\left(-\frac{\varepsilon^2}{4t_k}\right) \to 0 \text{ as } k \to \infty. \tag{B.167}$$

In the case $\sigma = \mathrm{id}$ we have $(-1)^{|\sigma|} = 1$ and the corresponding term in (B.164) is positive. Hence, (B.165) follows from (B.164) and (B.167).

Proof of (ii). Assume first that $x_k \to x \in \partial M$. There is $\varepsilon > 0$ such that $|x - y^\sigma| > \varepsilon$ for all permutations σ. We can assume that all x_k are in the ball $B_{\varepsilon/2}(x)$. The heat kernel $p_t^{\mathbb{R}^n}(z, y^\sigma)$ is continuous in z outside $B_{\varepsilon/2}(y^\sigma)$ uniformly in t (cf. Exercise 1.5), which implies that $p_t(z,y)$ is continuous in $z \in B_{\varepsilon/2}(x)$ uniformly in $t \in (0, +\infty)$. As it was observed above, $p_t(x,y) = 0$ whence (B.166) follows.

Let now $|x_k| \to \infty$. Then all terms in (B.164) go to 0 uniformly in t (cf. Exercise 1.5), whence we obtain again (B.166).

9.4. (a) Similarly to the proof of Theorem 9.15, we have, using $\frac{\Delta_\mu h}{h} = \Phi$,

$$\begin{aligned} \frac{1}{h}\Delta_\mu(hf) &= \frac{1}{h}\left(h\Delta_\mu f + 2\langle \nabla h, \nabla f\rangle_{\mathbf{g}} + f\Delta_\mu h\right) \\ &= \Delta_\mu f + 2\langle \frac{\nabla h}{h}, \nabla f\rangle_{\mathbf{g}} + f\frac{\Delta_\mu h}{h} \\ &= \Delta_{\widetilde{\mu}} f + \Phi f, \end{aligned}$$

whence (9.44) follows by replacing f by $h^{-1}f$.

(*b*) Multiplying (9.44) by f and integrating, we obtain

$$\begin{aligned}\int_M \left(|\nabla f|^2 + \Phi f^2\right) d\mu &= -\int_M h f \Delta_{\widetilde{\mu}} \left(h^{-1} f\right) d\mu \\ &= -\int_M h^{-1} f \Delta_{\widetilde{\mu}} \left(h^{-1} f\right) d\widetilde{\mu} = \int_M \left|\nabla \left(h^{-1} f\right)\right|^2 d\widetilde{\mu} \geq 0,\end{aligned}$$

where we have applies the Green formula both on $(M, \mathbf{g}, \mu)$ and $(M, \mathbf{g}, \widetilde{\mu})$.

9.5. Setting $h(x) = |x|^\beta$ for some real β, we obtain $\Delta h = \Phi h$ where

$$\Phi = \frac{\beta^2 + (n-2)\beta}{|x|^2}.$$

Choosing $\beta = 1 - n/2$, we obtain

$$\Phi = -\frac{(n-2)^2}{4|x|^2}.$$

Substituting this function into (9.45), we obtain (9.46).

9.6. By Theorem 9.20, both u and v satisfy equation (9.48). Hence, the difference $w = u - v$ satisfies in $\mathbb{R} \times M$ the heat equation $\frac{\partial w}{\partial t} - \Delta_\mu w = 0$ and, hence, w is C^∞ function on $\mathbb{R} \times M$ by Theorem 7.4.

9.7. As in the proof of Corollary 9.21, it suffices to show that the following equation

$$\frac{\partial u_t}{\partial t} = \Delta_x u_t = \Delta_y u_t$$

holds in $\mathbb{R} \times \Omega \times \Omega$. It will follows if we prove that, for any $\varphi \in \mathcal{D}(\mathbb{R} \times \Omega \times \Omega)$ and any $y \in \Omega$,

$$\int_{\mathbb{R}\times\Omega} \left(\partial_t \varphi + \Delta_x \varphi\right) u_t(x, y)\, d\mu(x) dt = 0. \tag{B.168}$$

Since both $p_t(\cdot, y)$ and $p_t^\Omega(\cdot, y)$ are regular fundamental solutions at y, Theorem 9.20 yields

$$\int_{\mathbb{R}\times\Omega} \left(\partial_t \varphi + \Delta_x \varphi\right) p_t(x, y)\, d\mu(x) dt = -\varphi(0, y, y)$$

and the same identity for p_t^Ω, which implies (B.168).

9.8. (*a*) The condition $u(t, \cdot) \overset{L^1_{loc}}{\longrightarrow} f$ implies that, for any compact set $K \subset M$,

$$\int_K |u(t, \cdot)|\, d\mu \to \int_K |f|\, d\mu \text{ as } t \to 0, \tag{B.169}$$

whence it follows that, for any $T > 0$,

$$\int_0^T \int_K |u(t, x)|\, d\mu(x)\, dt < \infty.$$

Therefore, the function u, extended by 0 to $t \leq 0$, belongs to $L^1_{loc}(\mathbb{R} \times M)$ and, hence, can be considered as a distribution on $\mathbb{R} \times M$.

The equation (9.56) is equivalent to the identity

$$-\int_{\mathbb{R}\times M}(\partial_t\varphi+\Delta_\mu\varphi)\,u\,d\mu dt=\int_M\varphi(0,\cdot)\,f\,d\mu, \tag{B.170}$$

which should be satisfied for any $\varphi\in\mathcal{D}(\mathbb{R}\times M)$. Since $u\equiv 0$ for $t\leq 0$, the integral in the left hand side of (B.170) is equal to

$$\int_0^\infty\int_M(\partial_t\varphi+\Delta_\mu\varphi)\,u\,d\mu dt=\lim_{\varepsilon\to 0+}\int_\varepsilon^\infty\int_M(\partial_t\varphi+\Delta_\mu\varphi)\,u\,d\mu dt.$$

As in the proof of Theorem 9.20, we obtain, for any $\varepsilon>0$,

$$\int_\varepsilon^\infty\int_M(\partial_t\varphi+\Delta_\mu\varphi)\,u\,d\mu dt=-\int_M\varphi(\varepsilon,\cdot)\,u(\varepsilon,\cdot)\;d\mu.$$

We are left to verify that

$$\int_M\varphi(\varepsilon,\cdot)\,u(\varepsilon,\cdot)\;d\mu\to\int_M\varphi(0,\cdot)\,f\,d\mu\;\;\text{as }\varepsilon\to 0. \tag{B.171}$$

By hypothesis, we have $u(\varepsilon,\cdot)\overset{L^1_{loc}}{\longrightarrow}f$ as $\varepsilon\to 0$, which implies

$$\int_M\varphi(0,\cdot)\,u(\varepsilon,\cdot)\;d\mu\to\int_M\varphi(0,\cdot)\,f\,d\mu\;\;\text{as }\varepsilon\to 0. \tag{B.172}$$

Let $K\subset M$ be the projection onto M of $\operatorname{supp}\varphi$ in $\mathbb{R}\times M$. Then K is compact, and we have

$$\begin{aligned}&\left|\int_M\varphi(\varepsilon,\cdot)\,u(\varepsilon,\cdot)\;d\mu-\int_M\varphi(0,\cdot)\,u(\varepsilon,\cdot)\;d\mu\right|\\ \leq\;&\sup_{x\in K}|\varphi(\varepsilon,x)-\varphi(0,x)|\int_K|u(\varepsilon,\cdot)|\,d\mu,\end{aligned}$$

which tends to 0 as $\varepsilon\to 0$ by the continuity of φ and (B.169). Together with (B.172), this proves (B.171).

(*b*) If $f=0$ in M and also $F=0$ in $\mathbb{R}\times M$ and we obtain from (9.56) $\frac{\partial u}{\partial t}=\Delta_\mu u$ in $\mathbb{R}\times M$. Hence, $u\in C^\infty(\mathbb{R}\times M)$ by Theorem 7.4.

(*c*) It suffices to prove that

$$u(t,\cdot)\overset{C^\infty(U)}{\longrightarrow}f\;\;\text{as }t\to 0 \tag{B.173}$$

for any relatively compact open set $U\subset M$, where $t>0$. Let φ be a cutoff function of $\overline{U}$ in M. Since $\varphi f\in C_0^\infty(M)$, we obtain by Exercise 7.19 that

$$P_t(\varphi f)\overset{C^\infty(M)}{\longrightarrow}\varphi f\;\;\text{as }t\to 0,$$

and, in particular,

$$P_t(\varphi f)\overset{C^\infty(U)}{\longrightarrow}f\;\;\text{as }t\to 0. \tag{B.174}$$

On the other hand, the function

$$v(t,\cdot):=u(t,\cdot)-P_t(\varphi f)$$

solves the heat equation in $\mathbb{R}_+ \times M$ with the initial condition

$$v(t,\cdot) \overset{L^1_{loc}}{\longrightarrow} f - \varphi f \text{ as } t \to 0.$$

Since $f - \varphi f = 0$ in U, we obtain by part (b) that

$$v(t,\cdot) \overset{C^\infty(U)}{\longrightarrow} 0 \text{ as } t \to 0. \tag{B.175}$$

Adding up (B.174) and (B.175), we obtain (B.173).

9.9. Consider the function

$$u(t,\cdot) = P_t 1_{\Omega^c} = \int_{\Omega^c} p_t(\cdot,y)\, d\mu(y),$$

which solves the heat equation in $\mathbb{R}_+ \times \Omega$ and satisfies the initial condition

$$u(t,\cdot) \overset{L^1_{loc}(\Omega)}{\longrightarrow} 0$$

(see Exercises 7.33 and (7.34). Extending $u(t,\cdot)$ to $t \le 0$ by setting $u(t,\cdot) = 0$ and applying Exercise 9.8 in manifold Ω, we obtain that $u \in C^\infty(\mathbb{R} \times \Omega)$. By the Taylor formula, we have for any positive integer N, $t > 0$ and $x \in \Omega$,

$$u(t,x) = \sum_{k=0}^{N-1} u^{(k)}(0,x)\frac{t^k}{k!} + u^{(N)}(\xi,x)\frac{\xi^N}{N!},$$

where $u^{(k)} = \frac{\partial^k u}{\partial t^k}$ and $\xi \in (0,t)$. Clearly, $u^{(k)}(0,x) = 0$. Since $u^{(N)}$ is uniformly bounded in $[0,1] \times K$, we obtain, for some constant C, that

$$\sup_{x \in K} u(t,x) \le Ct^N \text{ for all } t \in [0,1],$$

whence the claim follows.

Another solution can be obtained using Exercise 15.1.

9.10. The identity (9.58) implies that $r_\alpha(x,y)$ is a non-negative measurable function in x, y. By Fubini's theorem and (7.50), we obtain

$$\int_M \int_0^\infty e^{-\alpha t} p_t(x,y)\, dt d\mu(x) = \int_0^\infty \left(\int_M p_t(x,y)\, d\mu(x)\right) e^{-\alpha t} dt \le \int_0^\infty e^{-\alpha t} dt = \alpha^{-1},$$

which implies that the function $x \mapsto r_\alpha(x,y)$ belongs to $L^1(M)$ and

$$\int_M r_\alpha(x,y)\, d\mu(x) \le \alpha^{-1}.$$

Since $r_\alpha(x,y)$ is symmetric in x,y, the same applies to the function $y \mapsto r_\alpha(x,y)$. To prove (9.59), fix $y \in M$ and show that, for any $\varphi \in \mathcal{D}(M)$,

$$(-\Delta_\mu r_\alpha + \alpha r_\alpha, \varphi) = \varphi(y). \tag{B.176}$$

Indeed, we have

$$\begin{aligned}
(-\Delta_\mu r_\alpha + \alpha r_\alpha, \varphi) &= (r_\alpha, -\Delta_\mu \varphi + \alpha\varphi) = \int_M r_\alpha(\cdot, y)(-\Delta_\mu\varphi + \alpha\varphi)\, d\mu \\
&= \int_M \left(\int_0^\infty e^{-\alpha t} p_t(\cdot, y)\, dt\right)(-\Delta_\mu\varphi + \alpha\varphi)\, d\mu \\
&= \int_0^\infty \int_M e^{-\alpha t} p_t(\cdot, y)(-\Delta_\mu\varphi + \alpha\varphi)\, d\mu dt \\
&= \int_0^\infty \left(\int_M (-\Delta_\mu + \alpha\, \mathrm{id})\, p_t(\cdot, y)\, \varphi d\mu\right) e^{-\alpha t} dt \\
&= \int_0^\infty \left(\int_M \left(-\frac{\partial}{\partial t} + \alpha\, \mathrm{id}\right) p_t(\cdot, y)\, \varphi d\mu\right) e^{-\alpha t} dt \\
&= -\int_0^\infty \int_M \frac{\partial}{\partial t}\left(e^{-\alpha t} p_t(\cdot, y)\right) \varphi d\mu dt \\
&= -\int_0^\infty \left(\frac{\partial}{\partial t}\int_M e^{-\alpha t} p_t(\cdot, y)\, \varphi d\mu\right) dt \\
&= -\left[\int_M e^{-\alpha t} p_t(\cdot, y)\, \varphi d\mu\right]_0^\infty \\
&= -\left[e^{-\alpha t} P_t\varphi(y)\right]_0^\infty
\end{aligned}$$

(here the derivative $\partial/\partial t$ and the integral can be interchanged because the function $p_t(\cdot, y)\,\varphi$ under the integral is smooth and compactly supported). Since $P_t\varphi$ is a bounded function and $P_t\varphi(y) \to \varphi(y)$ as $t \to 0$, (B.176) follows.

Solutions to Chapter 10

10.1. By hypothesis, for any point $x \in S$ there is a ball $B(x, r_x)$ of a positive radius r_x such that the only point of S inside this ball is x. We claim that all balls $\{B(x, r_x/2)\}_{x\in S}$ are disjoint. Indeed, if x, y are two distinct points from S then $y \notin B(x, r_x)$ and $x \notin B(y, r_y)$ whence

$$d(x, y) \geq r_x \text{ and } d(x, y) \geq r_y.$$

Hence,

$$d(x, y) \geq \frac{r_x}{2} + \frac{r_y}{2},$$

which implies that the balls $B(x, r_x/2)$ and $B(y, r_y/2)$ are disjoint.

Since X is a separable, there is a countable set $Y \subset X$ which is dense in X. Hence, each ball $B(x, r_x/2)$ contains a point from Y, and different balls contain different points. This obviously implies that the family of all balls $\{B(x, r_x/2)\}_{x\in S}$ is at most countable, whence the claim follows.

10.2. Let $\{v_k\}$ be an orthonormal basis in the Hilbert space $\mathcal{H}$ such that each v_k is either contained in $\operatorname{ran} E_U$ or is orthogonal to $\operatorname{ran} E_U$. Since E_U is a projector, in the first case we have $E_U v_k = v_k$ whereas in the second case $E_U v_k = 0$. Using Lemma 10.4 and the definition (10.3) of $m(U)$, we obtain

$$\operatorname{trace} E_U = \sum_k (E_U v_k, v_k) = \sum_{v_k \in \operatorname{ran} E_U} 1 = \dim \operatorname{ran} E_U = m(U).$$

10.3. Left to the reader

10.4. Left to the reader

10.5. Using (10.31), we obtain

$$(P_t f, f) = \left(P_{t/2} f, P_{t/2} f\right) = \|P_{t/2} f\|^2 \leq \exp(-\lambda_{\min} t) \|f\|^2,$$

which was to be proved.

10.6. (a) Since

$$\lambda_{\min}(\Omega) = \inf_{f \in C_0^\infty(\Omega) \setminus \{0\}} \mathcal{R}(f)$$

and $C_0^\infty(\Omega_1) \subset C_0^\infty(\Omega_2)$, we obtain $\lambda_{\min}(\Omega_1) \geq \lambda_{\min}(\Omega_2)$.

(b) It is obvious that $f \in C_0^\infty(\Omega)$ if and only if $f_k := f|_{\Omega_k} \in C_0^\infty(\Omega_k)$ for all k. Clearly, we have

$$\mathcal{R}(f) = \frac{\sum_k \int_{\Omega_k} |\nabla f_k|^2 \, d\mu}{\sum_k \int_{\Omega_k} f_k^2 \, d\mu} \geq \inf_k \mathcal{R}(f_k)$$

(note that some of functions f_k may identically vanish; in this case set $\mathcal{R}(f_k) = +\infty$). Taking inf in f, we obtain

$$\lambda_{\min}(\Omega) \geq \inf_k \lambda_{\min}(\Omega_k). \tag{B.177}$$

On the other hand, since $\Omega \supset \Omega_k$, we have also $\lambda_{\min}(\Omega) \leq \lambda_{\min}(\Omega_k)$ whence the opposite inequality in (B.177) follows.

(c) By part (a), the sequence $\{\lambda_{\min}(\Omega_k)\}$ decreases and

$$\lambda_{\min}(\Omega_k) \geq \lambda_{\min}(\Omega). \tag{B.178}$$

To prove the opposite inequality, observe that, for any $f \in C_0^\infty(\Omega)$, there is a set Ω_k that contains $\operatorname{supp} f$ and, hence,

$$\lambda_{\min}(\Omega_k) \leq \mathcal{R}(f).$$

Taking infimum over all such f, we obtain

$$\lim_{k\to\infty} \lambda_{\min}(\Omega_k) = \inf_k \lambda_{\min}(\Omega_k) \leq \lambda_{\min}(\Omega),$$

which together with (B.178) proves the claim.

10.7. It follows from (10.23) and Exercise 3.5, that

$$\left(BA^{n/2}\right)^{-1} d\mu \leq d\widetilde{\mu} \leq \left(BA^{n/2}\right) d\mu$$

and

$$A^{-1} |\nabla\varphi|^2_{\mathbf{g}} \leq |\nabla\varphi|^2_{\widetilde{\mathbf{g}}} \leq A |\nabla\varphi|^2_{\mathbf{g}}.$$

By Theorem 10.8, we have

$$\widetilde{\lambda}_{\min}(M) = \inf_{\varphi \in C_0^\infty(M)\setminus\{0\}} \frac{\int |\nabla\varphi|^2_{\widetilde{\mathbf{g}}}\, d\widetilde{\mu}}{\int \varphi^2 d\widetilde{\mu}},$$

whence it follows that

$$\left(A^{n+1}B^2\right)^{-1}\lambda_{\min}(M) \le \widetilde{\lambda}_{\min}(M) \le \left(A^{n+1}B^2\right)\lambda_{\min}(M).$$

10.8. Replacing in (10.25) f by f^2, we obtain

$$h(M) \le \frac{\int_M \left|\nabla\left(f^2\right)\right| d\mu}{\int_M f^2\, d\mu} = 2\frac{\int_M |f|\,|\nabla f|\, d\mu}{\int_M f^2\, d\mu} \le 2\frac{\|f\|_{L^2}\|\nabla f\|_{L^2}}{\|f\|^2_{L^2}} = 2\frac{\|\nabla f\|_{L^2}}{\|f\|_{L^2}}.$$

Taking inf in f, we obtain

$$h(M) \le 2\left(\lambda_{\min}(M)\right)^{1/2},$$

whence (10.26) follows.

10.9. Left to the reader

10.10. (a) Consider a function φ such that $\varphi \equiv 1$ on one of the components of M and $\varphi \equiv 0$ on the other components. Since $\varphi \in C_0^\infty(M)$ and $\Delta_\mu\varphi = 0$, we see φ is an eigenfunction of $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ with eigenvalue 0. Obviously, there are m linearly independent eigenfunctions as above, whence it follows that $\lambda_k(M) = 0$ for all $k \le m$.

To prove that $\lambda_{m+1}(M) > 0$, we need to show that any eigenfunction u of 0 is a linear combination of the above eigenfunctions. For that, it suffices to verify that $u = \text{const}$ on any component of M. Since $u \in L^2$ and $\Delta_\mu u = 0$, Corollary 7.3 implies that $u \in C^\infty(M) = C_0^\infty(M)$. By the Green formula,

$$\int_M |\nabla u|^2\, d\mu = -\int_M u\Delta_\mu u d\mu = 0,$$

which implies $\nabla u \equiv 0$ on M.

For any two points x, y on the same component, there is a smooth path γ connecting x and y. Then, by (3.103) and (3.17),

$$\frac{d}{dt}u(\gamma(t)) = \dot{\gamma}(t)(u) = \langle\nabla u, \dot{\gamma}(t)\rangle = 0,$$

so that $u(\gamma(t)) = \text{const}$. It follows that $u(x) = u(y)$ and, hence, $u = \text{const}$ on this component, which finishes the proof.

(b) This follows from $\lambda_{m+1}(M) > 0$ and $\lambda_m(M) = 0$.

10.11. By Exercise 10.10, we have $\lambda_1(M) = 0$ and $\lambda_2(M) > 0$. The first eigenfunction φ_1 is constant, and the condition $\|\varphi_1\|_{L^2} = 1$ yields $\varphi_1 \equiv \frac{1}{\sqrt{\mu(M)}}$. Therefore, the eigenfunction expansion (10.33) yields

$$p_t(x,y) = \frac{1}{\mu(M)} + \sum_{k=2}^{\infty} e^{-\lambda_k t}\varphi_k(x)\varphi_k(y),$$

where $\lambda_k = \lambda_k(M)$, whence

$$\sup_{x,y\in M}\left|p_t(x,y)-\frac{1}{\mu(M)}\right|\leq\sum_{k=2}^{\infty}e^{-\lambda_k t}\sup_{x,y\in M}|\varphi_k(x)\varphi_k(y)|.$$

Since $\lambda_2 > 0$, the right hand side here tends to 0 as $t\to\infty$ by Remark 10.15.

An alternative solution follows from Exercise 11.21.

10.12. Note that $\varphi_1(x) > 0$ by Theorem 10.11. By the identity (10.33) of Theorem 10.13, we have

$$\frac{p_t^{\Omega}(x,y)}{e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y)}-1=\frac{1}{\varphi_1(x)\varphi_1(y)}\sum_{k=2}^{\infty}e^{-(\lambda_k-\lambda_1)t}\varphi_k(x)\varphi_k(y).$$

By Theorem 10.23, we have $\lambda_2 > \lambda_1$. Consequently, there is $\varepsilon > 0$ such that

$$\lambda_k-\lambda_1\geq\varepsilon\lambda_k \text{ for all } k\geq 2.$$

Hence, we have

$$\left|\sum_{k=2}^{\infty}e^{-(\lambda_k-\lambda_1)t}\varphi_k(x)\varphi_k(y)\right|\leq\sum_{k=2}^{\infty}e^{-\lambda_k\varepsilon t}\sup_{x,y\in M}|\varphi_k(x)\varphi_k(y)|.$$

Since $\lambda_2 > 0$, the right hand side tends to 0 as $t\to\infty$ by Remark 10.15, which finishes the proof.

10.13. It follows from (10.50) that, for all $t > 0$,

$$\sup_{x\in\Omega}|\varphi_k(x)|\leq Ce^{t\lambda_k}\left(1+t^{-\sigma}\right).$$

Choosing $t = 1/\lambda_k$, we obtain the required estimate.

10.14. (*a*) Using $P_t\varphi_k = e^{-t\mathcal{L}}\varphi_k = e^{-t\lambda_k}\varphi_k$ and the symmetry of P_t, we obtain

$$(P_tf,\varphi_k)_{L^2}=(f,P_t\varphi_k)=e^{-t\lambda_k}(f,\varphi_k)=e^{-t\lambda_k}a_k.$$

Hence, P_tf has the following expansion in the basis φ_k:

$$P_tf=\sum_k e^{-\lambda_k t}a_k\varphi_k. \tag{B.179}$$

Let us show that this series converges also in $L^2_{loc}(\mathbb{R}_+\times M)$. Indeed, for any $t > 0$ and positive integer N, we have

$$\|P_tf-\sum_{k=1}^{N}e^{-\lambda_k t}a_k\varphi_k\|_{L^2}^2=\sum_{k>N}e^{-2\lambda_k t}a_k^2\leq\sum_{k>N}a_k^2$$

so that the $L^2(M)$-convergence of the series (B.179) is uniform in t. It follows that the series converges in $L^2([\alpha,\beta]\times M)$ for any bounded interval $[\alpha,\beta]\subset\mathbb{R}_+$. Finally, since all the terms $e^{-\lambda_k t}\varphi_k(x)$ in (B.179) solve the heat equation, the convergence is in $C^\infty(\mathbb{R}_+\times M)$ by Theorem 7.4.

(*b*) As in the proof of Theorem 10.13, we have, for any $x\in M$ and $t>0$,

$$(p_{t,x},\varphi_k)_{L^2}=P_t\varphi_k(x)=e^{-t\mathcal{L}}\varphi_k(x)=e^{-t\lambda_k}\varphi_k(x),$$

which implies

$$p_{t,x} = \sum_k e^{-\lambda_k t} \varphi_k(x) \varphi_k,$$

where the series converges in $L^2(M)$. Similarly to part (a), we obtain

$$\|p_{t,x} - \sum_{k=1}^N e^{-\lambda_k t} \varphi_k(x) \varphi_k\|^2_{L^2(M)} = \sum_{k>N} e_k^{-2\lambda_k t} \varphi_k^2(x),$$

whence, integrating also in $x \in M$,

$$\|p_t(x,y) - \sum_{k=1}^N e^{-\lambda_k t} \varphi_k(x) \varphi_k(y)\|^2_{L^2(M\times M)} = \sum_{k>N} e_k^{-2\lambda_k t}.$$

By hypothesis, the right hand side is finite and tends to 0 as $N \to \infty$ locally uniformly in t. Integrating the previous line also in t over a bounded interval $[\alpha, \beta] \subset \mathbb{R}_+$ and passing to the limit as $N \to \infty$, we obtain that

$$p_t(x,y) = \sum_k e^{-\lambda_k t} \varphi_k(x) \varphi_k(y),$$

where the series converges in $L^2([\alpha,\beta] \times M \times M)$. Finally, the convergence in the sense of $C^\infty(\mathbb{R}_+ \times M \times M)$ follows in the same way as in the last paragraph of the proof of Theorem 10.13.

10.15. (a) Let $\{v_k\}$ be any orthonormal basis in L^2. Since the operators P_t and R^s are bounded, all v_k belong to their domains. By Exercise 5.11, we have the identity

$$(R^s v_k, v_k) = \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} e^{-t} (P_t v_k, v_k)\, dt.$$

Summing up in all k and using Lemma 10.4, we obtain (10.60).

(b) By Lemma 10.14, we have, for all $0 < t < 1$,

$$\begin{aligned} \operatorname{trace} P_t &= \operatorname{trace} P_{t/2}^2 = \int_M \int_M p_{t/2}^2(x,y)\, d\mu(x)\, d\mu(y) \\ &= \int_M p_t(x,x)\, d\mu(x) \\ &\le Ct^{-\nu} \mu(M). \end{aligned}$$

It follows from (a) that

$$\operatorname{trace} R_\alpha^s \le \operatorname{const} \int_0^\infty \frac{t^{s-\nu-1}}{\Gamma(k)} e^{-t} dt.$$

The integral always converges at ∞ and converges at 0 provided $s > \nu$, whence the claim follows.

10.16. (*a*) We have

$$p_t^\Omega(x,x) = \int_\Omega p_{t/2}^\Omega(x,y)^2\, d\mu(y) \le \int_\Omega p_{t/2}(x,y)^2\, d\mu(y).$$

Since $\overline{\Omega}$ is compact, we obtain by Theorem 7.7 for all $x \in \Omega$ and $0 < t < 1$,

$$\int_\Omega p_{t/2}(x,y)^2\, d\mu(y) \le \sup_{x\in\overline{\Omega}} \|p_{t/2,x}\|^2_{L^2(M)} \le Ct^{-2\sigma},$$

where σ is the smallest integer larger than $n/4$, whence

$$p_t^\Omega(x,x) \le Ct^{-2\sigma}.$$

Consider the resolvent $R_\Omega = \left(\mathrm{id} + \mathcal{L}^\Omega\right)^{-1}$. By Exercise 10.15, we obtain that $\operatorname{trace} R_\Omega^s < \infty$ provided $s > 2\sigma$. On the other hand, since the eigenvalues of R_Ω^s are $(1+\lambda_k)^{-s}$, we have by (10.14)

$$\operatorname{trace} R_\Omega^s = \sum_{k=1}^\infty (1+\lambda_k)^{-s}.$$

Hence, the series here converges for $s > 2\sigma$, whence (10.61) follows.

(*b*) We need to prove that

$$\sum_{k=1}^\infty |c_k| \sup|\varphi_k| < \infty.$$

This will imply that the sum $\sum_k c_k\varphi_k$ is a continuous function, which, being equal to f almost everywhere, must coincide with f pointwise.

In the view of the estimate (10.57) of Exercise 10.13, it suffices to show that

$$\sum_{k=1}^\infty \lambda_k^\sigma |c_k| < \infty. \tag{B.180}$$

Restricting the summation to those k where $\lambda_k \neq 0$ and using the Cauchy-Schwarz inequality, we obtain, for any $s > 0$,

$$\sum_{k=1}^\infty \lambda_k^\sigma |c_k| \le \left(\sum_{k=1}^\infty \lambda_k^{2\sigma+s} c_k^2\right)^{1/2} \left(\sum_{k:\lambda_k>0}^\infty \lambda_k^{-s}\right)^{1/2}. \tag{B.181}$$

By part (*a*), the last term here is finite provided s is large enough. To estimate the middle term, observe that $(-\Delta_\mu)^m f \in C_0^\infty(\Omega)$ for any positive integer m, whence it follows that $f \in \operatorname{dom}\left(\mathcal{L}^\Omega\right)^m$ and

$$\left(\left(\mathcal{L}^\Omega\right)^m f, \varphi_k\right) = \left(f, \left(\mathcal{L}^\Omega\right)^m \varphi_k\right) = \lambda_k^m (f,\varphi_k) = \lambda_k^m c_k.$$

By the Parseval identity, we have

$$\sum_{k=1}^{\infty} \lambda_k^{2m} c_k^2 = \| \left(\mathcal{L}^{\Omega}\right)^m f\|_{L^2(\Omega)}^2 < \infty. \tag{B.182}$$

Hence, the middle term in (B.181) is finite for any $s > 0$, whence (B.180) follows.

REMARK. Let $\{f_i\}_{i\in I}$ be a family of functions from $C_0^\infty(\Omega)$ depending on some parameter i. It follows from the above argument that the Fourier series of f_i converges to $f_i(x)$ absolutely and uniformly both in x and i, provided each norm $\|\Delta_\mu^m f_i\|_{L^2(\Omega)}$ is bounded uniformly in i.

10.17. Let $f \in C(M)$. By Exercise 10.14, for any $t > 0$, the function $P_t f$ can be uniformly approximated by linear combinations of φ_k (this also follows from Exercise 10.16 because $P_t f \in C_0^\infty(M)$). Since $P_t f \rightrightarrows f$ as $t \to 0$ (cf. Theorem 7.16), the same applies to f.

10.18. (i) If φ is an eigenfunction of Δ_μ on $\mathbb{S}^1$ with an eigenvalue λ then $\varphi'' + \lambda\varphi = 0$. This implies that $\lambda = k^2$ where k is a non-negative integer, and the corresponding eigenfunction φ is given by

$$\begin{cases} \varphi = \text{const}, & \text{if } k = 0, \\ \varphi = C_1 \cos kx + C_2 \sin kx, & \text{if } k > 0. \end{cases}$$

Hence, we obtain an orthonormal basis of eigenfunctions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, ..., \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}, ...,$$

whence, by (10.33),

$$\begin{aligned} p_t(x,y) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t} \cos kx \cos ky + \frac{1}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t} \sin kx \sin ky \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t} \cos k(x-y). \end{aligned}$$

(ii) Set

$$q_t(x,y) = \sum_{n\in\mathbb{Z}} \widetilde{p}_t(x + 2\pi n, y)$$

and observe that the series converges in any reasonable sense because $\widetilde{p}_t(x,y)$ decays quickly in $|x-y|$. Using the fact that $\widetilde{p}_t(x,y)$ satisfies the heat equation in t, x for any fixed y, it is easy to show that so does $q_t(x,y)$. Next, we have

$$\int_{\mathbb{S}^1} q_t(x,y)\,dx = \int_0^{2\pi} q_t(x,y)\,dx = \int_{-\infty}^{\infty} \widetilde{p}_t(x,y)\,dx = 1,$$

and

$$\int_{y-\varepsilon}^{y+\varepsilon} q_t(x,y)\,dx \ge \int_{y-\varepsilon}^{y+\varepsilon} \widetilde{p}_t(x,y)\,dx \to 1 \text{ as } \varepsilon \to 0.$$

Hence, $q_t(x,y)$ is a regular fundamental solution to the heat equation on $\mathbb{S}^1$. Since $\mathbb{S}^1$ is stochastically complete (cf. Exercise 8.7), we conclude by Corollary 9.6 that $q_t(x,y)$ is the heat kernel on $\mathbb{S}^1$.

(*iii*) Rewrite (10.62) as follows

$$p_t(x,y) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-k^2 t}\cos k(x-y). \tag{B.183}$$

In particular, for $x=y=0$ we obtain

$$p_t(0,0) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-k^2 t}. \tag{B.184}$$

From (10.63) at $x=y=0$, we obtain

$$p_t(0,0) = \sum_{n\in\mathbb{Z}} \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{\pi^2 n^2}{t}\right).$$

Comparing the above two lines, we obtain (10.64).

10.19. Let (r,θ) be the polar coordinates on $\mathbb{R}^{n+1}$. By (3.83) we have

$$\Delta_{\mathbb{R}^{n+1}} P = \frac{\partial^2 P}{\partial r^2} + \frac{n}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^n} P.$$

In particular, setting $r=1$ and using $\Delta_{\mathbb{R}^{n+1}} P = 0$, we obtain

$$-\Delta_{\mathbb{S}^n} f = \left.\frac{\partial^2 P}{\partial r^2} + n\frac{\partial P}{\partial r}\right|_{r=1}.$$

By the homogeneity of P, we have, for $x=(r,\theta)$,

$$P(x) = r^k P\left(\frac{x}{r}\right) = r^k f(\theta).$$

It follows that

$$\frac{\partial P}{\partial r} = k r^{k-1} f(\theta) \quad \text{and} \quad \frac{\partial^2 P}{\partial r^2} = k(k-1) r^{k-2} f(\theta)$$

whence

$$-\Delta_{\mathbb{S}^n} f = (k(k-1)+nk) f = \alpha f,$$

which was to be proved.

10.20. As was shown in Exercise 3.10, each Hermite polynomial

$$h_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}e^{-x^2}$$

satisfies the equation

$$\Delta_\mu h_k + 2k h_k = 0. \tag{B.185}$$

Let us show that $h_k \in \operatorname{dom}\mathcal{L}$, which will imply that h_k are eigenfunctions of $\mathcal{L}$ with eigenvalues $2k$. Indeed, since $h_k(x)$ is a polynomial in x, it obviously belongs to $L^2(\mathbb{R},\mu)$ because $d\mu = e^{-x^2}dx$. By (B.185), we have also $\Delta_\mu h_k \in L^2(\mathbb{R},\mu)$, which implies by Lemma 11.7 that $h_k \in \operatorname{dom}\mathcal{L}$.

Since $\deg h_k = k$ and, hence, the sequence $\{h_k\}$ spans all the polynomials, the completeness of this sequence in $L^2(\mathbb{R}, \mu)$ follows from the Weierstrass approximation theorem. Since the eigenvalues $\lambda_k = 2k$ do not have finite accumulation points, we conclude that they exhaust all the spectrum of $\mathcal{L}$.

Since

$$\operatorname{trace} e^{-t\mathcal{L}} = \sum_k e^{-2kt} < \infty,$$

we can apply the eigenfunction expansion formula for the heat kernel (cf. Exercise 10.14), that is

$$p_t(x, y) = \sum_{k=0}^{\infty} e^{-2kt} \varphi_k(x) \varphi_k(y),$$

where φ_k are normalized eigenfunctions. A computation yields

$$\|h_k\|_{L^2}^2 = \sqrt{\pi} 2^k k!\ ,$$

whence (10.65) follows.

10.21. (*a*) By (10.68), we have, for any $f \in W_0^1(M)$,

$$(\nabla f, \nabla \varphi_k)_{L^2} = \lambda_k (f, \varphi_k)_{L^2}\ , \tag{B.186}$$

and by (10.69),

$$(\nabla \varphi_i, \nabla \varphi_j)_{L^2} = 0, \quad i \neq j.$$

Since the inner product in $W_0^1(M)$ is given by

$$(u, v)_{W^1} = (u, v)_{L^2} + (\nabla u, \nabla v)_{L^2}\ ,$$

we obtain that $\{\varphi_k\}$ is an orthogonal sequence in $W_0^1(M)$ as well.

It follows from (B.186) that, for any $f \in W_0^1(M)$,

$$(f, \varphi_k)_{W^1} = (1 + \lambda_k)(f, \varphi_k)_{L^2}\ .$$

In particular, if $(f, \varphi_k)_{W^1} = 0$ for all k then also $(f, \varphi_k)_{L^2} = 0$ and, hence, $f = 0$. Therefore, $\{\varphi_k\}$ is a basis in $W_0^1(M)$.

(*b*) Expanding f in the basis $\{\varphi_k\}$ in $W_0^1(M)$ and using

$$(\varphi_k, \varphi_k)_{W^1} = 1 + \lambda_k,$$

we obtain

$$f = \sum_k \frac{(f, \varphi_k)_{W^1}}{(\varphi_k, \varphi_k)_{W^1}} \varphi_k = \sum_k \frac{(1 + \lambda_k)(f, \varphi_k)_{L^2}}{(1 + \lambda_k)(\varphi_k, \varphi_k)_{L^2}} \varphi_k = \sum_k a_k \varphi_k.$$

Hence, the series $f = \sum_k a_k \varphi_k$ converges also in $W_0^1(M)$, which implies (10.74). Then (10.75) follows from the Parseval identity, (10.74) and $(\nabla \varphi_k, \nabla \varphi_k)_{L^2} = \lambda_k$.

(*c*) By the symmetry of Δ_μ, if $f \in W_0^2(M) = \operatorname{dom} \Delta_\mu$ then

$$(\Delta_\mu f, \varphi_k)_{L^2} = (f, \Delta_\mu \varphi_k)_{L^2} = -\lambda_k (f, \varphi_k)_{L^2} = -\lambda_k a_k,$$

whence both (10.76) and (10.77) follow.

10.22. Consider the space $F = \operatorname{span}(f_1, ..., f_k)$, which is a k-dimensional subspace of $W_0^1(M)$ (functions f_i are linearly independent because they have disjoint supports). It is easy to check that, for any $f \in F \setminus \{0\}$, we have $\mathcal{R}(f) \leq a$. Hence, using the identity (10.67) of Theorem 10.18, we conclude that $\lambda_k(M) \leq a$.

10.23. By Theorem 10.20, the embedding operator $W_0^1(M) \hookrightarrow L^2(M)$ is compact. As it was shown in Section 5.5, trivial extension of functions from Ω to M determines an embedding $W_0^1(\Omega) \hookrightarrow W_0^1(M)$. Since a composition of a bounded operator with a compact one is a compact operator, we conclude that the composite embedding $W_0^1(\Omega) \hookrightarrow L^2(M)$ is a compact operator. Obviously, the range of this embedding is contained in $L^2(\Omega)$, which implies that the embedding $W_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is also compact. By Theorem 10.20, we conclude that $\mathcal{L}^\Omega$ has a discrete spectrum.

10.24. If $\{\varphi_i\}$ is an orthonormal basis of eigenfunctions in $L^2(M')$ and $\{\psi_j\}$ is that of $L^2(M'')$ then $f_{ij}(x,y) := \varphi_i(x)\psi_j(y)$ is an orthonormal basis in $L^2(M)$. Obviously, we have

$$\Delta_\mu f_{ij} = (\Delta_x + \Delta_y)(\varphi_i(x)\psi_j(y)) = -(\alpha_i + \beta_j) f_{ij}$$

so that $\alpha_i + \beta_j$ are the eigenvalues in M. Since the sequence $\{\alpha_i + \beta_j\}$ has no finite accumulation point, it represents all spectrum of M, which, hence, is discrete.

10.25. Consider two relatively compact open sets $\Omega' \Subset \Omega$ and let ψ be a cutoff function of $\overline{\Omega'}$ in Ω. Since the sequence $\{u_k\}$ is bounded in $W^1(\Omega)$, the sequence $\{u_k\psi\}$ is also bounded in $W^1(\Omega)$. Note that $u_k\psi \in W_0^1(\Omega)$ by Corollary 5.6), and the embedding $W_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact by Corollary 10.21. Therefore, there is a subsequence of $\{u_k\psi\}$ that converges in $L^2(\Omega)$; the corresponding subsequence of $\{u_k\}$ converges in $L^2(\Omega')$. Using the diagonal process, we can choose a subsequence of $\{u_k\}$ that converges in $L^2(\Omega)$ for all relatively compact open sets Ω, which was to be proved.

10.26. If f vanishes at a point then $f \equiv 0$ by the strong minimum principle of Corollary 8.14. Hence, assume in the sequel that $f > 0$ and prove that $\alpha \leq \lambda_{\min}(M)$. Exhausting M be a sequence of connected relatively compact open sets Ω and noticing that $\lambda_{\min}(\Omega) \to \lambda_{\min}(M)$ (cf. Exercise 10.6), it suffices to prove that $\alpha \leq \lambda_{\min}(\Omega)$ for any such Ω.

The function $u(t,x) = e^{-\alpha t} f(x)$ obviously satisfies the heat equation in $\mathbb{R}_+ \times \Omega$ and $u(t,\cdot) \to f$ as $t \to 0$ locally uniformly. Since $u \geq 0$, by Theorem 8.1 we conclude that $u \geq P_t^\Omega f$, which implies that

$$\left(P_t^\Omega f, f\right)_{L^2(\Omega)} \leq (u, f)_{L^2(\Omega)}$$

and, hence,

$$\left(P_t^\Omega f, f\right)_{L^2(\Omega)} \leq e^{-\alpha t} \|f\|_{L^2(\Omega)}^2. \tag{B.187}$$

Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis in $L^2(\Omega)$ that consists of the eigenfunctions of $\mathcal{L}^\Omega$, and let $\{\lambda_k\}_{k=1}^\infty$ be the sequence of the corresponding eigenvalues in the increasing order. Since $f \in L^2(\Omega)$, we can expand f in this basis, say,

$$f=\sum_k a_k\varphi_k.$$

It follows from Exercise 10.14 that

$$\left(P_t^\Omega f,f\right)_{L^2(\Omega)}=\sum_k e^{-\lambda_k t}a_k^2, \tag{B.188}$$

whence

$$\left(P_t^\Omega f,f\right)_{L^2(\Omega)}\geq e^{-\lambda_1 t}a_1^2. \tag{B.189}$$

By Theorem 10.11, we have $\varphi_1>0$ in Ω, which implies that

$$a_1=\int_\Omega f\varphi_1 d\mu>0$$

(we can assume that Ω is so big that $f\not\equiv 0$ in Ω). Comparing (B.187) and (B.189) and letting $t\to\infty$, we conclude that $\alpha\leq\lambda_1$.

10.27. (*a*) The spectrum of the operator $\mathcal{L}-\alpha\,\mathrm{id}$ is contained in the interval $[\lambda_{\min}(M)-\alpha,+\infty)$. By the hypothesis $\alpha<\lambda_{\min}(M)$, the spectrum of $\mathcal{L}-\alpha\,\mathrm{id}$ does not contain 0, which implies that it is invertible. The identity (10.85) proved exactly in the same way as Lemma 5.10.

(*b*) Set $v=u-1$ so that v is a function from $W_0^1(M)$ that must satisfy the equation

$$\Delta_\mu(v+1)+\alpha(v+1)=0,$$

that is,

$$\Delta_\mu v+\alpha v=-\alpha.$$

Since constants are in $L^2(M)$, we obtain that $\Delta_\mu v\in L^2(M)$ whence $v\in\mathrm{dom}(\mathcal{L})$. The latter allows to rewrite the above equation in the form

$$\mathcal{L}v-\alpha v=\alpha.$$

Since the operator $\mathcal{L}-\alpha\,\mathrm{id}$ is invertible, this equation has a unique solution

$$v=(\mathcal{L}-\alpha\,\mathrm{id})^{-1}\alpha.$$

Combining with (10.85) we obtain (10.86).

Since $P_t1\geq 0$, the conclusion that $u>0$ seems to be a trivial consequence of (10.86). However, one should make the following point clear. The identity (10.85) and its consequence (10.86) are understood weakly. In particular, (10.86) means that, for any $\varphi\in L^2(M)$,

$$(u,\varphi)=(1,\varphi)+\alpha\int_0^\infty e^{\alpha t}(P_t1,\varphi)\,dt.$$

If $\varphi\geq 0$ and $\varphi\not\equiv 0$ then $(1,\varphi)>0$ and $(P_t1,\varphi)\geq 0$ whence it follows that $(u,\varphi)>0$. This implies that $u>0$ a.e..

10.28. Assume first that u satisfies a strict inequality $\Delta_\nu u>0$ in Ω. Let z be a point where u attains its maximum in $\overline{\Omega}$. If $z\in\partial\Omega$ then (10.87) is trivially satisfied. Assume now that $z\in\Omega$. Let $x_1,x_2,...x_n$ be a coordinate system in a chart containing z. Recall that the Laplace operator Δ_μ is written in the local coordinates as follows:

$$\Delta_\mu u = \frac{1}{\rho}\frac{\partial}{\partial x^i}\left(\rho g^{ij}\frac{\partial u}{\partial x^j}\right),$$

where $\rho = \Upsilon\sqrt{\det g}$ and Υ is the density function of measure μ. It follows that

$$\Delta_\mu u = g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + b^j \frac{\partial u}{\partial x^j},$$

where

$$b^j = \frac{1}{\rho}\frac{\partial}{\partial x^i}\left(\rho g^{ij}\right).$$

By a linear change of the coordinates, the matrix $\left(g^{ij}\right)$ at the point z can be reduced to id (cf. the proof of Theorem 8.11), which yields

$$\Delta_\mu u(z) = \sum_j \frac{\partial^2 u}{\left(\partial x^j\right)^2}(z) + b^j(z)\frac{\partial u}{\partial x^j}(z).$$

Since z is the point of maximum of u, we have

$$\frac{\partial u}{\partial x^j}(z) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\left(\partial x^j\right)^2}(z) \leq 0,$$

whence $\Delta_\mu u(z) \leq 0$, which contradicts the hypothesis $\Delta_\mu u > 0$.

Consider now the general case when $\Delta_\mu u \geq 0$. Suppose that there exists a function $v \in C^2\left(\overline{\Omega}\right)$ such that $\Delta_\mu v > 0$ in Ω. Then, for any $\varepsilon > 0$, we have

$$\Delta_\mu\left(u + \varepsilon v\right) > 0,$$

and the first part of this proof applies to the function $u + \varepsilon v$ and yields

$$\sup_{\overline{\Omega}}\left(u + \varepsilon v\right) = \sup_{\partial\Omega}\left(u + \varepsilon v\right),$$

whence (10.87) follows when $\varepsilon \to 0$.

Let us show that such a function v always exists. For example, in $\mathbb{R}^n$ the following function will do:

$$v(x) = |x|^2 = x_1^2 + ... + x_n^2\ ,$$

because $\Delta v = 2n > 0$. On an arbitrary manifold, we can use the solvability of the weak Dirichlet problem to construct such a function. Indeed, let Ω' be a relatively compact open neighborhood of $\overline{\Omega}$ such that $M \setminus \overline{\Omega'}$ is still non-empty. By Theorem 10.22, $\lambda_1\left(\Omega'\right) > 0$. By Exercise 4.28, there exists a function $v \in W_0^1\left(\Omega'\right)$ such that $\Delta_\mu v = 1$ in Ω in the distributional sense. By Corollary 7.3, $v \in C^\infty\left(\Omega'\right)$. Hence, the function v satisfies all the above requirements, which finishes the proof.

10.29. Using (10.84), we obtain

$$\begin{aligned} p_t(x,y) &= \left(p_{t/2,x}, p_{t/2,y}\right) \le \|p_{t/2,x}\|_{L^2} \|p_{t/2,y}\|_{L^2} \\ &\le e^{-2\lambda_{\min}(M)(t-s)} \|p_{s/2,x}\|_{L^2} \|p_{s/2,y}\|_{L^2} \\ &= e^{-2\lambda_{\min}(M)(t-s)} \sqrt{p_s(x,x)\, p_s(y,y)}. \end{aligned}$$

Solutions to Chapter 11

11.1. All balls in $(\mathbb{R}^n, \mathbf{g}_{\mathbb{R}^n})$ are relatively compact by the Bolzano-Weierstrass theorem. To prove that $(\mathbb{R}^n, \mathbf{g})$ is complete with metric $\mathbf{g}$ is given by (11.1), first observe that the geodesic ball $B(0,r)$ centered at the origin, coincides with the Euclidean ball $B_r = \{|x| < r\}$, because by Exercise 3.37, $d(x,0) = |x|$.

Since the identity mapping between $(\mathbb{R}^n, \mathbf{g}_{\mathbb{R}^n})$ and $(\mathbb{R}^n, \mathbf{g})$ is a diffeomorphism, $B(0,r) = B_r$ is also relatively compact in the topology of $(\mathbb{R}^n, \mathbf{g})$. By the triangle inequality, any geodesic ball $B(x,r)$ is contained in $B(0,r')$ for $r' = r + |x|$, which implies that $B(x,r)$ is relatively compact.

11.2. Let $\{x_k\}$ be a Cauchy sequence with respect to the distance d. Then the sequence $\{x_k\}$ is bounded, that is, it is contained in a geodesic ball. Since the ball is relatively compact, the set of points $\{x_k\}$ is also relatively compact and hence contains a convergent subsequence. However, any Cauchy sequence containing a convergent subsequence, converges itself, which was to be proved.

11.3. For function $f \in C^1(M)$, the classical gradient ∇f coincides with the distributional one. Therefore, we obtain by Theorem 11.3 that

$$\sup |\nabla f| \le \|f\|_{Lip}.$$

The opposite inequality

$$\|f\|_{Lip} \le \sup_M |\nabla f|$$

was shown in the proof of Corollary 11.4.

11.4. (a) The fact that φ is Lipschitz means that, for all $X, Y \in I := I_1 \times ... \times I_m$,

$$|\varphi(X) - \varphi(Y)| \le \|\varphi\|_{Lip} |X - Y| = \|\varphi\|_{Lip} \left(\sum_{k=1}^m |X_k - Y_k|^2\right)^{1/2}.$$

Therefore, for any two points $x, y \in M$, setting

$$X = (f_1(x), ..., f_m(x)) \quad \text{and} \quad Y = (f_1(y), ..., f_m(y)),$$

we obtain

$$
\begin{aligned}
\left|\Phi(x)-\Phi(y)\right| &= \left|\varphi(X)-\varphi(Y)\right| \leq \|\varphi\|_{Lip}\left|X-Y\right| \\
&\leq \|\varphi\|_{Lip}\left(\sum_{k=1}^{m}\left|f_k(x)-f_k(y)\right|^2\right)^{1/2} \\
&\leq \|\varphi\|_{Lip}\left(\sum_{k=1}^{m}\|f_k\|_{Lip}^2 d^2(x,y)\right)^{1/2} \\
&= \|\varphi\|_{Lip}\left(\sum_{k=1}^{m}\|f_k\|_{Lip}^2\right)^{1/2} d(x,y),
\end{aligned}
$$

which proves (11.14).

(b) Prove that if $f \in Lip_0(M)$ and $\varphi \in Lip(\mathbb{R})$ such that $\varphi(0)=0$ then $\varphi \circ f \in Lip_0(M)$. By part ($a$), we have $\Phi := \varphi \circ f \in Lip(M)$. By condition $\varphi(0)=0$, function $\Phi(x)$ vanishes at any point x where $f(x)=0$. Hence, $\operatorname{supp}\Phi \subset \operatorname{supp} f$ whence it follows that $\operatorname{supp}\Phi$ is compact and $\Phi \in Lip_0(M)$.

11.5. Since the following functions in $\mathbb{R}^2$ are Lipschitz:

$$X+Y,\ \max(X,Y),\ \min(X,,Y),$$

we conclude by Exercise 11.4 that also the functions

$$f+g,\ \ \max(f,g),\ \min(f,g)$$

are Lipschitz. If in addition f is bounded on M then fg is Lipschitz because the function XY is Lipschitz when the domain of X is bounded.

Assume that f is bounded on $\operatorname{supp} g$, say

$$a \leq f \leq b \text{ on } \operatorname{supp} g.$$

Consider function $\widetilde{f} = \varphi \circ f$ where

$$\varphi(t) = \min(\max(t,a),b).$$

Since φ is a bounded Lipschitz function on $\mathbb{R}$, the function $\widetilde{f}$ is a bounded Lipschitz function on M, which implies by the above argument that $\widetilde{f}g$ is Lipschitz. Since $f=\widetilde{f}$ on $\operatorname{supp} g$, we have $fg=\widetilde{f}g$ so that fg is Lipschitz.

If $f+g \in Lip_0$ then both functions f,g are bounded. Hence, all the functions $f+g$, fg, $\max(f,g)$, $\min(f,g)$ are Lipschitz. Since they all have compact supports, they belong to Lip_0, which was to be proved.

11.6. Let $d(x)$ be the distance from x to K and $\delta = d(K,\Omega^c)$. Then the function

$$f(x) = \frac{(\delta/2 - d(x))_+}{\delta/2}$$

satisfies all the required properties.

11.7. (a) To prove that f is Lipschitz with constant C, it suffices to show that, for any smooth path $\gamma : [a,b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$,

$$|f(x) - f(y)| \leq C\ell(\gamma). \tag{B.190}$$

Consider the preimages $V_\alpha := \gamma^{-1}(U_\alpha)$ of the open sets U_α and observe that $\{V_\alpha\}$ is a family of open sets in $[a,b]$ that covers $[a,b]$.

We claim that, for any covering $\{V_\alpha\}$ of the interval $[a,b]$ by open sets, there exists a finite partition

$$a = t_0 < t_1 < ... < t_m = b \tag{B.191}$$

such that any interval $[t_{j-1}, t_j]$ is contained in one of the sets V_α. By splitting each V_α into its connected components, it suffices to assume that each V_α is connected, that is, V_α is an interval. Since $[a,b]$ is compact, we can further assume that the family $\{V_\alpha\}$ is finite, say $\{V_i\}_{i=1}^k$. Now we can prove the above claim by induction in k. If $k = 1$ then the trivial partition $a < b$ will do. Consider also the case $k = 2$ when

$$[a,b] \subset V_1 \cap V_2.$$

If $a, b \in V_i$ then $[a,b] \subset V_i$, which amounts to the case $k = 1$. Let $a \in V_1$ and $b \in V_2$. Since the interval $[a,b]$ is connected, the sets V_1, V_2 have a common point $s \in (a,b)$. Then the partition $a < s < b$ satisfies the requirements.

Assuming $k > 2$, let us prove the inductive step from $k-1$ to k. Since the following $k-1$ open sets

$$V_1, V_2, ..., V_{k-2}, V_{k-1} \cup V_k \tag{B.192}$$

cover $[a,b]$, by the inductive hypothesis, there exists a partition (B.191) such that any interval $[t_{j-1}, t_j]$ is contained in one of the sets (B.192). Suppose that, for some index j,

$$[t_{j-1}, t_j] \subset V_{k-1} \cap V_k$$

(if such j does not exist then the proof is finished). Then arguing as in the case $k = 2$, we split further $[t_{j-1}, t_j]$ into two intervals, which leads to a required partition.

Now, having constructed a partition (B.191) as above, set $x_j = \gamma(t_j)$ so that any two consecutive points x_{j-1}, x_j belong to the same set U_α. By hypothesis, we obtain

$$|f(x_{j-1}) - f(x_j)| \leq Cd(x_{j-1}, x_j),$$

whence

$$|f(x) - f(y)| \leq C\sum_j d(x_{j-1}, x_j) \leq C\ell(\gamma),$$

which finishes the proof.

(b) Using the notation of part (a), we need only to prove (B.190) when $x \in E_1$ and $y \in E_2$. The preimages $\gamma^{-1}(E_1)$ and $\gamma^{-1}(E_2)$ cover the interval $[a,b]$. Since they are closed sets and the interval $[a,b]$ is connected, they must have a common point, say t, which implies that $z = \gamma(t)$ belongs to both sets E_1 and E_2. Therefore, we obtain

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq Cd(x,z) + Cd(z,y) \leq C\ell(\gamma),$$

which was to be proved.

11.8. If $f \in C^1(M)$ then, for any open set $\Omega \Subset M$, we have $\sup_\Omega |\nabla f| < \infty$ which implies by Exercise 11.3 that f is Lipschitz in Ω and, hence, $f \in Lip_{loc}(M)$.

If $f \in Lip_{loc}(M)$ then $f \in Lip(\Omega)$ for any open $\Omega \Subset M$, and by Theorem 11.3 we have $\nabla f \in \vec{L}^\infty(\Omega)$ and, hence, $\nabla f \in \vec{L}^2(\Omega)$. Since also $f \in L^2(\Omega)$, it follows that $f \in W^1(\Omega)$ and, hence, $f \in W^1_{loc}(M)$.

11.9. Obviously, if $f \in Lip_0(M)$ then $f \in Lip_{loc}(M)$ and f has compact support. Conversely, assume that $f \in Lip_{loc}(M)$ and f has compact support. Let Ω be a relatively compact open set covering $\operatorname{supp} f$. Then f is Lipschitz in Ω. On the other hand, f is Lipschitz in the open set $\Omega' := M \setminus \operatorname{supp} f$ because $f \equiv 0$ in Ω'. Since Ω and Ω' cover M, we conclude by Exercise 11.7 that $f \in Lip(M)$. Since $\operatorname{supp} f$ is compact, we conclude $f \in Lip_0(M)$.

11.10. It suffices to prove that $\Phi(x)$ is Lipschitz on any open set $\Omega \Subset M$. Let $I_k = f_k(\Omega)$. Then I_k is a bounded subset of $\mathbb{R}$, which implies that the the product $I = I_1 \times ... \times I_m$ is a relatively compact subset of $\mathbb{R}^m$. Hence, φ is Lipschitz on I. Applying Exercise 11.4 to the manifold Ω, we conclude that Φ is Lipschitz on Ω, which was to be proved.

11.11. This follows from Exercise 11.10 and the fact that the following functions are locally Lipschitz in $\mathbb{R}^2$:

$$X + Y, \ XY, \ \max(X,Y), \ \min(X,Y).$$

(cf. Exercise 11.5).

11.12. Let $\Omega \Subset M$ be an open set and $\psi \in \mathcal{D}(M)$ be a cutoff function of Ω in M. By Exercises 11.9, 11.11, we have $\psi f \in Lip_0(M)$ whence, by Theorem 11.3, $\nabla(\psi f)$ belongs to $\vec{L}^\infty(M)$. Since $\psi f = f$ in Ω, we obtain that $\nabla f|_\Omega$ belongs to $\vec{L}^\infty(\Omega)$, whence the claim follows.

11.13. (a) By the argument of Exercise 11.12, it suffices to prove the product rule when $f, g \in Lip_0(M)$. By Corollary 11.4, $f \in Lip_0(M)$ implies $f \in W^1_0(M)$ and, hence, there is a sequence $\{f_k\} \subset \mathcal{D}(M)$ such that

$$f_k \xrightarrow{W^1} f. \tag{B.193}$$

Furthermore, we can assume that the sequence $\{f_k\}$ is uniformly bounded. Indeed, f is bounded and let $C := \sup|f|$. Let $\psi \in C^\infty(\mathbb{R})$ be a bounded function such that $\psi(t) = t$ if $|t| \leq C$, and that ψ' is also bounded[2]. Then

[2]To construct such a function ψ, let φ be a cutoff function of the interval $[-C, C]$, and then set

$$\psi(t) = \int_0^t \varphi(s)\, ds.$$

$\psi(f_k) \subset \mathcal{D}(M)$ and, by Theorem 5.7, (B.193) implies

$$\psi(f_k) \xrightarrow{W^1} \psi(f) = f.$$

Obviously, the sequence $\{\psi(f_k)\}$ is uniformly bounded, so we can rename $\psi(f_k)$ into f_k.

Let $\{g_k\}$ be a similar approximation for function $g \in Lip_0(M)$. Then we have

$$f_k g_k \xrightarrow{L^2} fg \tag{B.194}$$

because

$$\begin{aligned} \|f_k g_k - fg\|_{L^2} &\leq \|f_k(g_k - g)\|_{L^2} + \|(f_k - f)g\|_{L^2} \\ &\leq \|f_k\|_{L^\infty}\|g_k - g\|_{L^2} + \|g\|_{L^\infty}\|f_k - f\|_{L^2} \to 0. \end{aligned}$$

Using the fact that the functions $|\nabla f|$ and $|\nabla g|$ are bounded, which is due to Theorem 11.3, we obtain in the same way that

$$f_k \nabla g_k \xrightarrow{\vec{L}^2} f\nabla g \text{ and } g_k \nabla f_k \xrightarrow{\vec{L}^2} g\nabla f, \tag{B.195}$$

By Lemma 4.2, (B.194) implies

$$\nabla(f_k g_k) \xrightarrow{\vec{\mathcal{D}}'} \nabla(fg).$$

On the other hand, we have

$$\nabla(f_k g_k) = f_k \nabla g_k + g_k \nabla f_k$$

(see Exercise 3.3), which implies by (B.195)

$$\nabla(f_k g_k) \xrightarrow{\vec{\mathcal{D}}} f\nabla g + g\nabla f,$$

whence the claim follows.

(*b*) If $g \in C_0^\infty(M)$ then $fg \in Lip_0(M) \subset W_0^1(M)$ by Exercise 11.5 and Corollary 11.12, and product rule holds by part (*a*). For an arbitrary $g \in W_0^1(M)$, the boundedness of f and ∇f implies that

$$fg \in L^2(M) \text{ and } f\nabla g + g\nabla f \in \vec{L}^2(M)$$

so that $fg \in W^1(M)$.

By the definition of $W_0^1(M)$, there exists a sequence $\{g_k\}$ of functions from $C_0^\infty(M)$ such that $g_k \xrightarrow{W^1} g$. Hence, $fg_k \in W_0^1(M)$ and

$$fg_k \xrightarrow{L^2} fg. \tag{B.196}$$

Let us show that also

$$\nabla(fg_k) \xrightarrow{L^2} f\nabla g + g\nabla f. \tag{B.197}$$

Indeed, we have

$$\nabla(fg_k) = f\nabla g_k + g_k\nabla f$$

and

$$\begin{aligned}\|\nabla(fg_k)-(f\nabla g+g\nabla f)\|_{L^2} &\le \|f(\nabla g_k-\nabla g)\|_{L^2}+\|(g_k-g)\nabla f\|_{L^2}\\ &\le \|f\|_{L^\infty}\|\nabla(g_k-g)\|_{L^2}+\|\nabla f\|_{L^\infty}\|g_k-g\|_{L^2},\end{aligned}$$

which tends to 0 by the choice of the sequence $\{g_k\}$.

It follows from (B.196) and (B.197) that $fg_k \overset{W^1}{\to} fg$ whence $fg \in W_0^1(M)$. Besides, (B.196) and (B.197) imply by Lemma 4.2 that

$$\nabla(fg)=f\nabla g+g\nabla f,$$

which finishes the proof.

(c) Multiplying g by a cutoff function of $\operatorname{supp} f$, we can assume that $g\in W_0^1(M)$ (cf. Corollary 5.6). Then the claim follows from part (b).

11.14. The fact that $\psi(f)\in Lip_{loc}(M)$ follows from Exercise 11.10 because $\psi\in Lip_{loc}(\mathbb{R})$.

Hence, we only need to prove that chain rule. Using the argument from the solution of Exercise 11.12, we can assume that $f\in Lip_0(M)$.

If $\psi\equiv$ const then $\psi(f)\equiv$ const and the claim is obvious. So, subtracting a constant from ψ, we can assume that $\psi(0)=0$. Since f is bounded, say $|f|\le C$, the values of $\psi(t)$ for $|t|>C$ are not used in $\psi(f)$. Multiplying ψ by a cutoff function of the interval $[-C,C]$, we can assume that $\operatorname{supp}\psi$ is bounded; in particular, $|\psi'|$ is bounded.

Since by Corollary 11.4, $f\in W_0^1(M)$, the assumptions made about ψ allow to apply Theorem 5.7 (or Lemma 5.2) and to conclude that

$$\nabla\psi(f)=\psi'(f)\nabla f.$$

11.15. Let us show that any $u\in W^1(M)$ belongs also to $W_0^1(M)$. By Exercise 11.13, if $f\in Lip_0(M)$ then $fu\in W_0^1(M)$ and

$$\nabla(fu)=u\nabla f+f\nabla u.$$

Fix $o\in M$, set $B_r=B(o,r)$ and, for some $R>r>0$, choose f as follows:

$$f(x)=\min\left(1,\frac{(R-d(x,o))_+}{R-r}\right)=\begin{cases}1, & x\in B_r,\\ 0, & x\notin B_R,\\ \frac{R-d(x,o)}{R-r}, & x\in B_R\setminus B_r.\end{cases}$$

Note that $0\le f\le 1$. Since $d(\cdot,o)$ is a Lipschitz function with the Lipschitz constant 1, function f is Lipschitz with the Lipschitz constant $\frac{1}{R-r}$, which implies by Theorem 11.3 that $|\nabla f|\le\frac{1}{R-r}$. Noticing that $f=1$ and $\nabla f=0$ in B_r, we obtain

$$\|fu-u\|_{L^2}^2=\int_M(f-1)^2u^2d\mu\le\int_{M\setminus B_r}u^2d\mu \tag{B.198}$$

and

$$\begin{aligned} \|\nabla (fu) - \nabla u\|_{L^2} &\leq \|u\nabla f\|_{L^2} + \|(f-1)\nabla u\|_{L^2} \\ &\leq \frac{1}{R-r}\left(\int_{M\setminus B_r} u^2 d\mu\right)^{1/2} + \left(\int_{M\setminus B_r} |\nabla u|^2 d\mu\right)^{1/2} \end{aligned} \tag{B.199}$$

By choosing R and r large enough, the right hand sides of (B.198) and (B.199) can be made arbitrarily small, which means that u can be approximated in $W^1(M)$ by functions of the form fu. Since $fu \in W_0^1(M)$, it follows that also $u \in W_0^1(M)$.

11.16. (a) Replacing $u_k - u$ by u_k, we can assume that $u = 0$. Then the condition (11.17) means that

$$(u_k, \varphi)_{L^2} + (\nabla u_k, \nabla\varphi)_{L^2} \to 0$$

whence, by the definition of distributional gradient,

$$(u_k, \varphi)_{L^2} - (u_k, \Delta_\mu \varphi)_{L^2} \to 0.$$

Together with (11.18) this gives

$$(v, \varphi)_{L^2} - (v, \Delta_\mu \varphi)_{L^2} = 0,$$

which implies $\Delta_\mu v = v$ where Δ_μ is understood in the distributional sense. By Lemma 11.6, we conclude that $v = 0$, which was to be proved.

(b) To show that the hypothesis of completeness of M cannot be dropped, consider any incomplete manifold where there exists a non-zero function $v \in W^1$ such that $\Delta_\mu v = v$. For example, one can take $M = (0,1)$ and $v(x) = e^x$. Set $u_k = v$ for all k so that (11.18) holds. For any $\varphi \in C_0^\infty(M)$, we have

$$(u_k, \varphi)_{W^1} = (v, \varphi)_{L^2} + (\nabla v, \nabla \varphi)_{L^2} = (v, \varphi)_{L^2} - (\Delta_\mu v, \varphi)_{L^2} = 0,$$

so that (11.17) holds with $u = 0$.

11.17. (a) Obviously, mapping J is an isometry of the Hilbert spaces $L^2(M, \widetilde{\mu})$ and $L^2(M, \mu)$. Also, J maps $\mathcal{D}(M)$ onto itself. Identity (9.44), restricted to $f \in \mathcal{D}(M)$, can be rewritten in the following form

$$-\Delta_\mu + \Phi = J\left(-\Delta_{\widetilde{\mu}}\right) J^{-1}, \tag{B.200}$$

where all operators act in $\mathcal{D}(M)$. By Theorem 4.6, operator $\widetilde{\mathcal{L}}$ is a self-adjoint extension of $-\Delta_{\widetilde{\mu}}\big|_{\mathcal{D}}$ in $L^2(M, \widetilde{\mu})$. It follows from (B.200) that $J\widetilde{\mathcal{L}}J^{-1}$ is a self-adjoint extension of $-\Delta_\mu + \Phi|_{\mathcal{D}}$ in $L^2(M, \mu)$. If A is another self-adjoint extension of $-\Delta_\mu + \Phi|_{\mathcal{D}}$ in $L^2(M, \mu)$ then the operator $J^{-1}AJ$ is a self-adjoint extension of $-\Delta_{\widetilde{\mu}}\big|_{\mathcal{D}}$ in $L^2(M, \widetilde{\mu})$.

By hypothesis, the manifold $(M, \mathbf{g}, \widetilde{\mu})$ is complete. Hence, by Theorem 11.5, $-\Delta_{\widetilde{\mu}}\big|_{\mathcal{D}}$ is essentially self-adjoint, and its unique self-adjoint extension in $L^2(M, \widetilde{\mu})$ is $\widetilde{\mathcal{L}}$, whence we conclude $\widetilde{\mathcal{L}} = J^{-1}AJ$ and $A = J\widetilde{\mathcal{L}}J^{-1}$. Hence, $J\widetilde{\mathcal{L}}J^{-1}$ is a unique self-adjoint extension of $-\Delta_\mu + \Phi|_{\mathcal{D}}$, which was to be proved.

(b) It follows from $\mathcal{L}^\Phi = J\widetilde{\mathcal{L}}J^{-1}$ that

$$e^{-t\mathcal{L}^\Phi} = Je^{-t\widetilde{\mathcal{L}}}J^{-1},$$

which implies, for any $f \in L^2(M,\mu)$,

$$e^{-t\mathcal{L}^\Phi} f(x) = h(x)\int_M \widetilde{p}_t(x,y)\frac{f(y)}{h(y)}d\widetilde{\mu}(y) = \int_M \widetilde{p}_t(x,y)\,h(x)\,h(y)\,f(y)\,d\mu(y),$$

whence (11.20) follows.

11.18. Since $\Delta_\mu = \frac{d^2}{dx^2}$, the function $h(x) = e^{-\frac{1}{2}x^2}$ satisfies the equation

$$\Delta_\mu h = h'' = \left(x^2-1\right)h = \Phi h.$$

Consider measure $\widetilde{\mu}$ given by

$$d\widetilde{\mu} = h^2 d\mu = e^{-x^2}dx.$$

As was shown in Example 9.19, the heat kernel $\widetilde{p}_t$ of this measure is given by

$$\widetilde{p}_t(x,y) = \frac{e^t}{(2\pi\sinh 2t)^{1/2}}\exp\left(\frac{2xye^{-2t}-\left(x^2+y^2\right)e^{-4t}}{1-e^{-4t}}\right).$$

Hence, we obtain from the identity (11.20) of Exercise 11.17 that

$$\begin{aligned} p_t^\Phi(x,y) &= \widetilde{p}_t(x,y)\exp\left(-\frac{1}{2}x^2-\frac{1}{2}y^2\right) \\ &= \frac{e^t}{(2\pi\sinh 2t)^{1/2}}\exp\left(\frac{4xye^{-2t}-\left(x^2+y^2\right)\left(1+e^{-4t}\right)}{2\left(1-e^{-4t}\right)}\right). \end{aligned}$$

The rest follows from the elementary identity

$$\frac{4xye^{-2t}-\left(x^2+y^2\right)\left(1+e^{-4t}\right)}{2\left(1-e^{-4t}\right)} = -\frac{(x-y)^2}{2\sinh 2t}-\frac{x^2+y^2}{2}\tanh t.$$

11.19. Indeed, we have

$$\begin{aligned} \int_{r_1}^\infty \frac{rdr}{f(r)} &= \sum_{k=2}^\infty \int_{r_{k-1}}^{r_k}\frac{rdr}{f(r)} \\ &\geq \sum_{k=2}^\infty \frac{1}{f(r_k)}\int_{r_{k-1}}^{r_k} rdr \\ &\geq \frac{1}{2C}\sum_{k=2}^\infty \frac{r_k^2-r_{k-1}^2}{r_k^2}. \end{aligned} \tag{B.201}$$

We are left to observe that the series

$$\sum_{k=2}^\infty \frac{r_k^2-r_{k-1}^2}{r_k^2} = \sum_{k=2}^\infty\left(1-\frac{r_{k-1}^2}{r_k^2}\right)$$

diverges because so does the infinite product

$$\prod_{k=2}^{\infty} \frac{r_{k-1}^2}{r_k^2} = 0.$$

11.20. Left to the reader

11.21. Since the constants functions are in $L^2(M)$, zero is an eigenvalue of the Dirichlet Laplace operator $\mathcal{L}$ on M with eigenfunction

$$\varphi(x) \equiv \frac{1}{\sqrt{\mu(M)}}.$$

Using the spectral decomposition

$$P_t = e^{-t\mathcal{L}} = \int_{[0,+\infty)} e^{-\lambda t} dE_\lambda$$

and noticing that $e^{-\lambda t} \to 1_{\{\lambda=0\}}$ as $t \to \infty$, we obtain by Lemma 4.8 that, for any $f \in L^2(M)$,

$$P_t f \xrightarrow{L^2} \int_{\{0\}} dE_\lambda f = (f, \varphi)\, \varphi \quad \text{as } t \to \infty. \tag{B.202}$$

Choose $f = p_{s,x}$, for some $s > 0$ and $x \in M$. By Theorem 11.8, M is stochastically complete, which implies

$$(p_{s,x}, \varphi)\, \varphi = \frac{1}{\mu(M)} \int_M p_{s,x} d\mu = \frac{1}{\mu(M)}.$$

Using $P_t p_{s,x} = p_{t+s,x}$ (cf. Theorem 7.13), we obtain from (B.202) that

$$p_{t+s,x} \xrightarrow{L^2} \frac{1}{\mu(M)} \quad \text{as } t \to \infty.$$

By Theorem 7.4, we conclude that, in fact, the convergence is in C^∞, which finishes the proof.

11.22. Consider measure $\widetilde{\mu}$ given by

$$d\widetilde{\mu} = h^2 d\mu.$$

Then $v(r) = \widetilde{\mu}(B(x_0, r))$ and, by Theorem 11.8, the hypothesis (11.47) implies that the weighted manifold $(M, \widetilde{\mu})$ is stochastically complete. The heat semigroup $\widetilde{P}_t$ on $(M, \widetilde{\mu})$ is given by $\widetilde{P}_t = \frac{1}{h} \circ P_t \circ h$ (see Theorem 9.15). By the stochastic completeness we have $\widetilde{P}_t 1 = 1$ whence $P_t h = h$.

11.23. One of possible solutions is as follows. By approximation argument, we can assume $f \in C^\infty(0, +\infty)$. Then there exist a weighted model manifold with the volume function $V(r)$ such that $V(r) = f(r)$ for $r > 1$. The condition

$$\int^\infty \frac{r dr}{f(r)} = \infty \tag{B.203}$$

implies by Theorem 11.14 that M is parabolic. On the other hand, by Example 11.16, the parabolicity of the model manifold implies

$$\int^{\infty} \frac{dr}{f(r)} = \infty, \tag{B.204}$$

which was to be proved.

Of course, there is a more direct proof of the implication (B.203) $\Longrightarrow$ (B.204).

11.24. If M is stochastically incomplete then there exists a non-zero solution u to the equation $\Delta_\mu u = u$ such that $0 \leq u \leq 1$. In particular, $\Delta_\mu u \geq 0$ that is, u is a subharmonic function. Hence, $2 - u$ is a positive superharmonic function, which implies by hypothesis of parabolicity that $u = \text{const}$. The only constant solution to $\Delta_\mu u = u$ is $u \equiv 0$. This contradiction finishes the proof.

11.25. It follows from Lemma 6.4 that, for any $f \in C_0^\infty(\Omega)$,

$$\int_\Omega f^2 d\mu \leq \frac{1}{n\,(\operatorname{diam}\Omega)^2} \int_\Omega |\nabla f|^2 \, d\mu,$$

whence (11.62) follows by Theorem 10.8.

Any function $f \in C_0^\infty(B_1)$ determines a function $\widetilde{f} \in C_0^\infty(B_r)$ by

$$\widetilde{f}(x) = f\left(\frac{x}{r}\right).$$

We have

$$\int \widetilde{f}^2(x)\,dx = \int f^2\left(\frac{x}{r}\right)dx = r^n \int f^2(y)\,dy$$

and

$$\int |\nabla \widetilde{f}|^2 dx = \int r^{-2} |\nabla f|^2 \left(\frac{x}{r}\right) dx = r^{n-2} \int |\nabla f|^2(y)\,dy,$$

whence it follows that

$$\mathcal{R}(\widetilde{f}) = r^{-2}\mathcal{R}(f)$$

and, hence,

$$\lambda_{\min}(B_r) = r^{-2}\lambda_{\min}(B_1) = c_n r^{-2},$$

where $c_n := \lambda_{\min}(B_1)$. Note that $c_n > 0$ by the first part.

11.26. (a) The neighborhood U can be chosen sufficiently small, in particular, so that $\overline{U}$ is contained in a chart with coordinates $x^1, ..., x^n$. Then we have in U

$$|\nabla \psi|_{\mathbf{g}}^2 = g^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} \leq C\, |\nabla \psi|_{Eucl}^2$$

and

$$d\mu = \sqrt{\det g}\,dx \leq C dx,$$

so that the problem amounts to the case of the Euclidean metric and the Lebesgue measure. Hence, we can assume that $M = \mathbb{R}^n$ and o is the origin.

Let us construct a function $\psi \in Lip_0(U)$ such that $\psi = 1$ in a neighborhood of o and

$$\int_U |\nabla\psi|^2 \, dx < \varepsilon. \tag{B.205}$$

If $n \geq 3$ then choose $r > 0$ so that $B_{3r} \subset U$, and set

$$\psi(x) = \min\left(1, \frac{(2r - |x|)_+}{r}\right) = \begin{cases} 1, & |x| \leq r, \\ 0, & |x| \geq 2r, \\ \frac{2r - |x|}{r}, & r < |x| < 2r. \end{cases}$$

This function is Lipschitz with the Lipschitz constant $\frac{1}{r}$, which implies by Theorem 11.3 $|\nabla\psi| \leq \frac{1}{r}$ and, hence,

$$\int_U |\nabla\psi|^2 \, dx \leq \frac{1}{r^2}\mu(B_{2r}) = c_n r^{n-2}.$$

Choosing r sufficiently small, we obtain (B.205).

If $n = 2$ then choose $R > r > 0$ so that $B_{2R} \subset U$, and set

$$\psi(x) = \min\left(1, \frac{(\log R/|x|)_+}{\log R/r}\right) = \begin{cases} 1, & |x| \leq r, \\ 0, & |x| \geq R, \\ \frac{\log R/|x|}{\log R/r}, & r < |x| < R. \end{cases}$$

For this function, we have

$$\int_U |\nabla\psi|^2 \, d\mu = \int_r^R \pi s \left|\psi'(s)\right|^2 ds = \frac{\pi}{\log^2 R/r} \int_r^R s \frac{1}{s^2} ds = \frac{\pi}{\log R/r}.$$

Choosing the ratio R/r sufficiently large, we obtain (B.205).

Finally, having constructed ψ, consider its smooth approximation $\psi * \varphi_\varepsilon$ where φ is a mollifier in $\mathbb{R}^n$. By Lemma 2.4 and Theorem 2.13, $\psi * \varphi_\varepsilon \to \psi$ as $\varepsilon \to 0$ where the convergence is both uniform and in the norm of $W^1(U)$. Clearly, for small enough ε, function $\psi * \varphi_\varepsilon$ belongs to $C_0^\infty(U)$, is equal to 1 in a neighborhood of o, and its energy integral is sufficiently small. Renaming $\psi * \varphi_\varepsilon$ to ψ, we finish the proof.

(b) By Exercise 10.6, we have

$$\lambda_{\min}(M \setminus \{o\}) \geq \lambda_{\min}(M),$$

so we only have to prove the opposite inequality. By Theorem 10.8, it suffices to show that, for any non-zero $f \in C_0^\infty(M)$ and any $\varepsilon > 0$ there non-zero exists $g \in C_0^\infty(M \setminus \{o\})$ such that

$$\mathcal{R}(g) \leq \mathcal{R}(f) + \varepsilon.$$

Let U be a small neighborhood of o and ψ be a cutoff function of $\{o\}$ in U, which exists by part (a). Then function $g = (1 - \psi) f$ vanishes in a neighborhood of o and, hence, belongs to $C_0^\infty(M \setminus \{o\})$. For this function, we have

$$\|g\|_{L^2}^2 = \int_M (1 - \psi)^2 f^2 d\mu \geq \int_{M \setminus U} f^2 d\mu$$

and

$$\|\nabla g\|_{L^2} = \|(1-\psi)\nabla f - f\nabla\psi\|_{L^2} \le \|\nabla f\|_{L^2} + \sup|f|\,\|\nabla\psi\|_{L^2}.$$

By closing U sufficiently small, we obtain

$$\|g\|_{L^2} \ge \|f\|_{L^2} - \varepsilon,$$

and by choosing ψ with sufficiently small energy integral as in part (a), we obtain

$$\|\nabla g\|_{L^2} \le \|\nabla f\|_{L^2} + \varepsilon,$$

whence the claim follows.

(c) Let $M = \mathbb{S}^1$ so that $\lambda_{\min}(M) = 0$. The set $\mathbb{S}^1 \setminus \{o\}$ is isometric to the open interval $I = (0, 2\pi)$, while $\lambda_{\min}(I) > 0$.

11.27. Set $\rho(x) = d(x, x_0)$ and show that $e^{-\beta\rho} \in L^1(M)$ for any $\beta > \alpha$, which will imply the claim by Theorem 11.19. Indeed, fix $\gamma > 1$ and consider the balls $B_k = B\left(x_0, \gamma^k\right)$. Then we have

$$\int_{M\setminus B_0} e^{-\beta\rho} d\mu = \sum_{k=1}^{\infty} \int_{B_k\setminus B_{k-1}} e^{-\beta\rho} d\mu \le \sum_{k=1}^{\infty} e^{-\beta\gamma^{k-1}}\mu(B_k). \tag{B.206}$$

Using the hypothesis (11.64), we obtain, for large enough k,

$$\mu(B_k) \le \exp\left(\alpha'\gamma^k\right),$$

for any prescribed $\alpha' > \alpha$. Hence, for large enough k,

$$e^{-\beta\gamma^{k-1}}\mu(B_k) \le \exp\left(-\gamma^{k-1}\left(\beta - \alpha'\gamma\right)\right).$$

Since $\beta > \alpha$, we can choose α' close enough to α and γ close enough to 1 to ensure that $\beta - \alpha'\gamma > 0$, which implies the convergence of the series in (B.206). Hence, $e^{-\beta\rho} \in L^1(M)$, which was to be proved.

11.28. Note that a model based on $\mathbb{R}^n$ is complete by Exercise 11.1. The condition (11.65) implies that, for any $\varepsilon > 0$ and for all large enough r,

$$\frac{S'(r)}{S(r)} \le \alpha + \varepsilon.$$

Integrating this inequality, we obtain

$$S(r) \le Ce^{(\alpha+\varepsilon)r},$$

and, hence,

$$V(r) := \int_0^r S(t)\,dt \le C' + Ce^{(\alpha+\varepsilon)r},$$

where $V(r)$ is the volume function of M. This obviously implies

$$\limsup_{r\to\infty} \frac{1}{r}\log V(r) \le \alpha,$$

and we conclude by Exercise 11.27 that

$$\lambda_{\min}(M) \le \frac{\alpha^2}{4}.$$

To prove the lower bound for $\lambda_{\min}(M)$, observe that by Exercise 11.26,

$$\lambda_{\min}(M) = \lambda_{\min}(M'),$$

where $M' = M \setminus \{o\}$. Consider the function $\rho(x) = |x|$ in M'. In the polar coordinates (r, θ), we have $\rho = r$ and, by (3.93) or (8.42),

$$\Delta_\mu \rho = \rho'' + \frac{S'}{S}\rho' = \frac{S'}{S}.$$

By the first condition in (11.65), we obtain $\Delta_\mu \rho \geq \alpha'$ and, by Theorem 11.17,

$$\lambda_{\min}(M') \geq \frac{(\alpha')^2}{4},$$

which was to be proved.

Solutions to Chapter 12

12.1. As in the proof of Theorem 12.1, we can replace M by a relatively compact open set Ω. Besides, by the same approximation argument, we can assume that $f \in C_0^\infty(\Omega)$. Since

$$\|P_t^\Omega f\|_\infty \leq \|f\|_\infty =: a,$$

the range of all functions $P_t^\Omega f$, $t \geq 0$, is within a bounded interval $[0, a]$. Since Φ' is bounded on this interval, say by C, we obtain, for all $s, t \geq 0$,

$$\left|\Phi\left(P_t^\Omega f\right) - \Phi\left(P_t^\Omega f\right)\right| \leq C\left|P_t^\Omega f - P_t^\Omega f\right|.$$

Since $P_t^\Omega f \overset{L^1}{\to} P_t^\Omega f$ as $s \to t$ we conclude that $J(t)$ is continuous in $t \geq 0$.

It is suffices to show that $J'(t) \leq 0$ for $t > 0$. Setting $u = P_t^\Omega f$ and differentiating in t (which is justified as in the proof of Theorem 12.1), we obtain

$$\begin{aligned}
J'(t) &= \int_M \left(\Phi'(u)\frac{\partial u}{\partial t} + \Phi(u)\frac{\partial \xi}{\partial t}\right) e^\xi d\mu \\
&\leq \int_M \left(\Phi'(u)\Delta_\mu u - \frac{\Phi(u)}{4\delta}|\nabla \xi|^2\right) e^\xi d\mu \\
&= -\int_M \left(\Phi''(u)|\nabla u|^2 + \Phi'(u)\langle \nabla u, \nabla \xi\rangle + \frac{\Phi(u)}{4\delta}|\nabla \xi|^2\right) e^\xi d\mu \\
&\leq -\int_M \left(\Phi''(u)|\nabla u|^2 + \Phi'(u)|\nabla u||\nabla \xi| + \frac{\Phi(u)}{4\delta}|\nabla \xi|^2\right) e^\xi d\mu.
\end{aligned}$$

In the brackets we have a quadratic polynomial of $|\nabla u|$ and $|\nabla \xi|$, which is non-negative because by (12.10)

$$(\Phi')^2 \leq 4\Phi''\frac{\Phi(u)}{4\delta}.$$

Hence, $J' \leq 0$, which finishes the proof.

12.2. Using $|\nabla d(\cdot, A)| \leq 1$, we obtain

$$\frac{\partial \xi}{\partial t} + \frac{1}{2}|\nabla \xi|^2 \leq 0.$$

By Theorem 12.1, we conclude that the function

$$J(t) := \int_M (P_t f)^2 e^{\xi(t,\cdot)} d\mu$$

satisfies the inequality

$$J(t) \leq J(0) \exp(-2\lambda_{\min}(M)t).$$

If $x \in A$ then $\xi(0, x) = 0$ whence we obtain

$$J(0) = \int_A f^2 d\mu = \|f\|_2^2.$$

If $x \in B$ then $d(x, A) \geq d(B, A)$, which implies

$$J(t) \geq \int_B (P_t f)^2 e^{\xi(t,\cdot)} d\mu \geq \exp\left(\alpha d(A, B) - \frac{\alpha^2}{2}t\right) \int_B (P_t f)^2 d\mu.$$

Combining the above three lines, we obtain

$$\int_B (P_t f)^2 d\mu \leq \|f\|_2^2 \exp\left(\frac{\alpha^2}{2}t - \alpha d(A, B) - 2\lambda_{\min}(M)t\right).$$

Setting here $\alpha = d(A, B)/t$ we finish the proof.

12.3. (a) Left to the reader

(b) Replacing f and g by functions $f\mathbf{1}_A$ and $g\mathbf{1}_B$ where A and B are compact subsets of $\operatorname{supp} f$ and $\operatorname{supp} g$, respectively, we reduce the question to the case when f and g have compact supports. Next, approximating f and g by C_0^∞ functions with close L^2-norms and supports, we see that it suffices to prove (12.18) for $f, g \in C_0^\infty(M)$. Let us use the cos-wave operator $C_t = \cos\left(t\mathcal{L}^{1/2}\right)$ from Exercise 4.52. By Exercise 7.20, the function $C_t f$ is C^∞ smooth in $\mathbb{R} \times M$ and solves in $\mathbb{R} \times M$ the wave equation. By part (a), the support of $C_t f$ is contained in the closed $|t|$-neighborhood of $\operatorname{supp} f$. It follows that if $0 \leq s < r$ then $\operatorname{supp} C_s f$ is disjoint with $\operatorname{supp} g$ whence $(C_s f, g) = 0$. Therefore, the integration in s in the transmutation formula (B.100) can be reduced to $s \in [r, +\infty)$, that is,

$$(P_t f, g) = \int_r^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right)(C_s f, g)\, ds.$$

Using $|(C_s f, g)| \leq \|f\|_2 \|g\|_2$, we obtain (12.18).

12.4. By Theorem 8.18, it suffices to show that if u is a bounded solution of the heat equation in $(0, T) \times M$ and $u(t, \cdot) \to 0$ as $t \to 0$ then $u \equiv 0$. Applying the estimate (12.39) of Corollary 12.11 with $B = B(x, r_k)$, $A = B(x, r_k/2)$, $R = r_k/2$ and using (12.42) we obtain

$$\int_{B(x,r_k)} u^2(t, \cdot)\, d\mu \leq \exp\left(Cr_k^2\right) \|u\|_{L^\infty}^2 \max\left(\frac{r_k^2}{8t}, 1\right) \exp\left(-\frac{r_k^2}{8t} + 1\right).$$

Choose t small enough so that $\frac{1}{8t} > C$. Then letting $r_k \to \infty$, we obtain that the right hand side goes to 0, whence it follows that $u(\cdot,t) \equiv 0$ on M. Repeating the same argument with a shift of time, we obtain in the end that $u(\cdot,t) \equiv 0$ for all $t \in (0,T)$.

12.5. (*a*) Fix any compact subset K of B^c and consider the function

$$u(t,\cdot) = P_t(f 1_K).$$

By Theorem 7.10, u is a smooth solution to the heat equation in $(0,+\infty)\times M$. Moreover, since $f1_K \in L^2(M)$, we have by Theorem 4.9 that $u(t,\cdot) \overset{L^2(M)}{\to} f1_K$ as $t \to 0$ whence it follows that $u(t,\cdot) \overset{L^2(B)}{\to} 0$. Applying the inequality (12.39) of Corollary 12.11 (see also Remark 12.10) and using $\|u\|_{L^\infty} \le \|f\|_{L^\infty}$, we obtain

$$\int_A P_t(f1_K)^2\, d\mu(x) \le \mu(B)\, \|f\|^2_{L^\infty} \max\left(\frac{R^2}{2t},1\right) \exp\left(-\frac{R^2}{2t}+1\right).$$

Exhausting B^c by a sequence of compact sets K, we obtain (12.43).

(*b*) Applying (12.43) with $f = 1_{B^c}$ and then using the Cauchy-Schwarz inequality, we obtain (12.44).

Solutions to Chapter 13

13.1. (*a*) By Exercise 10.11, $p_t(x,y) \to \frac{1}{\mu(M)}$ as $t \to \infty$ whence

$$g(x,y) = \int_0^\infty p_t(x,y)\, dt = \infty.$$

(*b*) If $h(x)$ is a fundamental solution at x_0 then, for any $\varphi \in C_0^\infty(M)$,

$$\int_M h(x)(\Delta_\mu \varphi)\, d\mu = -\varphi(x_0).$$

If M is compact then setting here $\varphi \equiv 1$ we obtain a contradiction since the left hand side vanishes, while the right hand side is equal to 1.

13.2. (*a*) By Exercise 3.24, the function $h(x)$ is harmonic in $M \setminus \{o\}$. Let us show that $h \in L^1(B_R)$. Since $h(x)$ depends only on $r = |x|$, we will write $h(r)$ instead of $h(x)$. Then we have

$$\begin{aligned}\int_{B_R} h\, d\mu &= \int_0^R h(r) S(r)\, dr = \int_0^R S(r) \int_r^R \frac{dt}{S(t)} \\ &= \int_0^R \frac{dt}{S(t)} \int_0^t S(r)\, dr. \end{aligned} \tag{B.207}$$

In the bounded range of r, $S(r)$ is of the order r^{n-1} where $n = \dim M$. Therefore,

$$\frac{1}{S(t)} \int_0^t S(r)\, dr \simeq \frac{t^n}{t^{n-1}} = t,$$

whence the convergence of the integral (B.207) follows.

Finally, let us show that $-\Delta_\mu h = \delta_o$, that is, for any $\varphi \in C_0^\infty(B_R)$,

$$\int_M (\Delta_\mu \varphi)\, h\, d\mu = -\varphi(o).$$

Indeed, using the Green formula (3.97) from Exercise 3.25, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{B_R\setminus B_\varepsilon} (\Delta_\mu \varphi)\, h\, d\mu &= \int_{B_R\setminus B_\varepsilon} \varphi \Delta_\mu h\, d\mu + \int_{S_R} (\varphi_r h - \varphi h_r)\, d\mu_{S_R} \\ &\quad - \int_{S_\varepsilon} (\varphi_r h - \varphi h_r)\, d\mu_{S_\varepsilon}.\end{aligned}$$

Since $\Delta_\mu h = 0$ in $B_R \setminus B_\varepsilon$ and $\varphi = \varphi_r = 0$ on S_R, we obtain

$$\begin{aligned}\int_{B_R\setminus B_\varepsilon} (\Delta_\mu \varphi)\, h\, d\mu &= -\int_{S_\varepsilon} (\varphi_r h - \varphi h_r)\, d\mu_{S_\varepsilon} \\ &= -h(\varepsilon) \int_{S_\varepsilon} \varphi_r d\mu_{S_\varepsilon} + h_r(\varepsilon) \int_{S_\varepsilon} \varphi\, d\mu_{S_\varepsilon}. \qquad \text{(B.208)}\end{aligned}$$

As $\varepsilon \to 0$, we have

$$\left| \int_{S_\varepsilon} \varphi_r d\mu_{S_\varepsilon} \right| \le C S(\varepsilon),$$

where C is the supremum of $|\nabla \varphi|$ in a neighborhood of 0. Since

$$h(\varepsilon) S(\varepsilon) = S(\varepsilon) \int_\varepsilon^R \frac{dr}{S(r)} \simeq \begin{cases} \varepsilon, & n > 2, \\ \varepsilon \log \frac{R}{\varepsilon}, & n = 2, \end{cases},$$

it follows that

$$\lim_{\varepsilon \to 0} h(\varepsilon) \int_{S_\varepsilon} \varphi_r d\mu_{S_\varepsilon} = 0.$$

On the other hand, using that $h_r(\varepsilon) = -\frac{1}{S(\varepsilon)}$ and

$$\int_{S_\varepsilon} \varphi d\mu_{S_\varepsilon} \sim \varphi(o) S(\varepsilon) \text{ as } \varepsilon \to 0,$$

we obtain that

$$\lim_{\varepsilon \to 0} h_r(\varepsilon) \int_{S_\varepsilon} \varphi\, d\mu_{S_\varepsilon} = -\varphi(o).$$

It follows from (B.208) that

$$\int_M (\Delta_\mu \varphi)\, h\, d\mu = \lim_{\varepsilon \to 0} \int_{B_R\setminus B_\varepsilon} (\Delta_\mu \varphi)\, h\, d\mu = -\varphi(o),$$

which was to be proved.

(*b*) In $\mathbb{R}^n$ we have $S(r) = \omega_n r^{n-1}$ (cf. (3.91)). Taking $R = 1$, we obtain from part (*a*) the following fundamental solution

$$h(x) = \int_{|x|}^1 \frac{dr}{\omega_n r^{n-1}} = \begin{cases} \frac{1}{\omega_n (n-2)|x|^{n-2}} - \frac{1}{\omega_n (n-2)}, & n \ne 2, \\ \frac{1}{2\pi} \ln \frac{1}{|x|}, & n = 2. \end{cases}$$

Adding a constant in the case $n \neq 2$, we see that also the function

$$h(x) = \frac{1}{\omega_n (n-2) |x|^{n-2}}$$

is a fundamental solution at the origin (cf. (13.5)).

More generally, if the radius of a model manifold is ∞ and

$$\int^{\infty} \frac{dr}{S(r)} < \infty$$

then also the function

$$h(x) = \int_{|x|}^{\infty} \frac{dr}{S(r)} \tag{B.209}$$

is a fundamental solution at the pole because it differs from $\int_{|x|}^{R} \frac{dr}{S(r)}$ by an additive constant.

In $\mathbb{H}^n$ we have $S(r) = \omega_n \sinh^{n-1} r$ whence we obtain the fundamental solution at the pole:

$$h(x) = \frac{1}{\omega_n} \int_{|x|}^{\infty} \frac{dr}{\sinh^{n-1} r}.$$

13.3. By Corollary 8.12, we have $p_t(x,y) > 0$ for all $x, y \in M$, whence we conclude by (13.3) that $g(x,y) > 0$.

13.4. By computation (13.15) from the proof of Lemma 13.5, we have the identity

$$Gp_t(x_0, \cdot)|_x = \int_M g(x,y)\, p_t(x_0,y)\, d\mu(y) = \int_t^{\infty} p_s(x,x_0)\, ds,$$

for all $x, x_0 \in M$. By switching x_0 and x, we obtain

$$P_t g(x_0, \cdot)|_x = \int_M p_t(x,y)\, g(x_0,y)\, d\mu(y) = \int_t^{\infty} p_s(x_0,x)\, ds.$$

Observing that $p_s(x_0,x) = p_s(x,x_0)$, we obtain the desired identity $P_t g(x_0,\cdot) = Gp_t(x_0,\cdot)$.

13.5. Function $g(x,y)$ is always measurable on $M \times M$ by (13.3). As it was shown in the proof of Theorem 13.4, for any compact $K \subset M$, the function

$$x \mapsto \int_K g(x,y)\, d\mu(y)$$

is locally bounded. This implies that $g \in L^1_{loc}(M \times M)$ and, hence, g can be considered as a distribution on $M \times M$. The Laplace operator on $M \times M$ is $\Delta_x + \Delta_y$, where Δ_x and Δ_y are both the Laplace operator on M with respect to the variables x and y, respectively. Since away from diag

$$\Delta_x g(x,y) = \Delta_y g(x,y) = 0,$$

we conclude that the function $g(x,y)$ is harmonic in (x,y) in $M \times M \setminus g$ and, hence, g is C^∞ smooth in this domain.

13.6. Since the Green operator G in the inverse to the Dirichlet Laplace operator $\mathcal{L}$, it follows

$$\operatorname{spec} G \subset [0, \frac{1}{\lambda_{\min}}],$$

whence

$$\|G\| \leq \frac{1}{\lambda_{\min}}.$$

Alternatively, this can be seen directly from the definition (13.1) which implies that

$$\|G\| \leq \int_0^\infty \|P_t f\| dt \leq \int_0^\infty e^{-\lambda_{\min} t} \|f\| dt = \frac{1}{\lambda_{\min}} \|f\|,$$

where we have also used (10.31).

13.7. Using the estimate (13.19) of Exercise 13.6 and the fact that $1 \in L^2(M)$, we obtain

$$\int_M \int_M g(x,y)\, d\mu(x)\, d\mu(y) = (G1, 1) \leq \frac{1}{\lambda_{\min}} \|1\|_{L^2}^2 < \infty,$$

whence the claim follows.

13.8. This follows from the fact that $p_t^{\Omega_k}(x,y) \uparrow p_t(x,y)$ (cf. Exercise 7.40) and (13.3), (13.4).

13.9. Use the resolvent $R_\alpha f$ and observe that

$$R_\alpha f = \int_0^\infty e^{-\alpha t} (P_t f)\, dt \nearrow \int_0^\infty (P_t f)\, dt = Gf$$

as $\alpha \downarrow 0$. By Exercise 5.23, we have, for any $\alpha > 0$,

$$\begin{aligned} R_\alpha f &\leq R_\alpha^\Omega f + \operatorname*{esup}_{M \setminus K} R_\alpha f \\ &\leq G^\Omega f + \operatorname*{esup}_{M \setminus K} Gf. \end{aligned}$$

Letting $\alpha \to 0$ we obtain (13.20).

13.10. (a) Let V be an open neighborhood of x_0 in Ω such that $\varphi \equiv 1$ in V, and let Ω' be a relatively compact open neighborhood of $\overline{\Omega}$ such that $M \setminus \overline{\Omega'}$ is non-empty. By Theorem 10.22, $\lambda_{\min}(\Omega') > 0$, and by Theorem 13.4, $g^{\Omega'}$ is finite. Since the function $g^{\Omega'}(x_0, \cdot)$ is continuous in $\Omega' \setminus \{x_0\}$, it follows that

$$C := \sup_{\Omega \setminus V} g^{\Omega'}(x_0, \cdot) < \infty.$$

Setting for simplicity $v = g^\Omega(x_0, \cdot)$ and using $g^\Omega \leq g^{\Omega'}$, we obtain that $v \leq C$ in $\Omega \setminus V$. It follows that

$$(1 - \varphi)\, v \equiv (1 - \varphi) \min(v, C).$$

Indeed, in V we have $\varphi = 1$ so that the both side vanish, while outside V we have $v = \min(v, C)$. By Corollary 13.6, we have $\min(v, C) \in W_0^1(\Omega)$, whence also $(1 - \varphi) \min(v, C)$ and, hence, $(1 - \varphi)\, v$ are in $W_0^1(\Omega)$.

REMARK. The hypothesis that $M \setminus \overline{\Omega}$ is non-empty is used twice: to ensure that $\lambda_{\min}(\Omega) > 0$ and to ensure the boundedness of $g^{\Omega}(x_0, \cdot)$ away from V via comparison with $g^{\Omega'}(x_0, \cdot)$. However, the latter is true always, as one will see from Exercise 13.31. Hence, the hypothesis of non-emptiness of $M \setminus \overline{\Omega}$ can be relaxed to the assumption $\lambda_{\min}(\Omega) > 0$.

(b) Let φ be a cutoff function of $\{x_0\}$ in U. Set

$$u = g^{\Omega}(x_0, \cdot) - g^{U}(x_0, \cdot)$$

so that u is a harmonic function in U. Observing that

$$u = \varphi u + (1 - \varphi) u,$$

where $\varphi u \in C_0^{\infty}(\Omega)$ and $(1-\varphi) u \in W_0^1(\Omega)$ by part (a), we obtain that $u \in W_0^1(\Omega)$.

13.11. Consider function

$$\psi(s) = (\min(s, b) - a)_+$$

so that

$$v(x) = a + \psi(g(x_0, x)).$$

Hence, it suffices to prove that the function $u = \psi(g(x_0, \cdot))$ belongs to $W^1(M)$ and

$$\|\nabla u\|_{L^2}^2 \le b - a. \tag{B.210}$$

As in the proof of Corollary 13.6, construct a sequence $\{\psi_k\}$ of smooth functions on $[0, +\infty)$ satisfying (13.12) and such that $0 \le \psi_k' \le 1$ and $\psi_k(s) \uparrow \psi(s)$ as $k \to \infty$. Indeed, it suffices to choose $\psi_k' \in C_0^{\infty}(a, b)$ so that $\psi_k' \uparrow 1$ on (a, b) as $k \to \infty$, and then define ψ_k by integration of ψ_k'. By Lemma 13.5, we conclude that $u_k := \psi_k(g(x_0, \cdot)) \in W_0^1$ and

$$\|\nabla u_k\|_{L^2}^2 \le \int_0^{\infty} \left|\psi_k'(s)\right|^2 ds \le \sup \psi_k' \sup \psi_k \le \sup \psi = b - a.$$

Passing to the limit as $k \to \infty$, we obtain (B.210).

13.12. Let $\{\Omega_k\}$ be a compact exhaustion of M such that $x_0 \in \Omega_k$ for all k. Since M is non-compact, we have by Theorem 10.22 $\lambda_{\min}(\Omega_k) > 0$. Setting

$$u_k = \min\left(g^{\Omega_k}(x_0, \cdot), c\right)$$

we obtain by Corollary 13.6 that $u_k \in W_0^1(\Omega_k)$ and

$$\|\nabla u_k\|_{L^2(\Omega_k)}^2 \le c.$$

Fix some index m. By Exercise 13.8, $u_k(x) \uparrow u(x)$ as $k \to \infty$ for all $x \in \Omega_m$. Since u is bounded in Ω_m and, hence, $u \in L^2(\Omega_m)$, we obtain by Exercise 4.18 that $u \in W^1(\Omega_m)$ and

$$\|\nabla u\|_{L^2(\Omega_m)}^2 \le c.$$

Letting $m \to \infty$, we finish the proof.

13.13. By Theorem 10.13, the spectrum of the Dirichlet Laplace operator $\mathcal{L}^\Omega$ is discrete; let it be the sequence $\{\lambda_k\}_{k=1}^\infty$ in the increasing order so that $\lambda_{\min}(\Omega) = \lambda_1$.

Let φ be the eigenfunction of λ_1, normalized in $L^2(\Omega)$. By Theorem 10.11, we can assume that $\varphi(x) > 0$ in Ω. If $\lambda_1 = 0$ then by Exercise 10.12

$$p_t^\Omega(x, y) \to \varphi(x)\varphi(y) \text{ as } t \to \infty.$$

which implies $g^\Omega(x, y) \equiv \infty$. Hence, the estimate (13.21) is trivially satisfied.

Assume in the sequel that $\lambda_1 > 0$. Then, by Theorem 13.4, the Green function g^Ω is finite. By Exercise 10.13, the function φ is bounded. Let us renormalize φ so that

$$\sup_{x\in\Omega} \varphi(x) = 1.$$

For any $x \in \Omega$, we have then

$$G^\Omega\varphi(x) = \int_\Omega g^\Omega(x, y)\varphi(y)\, d\mu(y) \le \int_\Omega g^\Omega(x, y)\, d\mu(y). \tag{B.211}$$

On the other hand, we have

$$\mathcal{L}^\Omega\varphi = \lambda_1\varphi.$$

Since by Theorem 13.4, G^Ω is the inverse of $\mathcal{L}^\Omega$ in $L^2(\Omega)$, applying here G^Ω, we obtain

$$\varphi = \lambda_1 G^\Omega\varphi.$$

Combining with (B.211), we obtain

$$\varphi(x) \le \lambda_1 \int_\Omega g^\Omega(x, y)\, d\mu(y).$$

Finally, taking sup in $x \in \Omega$, we obtain (13.21).

13.14. Note that by Theorem 10.13 the spectrum of Ω is discrete, and $\lambda_1 > 0$ by Theorem 10.22. Then the Green function g^Ω is finite by Theorem 13.4. By Exercise 13.7, we have $g^\Omega \in L^1(\Omega \times \Omega)$ so that g^Ω can be considered as a distribution on $\Omega \times \Omega$. We need to prove that, for any $f \in C_0^\infty(\Omega \times \Omega)$

$$\left(g^\Omega, f\right) = \sum_{k=1}^\infty \frac{1}{\lambda_k}\left(\varphi_k \otimes \varphi_k, f\right), \tag{B.212}$$

where the brackets stand for the pairing of functions in $\Omega \times \Omega$ and

$$\left(\varphi_k \otimes \varphi_k\right)(x, y) = \varphi_k(x)\varphi_k(y).$$

Fix a function $f(x, y) \in C_0^\infty(\Omega \times \Omega)$ and consider for any $x \in \Omega$ the Fourier expansion

$$f(x, y) = \sum_{k=1}^\infty c_k(x)\varphi_k(y), \tag{B.213}$$

where

$$c_k(x) = \int_M f(x,y)\varphi_k(y)\, d\mu(y).$$

By Exercise 10.16, for any fixed $x \in \Omega$, the series in (B.213) converges to $f(x,y)$ absolutely and uniformly in $y \in \Omega$. Moreover, as it is remarked in Solution to Exercise 10.16, the convergence is uniform both in x and y in Ω provided the norm

$$\|\Delta_\mu^m f(x,\cdot)\|_{L^2(\Omega)}$$

is uniformly bounded in x for any positive integer m, which is clearly the case because $f \in C_0^\infty(\Omega \times \Omega)$.

Since the Green operator G^Ω is a bounded inverse to $\mathcal{L}^\Omega$, we have for any function $\psi \in L^2$ with Fourier series $\psi = \sum_k a_k\varphi_k$ that

$$\int_\Omega g^\Omega(\cdot,z)\psi(z)\, d\mu(z) = G^\Omega\psi = \sum_k a_k G^\Omega\varphi_k = \sum \frac{a_k}{\lambda_k}\varphi_k,$$

where the convergence is in $L^2(\Omega)$. Applying this for $\psi = f(x,\cdot)$ with a fixed $x \in \Omega$, we obtain the identity

$$\int_\Omega g^\Omega(y,z)f(x,z)\, d\mu(z) = \sum_{k=1}^\infty \frac{c_k(x)}{\lambda_k}\varphi_k(y), \tag{B.214}$$

where the convergence is in $L^2(\Omega)$ with respect to the variable y. However, since the series $\sum_{k=1}^\infty \frac{c_k(x)}{\lambda_k}\varphi_k(y)$ converges uniformly in x and y, its sum is a continuous function of x and y. Since also $G^\Omega f(x,\cdot)$ is also continuous by Theorem 13.4, we conclude that the identity (B.214) holds pointwise, and the convergence is uniform jointly in $x, y \in \Omega$.

Setting in (B.214) $x = y$ and integrating in x, we obtain

$$\int_\Omega\int_\Omega g^\Omega(x,z)f(x,z)\, d\mu(x)\, d\mu(z) = \sum_{k=1}^\infty \frac{1}{\lambda_k}\int_\Omega\int_\Omega f(x,y)\varphi_k(x)\varphi_k(y)\, d\mu(x)\, d\mu(y),$$

which is equivalent to (B.212).

13.15. (*a*) If $P_t f_k \le f_k$ and by the monotone convergence theorem $P_t f_k \to P_t f$, it follows that $P_t f \le f$.

(*b*) For any $i \in I$, we have $P_t f \le P_t f_i \le f_i$, whence

$$P_t f \le \inf_{i\in I} f_i = f.$$

13.16. (*a*) By definition, the inequality $P_t f \le f$ is satisfied almost everywhere. Since the both functions $P_t f$ and f are continuous, it follows that this inequality is satisfied pointwise. To prove that $P_t f(x) \to f(x)$ pointwise, fix a point $x \in M$ and choose a cutoff function φ of the point x. Since $f\varphi \in C_b(M)$, we have by Theorem 7.16 that $P_t(f\varphi) \to f\varphi$ pointwise as $t \to 0$. In particular, it follows that

$$P_t(f\varphi)(x) \to f\varphi(x) = f(x).$$

Since
$$P_t(f\varphi)(x) \le P_t f(x) \le f(x),$$
it follows that $P_t f(x) \to f(x)$.

(*b*) Let $\inf f = m$. Since M is stochastically complete, we have $P_t 1 = 1$, whence
$$P_t f \ge P_t m = m.$$
If $f(x) = m$ at some point x then by part (*a*) we have $P_t f(x) \le f(x) = m$, whence $P_t f(x) = m$. Since $P_t f$ solves the heat equation in $\mathbb{R}_+ \times M$ (cf. Exercise 7.30), we conclude by the strong parabolic minimum principle of Theorem 8.11, that $P_t f \equiv m$. Letting $t \to 0$, we obtain $f \equiv m$.

(*c*) Use the same argument as in the proof of Corollary 8.16.

13.17. Fix $x_0 \in M$. By (13.15) and Exercise 13.4, we have
$$P_t g(x_0, \cdot) = G p_t(x_0, \cdot) = \int_t^\infty p_s(x_0, \cdot)\, ds \le g(x_0, \cdot).$$
Hence, $g(x_0, \cdot)$ is superaveraging.

13.18. (*a*) By (13.24), the function $v = u - f$ belongs to $W_0^1(\Omega)$. Extending v by setting $v = 0$ outside Ω, we obtain $v \in W_0^1(M)$. Obviously, we have the identity $\widetilde{f} = f + v$ on M. Hence, $f \in W_0^1(M)$ implies $\widetilde{f} \in W_0^1(M)$.

(*b*) Since $f \ge 0$ and $\lambda_{\min}(\Omega) > 0$, by Theorem 5.13 we obtain from (13.24) that $u \ge 0$. Hence, $\widetilde{f} \ge 0$.

Since f is superaveraging, we have $\Delta_\mu f \le 0$, whence
$$-\Delta_\mu(u - f) = -\Delta_\mu u + \Delta_\mu f \le 0 \text{ in } \Omega.$$
Since $u - f \in W_0^1(\Omega)$, Theorem 5.13 yields that $u - f \le 0$. Therefore, $\widetilde{f} \le f$ in M. In particular, we have $\widetilde{f} \in L^2(M)$.

Let us show that $P_t\widetilde{f} \le \widetilde{f}$. Outside Ω this is true because
$$P_t\widetilde{f} \le P_t f \le f = \widetilde{f}.$$
To prove that $P_t\widetilde{f} \le \widetilde{f}$ inside Ω, observe that the both function $w_1(t, \cdot) = P_t\widetilde{f}$ and $w_2(t, \cdot) = \widetilde{f} = u$ solve the heat equation in $\mathbb{R}_+ \times \Omega$:
$$\frac{dw}{dt} - \Delta_\mu w = 0,$$
which is understood in the sense of Theorem 5.16, that is, $w(t, \cdot)$ is a path in $W^1(\Omega)$, $\frac{dw}{dt}$ is the strong derivative in $L^2(\Omega)$, and $\Delta_\mu w$ is understood in the distributional sense. Moreover, the both functions satisfy the same initial condition
$$w(t, \cdot) \overset{L^2(\Omega)}{\longrightarrow} u \text{ as } t \to 0.$$
Furthermore, these functions satisfy the conditions
$$w_2(t, \cdot) = f \mod W_0^1(\Omega)$$
while
$$w_1(t, \cdot) = P_t\widetilde{f} \le P_t f \le f,$$

whence it follows that

$$w_1(t,\cdot) - w_2(t,\cdot) \le 0 \mod W_0^1(\Omega).$$

Hence, we obtain by Theorem 5.16 that $w_1 \le w_2$, that is, $P_t\widetilde{f} \le u$ in Ω, which finishes the proof.

13.19. Assume first that $h \in W_0^2$. Then by the Green formula of Lemma 4.4,

$$(-\Delta_\mu P_t f, h) = (P_t f, -\Delta_\mu h) \le (f, -\Delta_\mu h) = (\nabla f, \nabla h),$$

where we have used that[3] $-\Delta_\mu h \ge 0$ and $P_t f \le f$.

For a general $h \in W_0^1$, the function $P_s h$ belongs to W_0^2 for any $s > 0$ and is superaveraging, because for any $t > 0$

$$P_t(P_s h) = P_s(P_t h) \le P_s h.$$

By the above argument, we have

$$(-\Delta_\mu P_t f, P_s h) \le (\nabla f, \nabla P_s h).$$

Since $P_s h \overset{W^1}{\to} h$ as $s \to 0$ (cf. Exercise 4.45), we can pass here to the limit and obtain (13.25).

13.20. The condition $\Delta_\mu u_k = 0$ is equivalent to

$$(\nabla u_k, \nabla v)_{L^2} = 0 \text{ for any } v \in W_0^1(\Omega_k) \tag{B.215}$$

(cf. the proof of Theorem 4.5). Setting here $v = u_k - f$, we obtain

$$\|\nabla u_k\|^2 = (\nabla u_k, \nabla f) \le \|\nabla u_k\| \|\nabla f\|, \tag{B.216}$$

whence

$$\|\nabla u_k\| \le \|\nabla f\|. \tag{B.217}$$

Since $f \in W_0^1(M)$, there is a sequence of functions $f_k \in C_0^\infty(M)$ such that $f_k \overset{W^1}{\longrightarrow} f$. Passing to a subsequence, we can assume that $\operatorname{supp} f_k \subset \Omega_k$. Choosing $v = f_k$ in (B.215) and using (B.217), we obtain

$$\begin{aligned}(\nabla u_k, \nabla f) &= (\nabla u_k, \nabla(f - f_k)) + (\nabla u_k, \nabla f_k) \\ &\le \|\nabla u_k\| \|\nabla(f - f_k)\| + 0 \\ &\le \|\nabla f\| \|\nabla(f - f_k)\|.\end{aligned}$$

Combining with (B.216), we conclude that

$$\|\nabla u_k\|^2 \le \|\nabla f\| \|\nabla(f - f_k)\| \to 0 \text{ as } k \to \infty,$$

which was to be proved.

13.21. Since $\lambda_{\min}(\Omega_k) > 0$, the weak Dirichlet problem in Ω_k

$$\begin{cases} \Delta_\mu u_k = 0, \\ u_k = h \mod W_0^1(\Omega_k), \end{cases}$$

[3] If $h \in W_0^2$ is superaveraging then the fact that $\Delta_\mu h \le 0$ follows also from

$$\Delta_\mu h = L^2\text{-}\lim_{t\to 0} \frac{P_t h - h}{t}$$

(cf. Exercise 4.40).

has a unique solution $u_k \in W^1(\Omega_k)$. Consider the following function

$$h_k = \begin{cases} h & \text{in } M \setminus \Omega_k, \\ u_k & \text{in } \Omega_k. \end{cases}$$

By Exercise 13.18, $h_k \in W_0^1(M)$ and h_k is superaveraging. Since $-\Delta_\mu P_t f \geq 0$, we obtain

$$\int_{M \setminus \Omega_k} (-\Delta_\mu P_t f)\, h d\mu \leq \int_M (-\Delta_\mu P_t f)\, h_k d\mu.$$

By Exercise 13.19 we have, for all $t > 0$ and $k \in \mathbb{N}$,

$$(-\Delta_\mu P_t f, h_k) \leq (\nabla f, \nabla h_k) \leq \|\nabla f\| \|\nabla h_k\|,$$

and by Exercise 13.20,

$$\|\nabla h_k\| \to 0 \text{ as } k \to \infty.$$

Combining the above three lines, we finish the proof.

13.22. By Theorem 13.10, there is a constant C_n such that, for any positive harmonic function h in $B(0,1)$,

$$\sup_{B(0,1/2)} h \leq C_n \inf_{B(0,1/2)} h\ . \tag{B.218}$$

If f is a positive harmonic function in $B(x_0, r)$ then the function

$$h(x) = f(x_0 + rx)$$

is positive and harmonic in $B(0,1)$. Hence, (13.33) follows from (B.218).

13.23. Let f be a positive harmonic function in $\mathbb{R}^n$. Renaming $f - \inf f$ by f, we can assume that $\inf f = 0$. Applying the Harnack inequality (13.33) of Exercise 13.22 in any ball $B(x,r)$ with fixed x and $r \to \infty$, we obtain $\sup f = 0$, whence $f \equiv 0$.

13.24. If M is compact then, by Exercise 13.1, $g(x,y) \equiv \infty$ so that the claim is satisfied. Assume in the sequel that M is non-compact. It suffices to prove that if $g(x_0, y_0) < \infty$ for some $x_0, y_0 \in M$, then $g(x, y_0) < \infty$ for all $x \neq y_0$. Then using the symmetry of the Green function and applying further this claim, we obtain $g(x,y) < \infty$ for all $x \neq y$.

Let $\{\Omega_k\}$ be a compact exhaustion sequence of M such $x_0, y_0 \in \Omega_1$. Since M is connected, Ω_k can also be chosen to be connected. By Theorem 10.22, we have $\lambda_{\min}(\Omega_k) > 0$, and by Theorem 13.4, the Green function g^{Ω_k} is finite. By Exercise 13.8, $g^{\Omega_k}(x,y) \uparrow g(x,y)$ as $k \to \infty$ for all $x, y \in M$. Fix some $m \geq 1$ and consider for any $k > m$ the following function

$$u_{km}(x) = g^{\Omega_k}(x, y_0) - g^{\Omega_m}(x, y_0)\,.$$

Clearly, $\{u_k\}$ is an increasing sequence of harmonic functions in Ω_m, which is bounded at point x_0. By the Harnack principle (Corollary 13.13), we have

$$\lim_{k \to \infty} u_k(x) < \infty \text{ for all } x \in \Omega_m,$$

whence it follows that

$$g(x,y_0)=\lim_{k\to\infty}u_k(x)+g^{\Omega_m}(x,y_0)<\infty$$

for all $x\in\Omega_m\setminus\{y_0\}$. Letting $m\to\infty$ we obtain that $g(x,y_0)<\infty$ for all $x\neq y_0$.

13.25. Set $\lambda=\lambda_{\min}(M)$. By Exercise 12.3, we have the estimate (12.18) but without the term $e^{-\lambda t}$. By Theorem 13.16, there is a positive solution h of the equation $\Delta_\mu h+\lambda h=0$ on M. Consider the new measure $\widetilde{\mu}$ defined by $d\widetilde{\mu}=h^2\,d\mu$. By Theorem 9.15, the heat semigroup $\widetilde{P}_t$ of the weighted manifold $(M,\widetilde{\mu})$ is related to P_t by

$$\widetilde{P}_t=e^{\lambda t}\frac{1}{h}\circ P_t\circ h.$$

Applying the estimate (12.18) of Exercise 12.3 to $\widetilde{P}_t$, we obtain that, for all $f,g\in C_0^\infty(M)$,

$$\left|(\widetilde{P}_tf,g)_{L^2(\widetilde{\mu})}\right|\leq\|f\|_{L^2(\widetilde{\mu})}\|g\|_{L^2(\widetilde{\mu})}\int_r^\infty\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{s^2}{4t}\right)ds.$$

Noticing that

$$(\widetilde{P}_tf,g)_{L^2(\widetilde{\mu})}=\int_M e^{\lambda t}\frac{1}{h}P_t(hf)\,g\;d\widetilde{\mu}=e^{\lambda t}\int_M P_t(hf)\,hg\,d\mu$$

and

$$\|f\|_{L^2(\widetilde{\mu})}=\|fh\|_{L^2(\mu)},$$

we can rename everywhere fh and gh by f and g, respectively, and obtain

$$e^{\lambda t}(P_tf,g)_{L^2(\mu)}\leq\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}\int_r^\infty\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{s^2}{4t}\right)ds,$$

which is equivalent to (13.36). Finally, this estimate extends to arbitrary $f,g\in L^2(M,\mu)$ in the same way as it was done in Exercise 12.3.

13.26. (a) By Theorem 10.22, $\lambda_1(\Omega)>0$ so that by Theorem 13.4 the Green function g^Ω is finite. Modifying the computation from the proof of that lemma, we have, for all $x\in\Omega$ and $s>0$,

$$\begin{aligned}G^\Omega 1(x)&=\int_\Omega g^\Omega(x,y)\,d\mu(y)\\&=\int_0^s\int_\Omega p_t^\Omega(x,y)\,d\mu(y)\,dt+\int_\Omega\int_s^\infty p_t^\Omega(x,y)\,dt\,d\mu(y)\\&\leq s+\int_\Omega\int_s^\infty\sqrt{p_s(x,x)\,p_s(y,y)}\exp(-\lambda(t-s))\,dt\,d\mu(y)\\&\leq s+C\lambda^{-1}\mu(\Omega),\end{aligned}$$

where $\lambda = \lambda_1(\Omega)$ and

$$C = \sup_{z \in \overline{\Omega}} p_s(z,z) < \infty.$$

Hence, $E := \|G^\Omega 1\|_\infty < \infty$. Then by Exercise 2.20, G^Ω is a bounded operator in any space L^r, $1 \le r \le +\infty$, and $\|G^\Omega\|_{L^r \to L^r} \le E$.

Let us give also independent proofs for the boundedness of G^Ω in L^∞, L^2, L^1. For any $f \in L^\infty(\Omega)$, we have

$$\|G^\Omega f\|_\infty \le \|G^\Omega 1\|_\infty \|f\|_\infty = E\|f\|_\infty$$

so that $\left\|G^\Omega\right\|_{L^\infty \to L^\infty} \le E$.

The fact that G^Ω is a bounded operator in $L^2(\Omega)$ follows from Theorem 13.4 because by $G^\Omega = \left(\mathcal{L}^\Omega\right)^{-1}$, the spectrum of G^Ω is contained in $[0,\lambda]$.

The boundedness in $L^1(\Omega)$ follows from

$$\begin{aligned} \|G^\Omega f\|_{L^1} &= \int_M \left| \int_M g^\Omega(x,y) f(y)\, d\mu(y) \right| d\mu(x) \\ &\le \int_M \left(\int_M g^\Omega(x,y)\, d\mu(x) \right) |f(y)|\, d\mu(y) \\ &\le E \int_M |f(y)|\, d\mu(y) = E\|f\|_{L^1}. \end{aligned}$$

(*b*) If $f \in L^2(\Omega)$ then the fact that $u = G^\Omega f$ solves the equation $-\Delta_\mu u = f$ follows from Lemma 13.1. Assume now that $f \in L^1(\Omega)$, which is the general case, because $L^p(\Omega) \subset L^1(\Omega)$ for any $p \ge 1$. Then the function $f_k = (f \wedge k) \vee (-k)$ is in $L^2(\Omega)$ and, hence, the function $u_k = G^\Omega f_k$ satisfies the equation $-\Delta_\mu u_k = f_k$. Since $f_k \overset{L^1}{\to} f$ and, hence, $u_k \overset{L^1}{\to} u$, we obtain in the limit $-\Delta_\mu u = f$.

13.27. We use the fact that if $\{u_k\}$ is an increasing sequence of L^1_{loc} functions such that $-\Delta_\mu u_k = f$ and if $u := \lim_{k\to\infty} u_k$ is finite then $u \in L^1_{loc}$ and $-\Delta_\mu u = f$. Indeed, the differences $v_k := u_k - u_1$ are harmonic functions, and the result follows from the Harnack principle for harmonic functions (Corollary 13.13).

Let $\{\Omega_k\}$ be a compact exhaustion sequence in M. Set $u_k = G^{\Omega_k} f$. Since $f \in L^1(\Omega_k)$, by Exercise 13.26 we have $u_k \in L^1(\Omega_k)$ and $-\Delta_\mu u_k = f$. By the above remark, we conclude that the function

$$u := Gf = \lim_{k \to \infty} u_k$$

belongs to L^1_{loc} and satisfies the equation $-\Delta_\mu u = f$.

13.28. By enlarging K, we can assume that x_0 is an interior point of K. Let us first consider the following special case. Let Ω be a relatively compact open set containing K and such that $M \setminus \overline{\Omega}$ is non-empty. Let $v \in W^1(\Omega)$ be a harmonic function in Ω such that

$$v(x) \le g^\Omega(x_0, x) \text{ for all } x \in \Omega \setminus K.$$

We claim that $v \leq 0$ in Ω. Choose a cutoff function φ of $\{x_0\}$ in K and a cutoff function ψ of K in Ω, set $C = \sup_K v$, and show that

$$v \leq (1-\varphi)\, g^{\Omega}(\cdot, x_0) + C\psi \text{ in } \Omega. \tag{B.219}$$

Indeed, in K we have $\psi = 1$, and (B.219) holds by $v \leq C$. In $\Omega \setminus K$ we have $1-\varphi = 1$, and (B.219) holds by $v \leq g^{\Omega}(\cdot, x_0)$. By Exercise 13.10, the function $(1-\varphi)\, g^{\Omega}(\cdot, x_0)$ belongs to $W_0^1(\Omega)$. Since also $C\psi \in W_0^1(\Omega)$, it follows from (B.219) that

$$v \leq 0 \mod W_0^1(\Omega).$$

Since $\lambda_{\min}(\Omega) > 0$, we conclude by the weak maximum principle of Theorem 5.13, that $v \leq 0$ in Ω.

Returning to the general case, let $\{\Omega_k\}$ be a compact exhaustion sequence of M such that $K \subset \Omega_k$ for all k. Since M is non-compact, $M \setminus \overline{\Omega}_k$ is non-empty for all k. We claim that, for any index k and any $\varepsilon > 0$, there exists m large enough such that

$$g - g^{\Omega_m} \leq \varepsilon \text{ in } \Omega_k \setminus K. \tag{B.220}$$

Indeed, the function $g^{\Omega_m} - g^{\Omega_{k+1}}$ is harmonic in $\Omega_{k+1} \setminus \{x_0\}$ and, as $m \to \infty$, converges pointwise monotonically to $g - g^{\Omega_{k+1}}$. Since the limit function is continuous in $\Omega_{k+1} \setminus \{x_0\}$, the convergence is locally uniform in $\Omega_{k+1} \setminus \{x_0\}$ by the Dini theorem.[4] In particular, the convergence is uniform on $\overline{\Omega}_k \setminus K$, whence (B.220) follows.

Using (B.220) and the hypothesis $u \leq g$ in $M \setminus K$, we obtain that, for the above m and k,

$$v := u - \varepsilon - g^{\Omega_m} + g^{\Omega_k} \leq g^{\Omega_k} \text{ in } \Omega_k \setminus K.$$

The function v is harmonic in Ω_k and belongs to $W^1(\Omega_k)$ because so do $u - \varepsilon$ and $g^{\Omega_m} - g^{\Omega_k}$ (cf. Exercise 13.10). We conclude by the above special case that $v \leq 0$ in Ω_k, that is,

$$u \leq \varepsilon + g^{\Omega_m} - g^{\Omega_k} \text{ in } \Omega_k.$$

Replacing g^{Ω_m} by its upper bound g, letting $k \to \infty$ and then $\varepsilon \to 0$, we obtain $u \leq 0$ in M, which was to be proved.

13.29. Set

$$u(x) = g(x, x_0) - h(x).$$

Then $u(x)$ is a harmonic function in M, and

$$\limsup_{x_k \to \infty} u(x_k) \geq 0,$$

for any sequence $\{x_k\}$ such that $x_k \to \infty$ in M. By Exercise 8.5, we conclude that $u \geq 0$ in M. Hence, $g(x, x_0) \geq h(x)$.

If we knew a priori that h is non-negative then the opposite inequality would hold by Theorem 13.17. In the general case, we use again that $h(x) \to$

[4]By the Harnack principle, we have in fact that $g^{\Omega_m} - g^{\Omega_{k+1}} \to g - g^{\Omega_{k+1}}$ in $C^{\infty}(\Omega_{k+1})$ (cf. the proof of Theorem 13.17), but here we need much less.

0, which means that, for any $\varepsilon > 0$ there is a compact $K \subset M$ such that $|h(x)| \leq \varepsilon$ in $M \setminus K$. It follows that

$$u(x) - \varepsilon \leq g(x, x_0) \text{ for all } x \in M \setminus K. \tag{B.221}$$

Since $u - \varepsilon$ is harmonic, we conclude by Exercise (13.28) that $u - \varepsilon \leq 0$ on M. Since ε is arbitrary, we obtain $u \leq 0$ and $g(x, x_0) \leq h(x)$, which finishes the proof.

13.30. $(i) \Rightarrow (ii)$ If the Green function is finite then, by Theorem 13.17, $g(x, \cdot) \in L^1_{loc}(M)$. Therefore, for any non-negative $\varphi \in C_0^\infty$, the function $f(x) = G\varphi$ is finite. By Exercise 13.27, $f \in L^1_{loc}$ and $-\Delta_\mu f = \varphi$. It follows that f is a smooth non-negative superharmonic function. If $\varphi \not\equiv 0$ then f is non-constant.

The implication $(ii) \Rightarrow (iii)$ follows from the fact that any positive superharmonic function is also superaveraging (see Exercise 7.29).

$(iii) \Rightarrow (i)$ Let f be a non-constant positive superaveraging function. Choose a constant c strictly between $\operatorname{einf} f$ and $\operatorname{esup} f$ and set $\widetilde{f} = \min(f, c)$. By Exercise 13.15, $\widetilde{f}$ is also superaveraging. Moreover, we claim that in fact a strict inequality holds

$$P_t \widetilde{f} < \widetilde{f} \text{ a.e.}$$

for any $t > 0$. Indeed, by the choice of c, the strict inequality $\widetilde{f} < f$ holds on a set of positive measure. It follows that, for all $t > 0$ and $x \in M$,

$$P_t \widetilde{f}(x) = \int_M p_t(x, y) \widetilde{f}(y)\, d\mu(y) < \int_M p_t(x, y) f(y)\, d\mu(y) = P_t f \leq f.$$

In the same way, the strict inequality $\widetilde{f} < c$ holds on a set of positive measure whence

$$P_t \widetilde{f} < P_t c \leq c.$$

It follows that

$$P_t \widetilde{f} < \min(f, c) = \widetilde{f}.$$

For simplicity of notation, let us now rename $\widetilde{f}$ to f so that $P_t f < f$. Consider a truncated Green function

$$g_T(x, y) = \int_0^T p_t(x, y)\, dt,$$

where $T \in (0, +\infty)$, and obtain some upper bound for g_T, which would be uniform in T. We have

$$\begin{aligned}\int_M g_T(\cdot, y)(f - P_t f)(y)\, d\mu(y) &= \int_0^T P_s(f - P_t f)\, ds \\ &= \int_0^T P_s f\ ds - \int_0^T P_s(P_t f)\, ds \\ &= \int_0^T P_s f\ ds - \int_0^T P_{s+t} f\ ds \\ &= \int_0^T P_s f\ ds - \int_t^{T+t} P_s f\ ds \\ &\le \int_0^t P_s f\ ds.\end{aligned}$$

Since the right hand side here is independent of T, we can let $T \to \infty$ and obtain

$$\int_M g(\cdot, y)(f - P_t f)(y)\, d\mu(y) < \infty.$$

Since the function $f - P_t f$ is strictly positive, it follows that the Green function $g(x, y)$ is finite, which was to be proved.

13.31. If M is compact then $g \equiv \infty$ and there is nothing to prove. Hence, we can assume that M is non-compact. By switching to a connected component, we can also assume that Ω is connected.

Fix once and for all a point $y \in \Omega$. Let us first prove (13.40) in the case $x \in \Omega$. For that consider in Ω the function

$$u(x) = g(x, y) - g^{\Omega}(x, y),$$

that is harmonic in Ω and is bounded by $g(x, y)$. In particular, we have

$$\limsup_{x \to \partial\Omega} u(x) \le a := \sup_{z \in \partial\Omega} g(z, y).$$

It follows from the maximum principle of Exercise 8.5) that $u \le a$ in Ω, which proves (13.40) in the case $x \in \Omega$.

If $x \in \partial\Omega$ then (13.40) is trivially satisfied, so we are left to treat the case $x \notin \overline{\Omega}$. In this case $g^{\Omega}(x, y) = 0$ and (13.40) amounts to

$$\sup_{x \in M \setminus \overline{\Omega}} g(x, y) \le \sup_{x \in \partial\Omega} g(x, y). \tag{B.222}$$

Let $\{\Omega_k\}_{k=1}^{\infty}$ be a compact exhaustion of M such that $\Omega \Subset \Omega_1$. We will show that, for any index k,

$$\sup_{x \in \Omega_k \setminus \overline{\Omega}} g^{\Omega_k}(x, y) \le \sup_{x \in \partial\Omega} g^{\Omega_k}(x, y),$$

whence (B.222) will follow by letting $k \to \infty$. By Theorem 10.22, we have $\lambda_{\min}(\Omega_k) > 0$, whence g^{Ω_k} is finite by Theorem 13.4. Set

$$b := \sup_{x \in \Omega_k \setminus \overline{\Omega}} g^{\Omega_k}(x, y)$$

and observe that $b < \infty$ because $g^{\Omega_k} \leq g^{\Omega_{k+1}}$ and $g^{\Omega_{k+1}}$ is continuous in $\Omega_k \setminus \overline{\Omega}$. Consider in Ω_k the function

$$u(x) = \min\left(g^{\Omega_k}(x, y), b\right).$$

Clearly, $u = g^{\Omega_k}(\cdot, y)$ in $\Omega_k \setminus \overline{\Omega}$ and, hence, function u is harmonic in $\Omega_k \setminus \overline{\Omega}$. By Corollary 13.6, we have $u \in W_0^1(\Omega_k)$. Clearly, $u \in C(\Omega_k \setminus \{y\})$. We conclude by Exercise 7.8, applied to manifold Ω_k and its subset $\Omega_k \setminus \overline{\Omega}$ (which is the exterior of a compact set $\overline{\Omega}$) that

$$\sup_{\Omega_k \setminus \overline{\Omega}} u = \sup_{\partial\Omega} u.$$

Since the left hand side is equal to b, we obtain

$$b = \sup_{\partial\Omega} u \leq \sup_{\partial\Omega} g^{\Omega_k}(\cdot, y),$$

which was to be proved.

13.32. We show that a fundamental solution exists at any given point $x_0 \in M$. Let $\{\Omega_k\}_{k=1}^{\infty}$ be a compact exhaustion sequence in M such that all Ω_k are connected. Since $M \setminus \overline{\Omega_k} \neq \emptyset$, by Theorems 10.22 and 13.4, the weighted manifold Ω_k has a finite Green function g^{Ω_k}, which is also a fundamental solution of the Laplacian in Ω_k.

We can assume that x_0 belongs to all sets Ω_k. Fix another point x_0' that also belongs to all Ω_k and set

$$c_k = g^{\Omega_k}\left(x_0', x_0\right).$$

Consider sequence of functions

$$f_k(x) = g^{\Omega_k}(x, x_0) - c_k$$

so that f_k is a fundamental solution in Ω_k at the point x_0, and in addition $f_k(x_0') = 0$. Fix some index $m \in \mathbb{N}$ and consider functions

$$h_k(x) = f_k(x) - f_m(x).$$

Clearly, for any $k > m$, the function $h_k(x)$ is harmonic in Ω_m and vanishes at x_0'. By the compactness principle (Theorem 13.12), there is a subsequence $\{h_{k_i}\}$ that converges to a harmonic function in Ω_m in the sense $C^\infty(\Omega_m)$. Therefore, the sequence $\{f_{k_i}\}$ converges to a fundamental solution in Ω_m in the sense of distributions in Ω_m. Using the diagonal process, the subsequence $\{f_{k_i}\}$ can be chosen so that it converges to a fundamental solution in any Ω_m. This implies that its limit is defined on all of M and is a fundamental solution on M.

Note that if M is non-parabolic then by Exercise 13.30 the Green function is finite and, hence, is a (positive) fundamental solution. The above argument proves the existence of a signed fundamental solution on parabolic manifolds.

13.33. Assume the contrary that M is stochastically complete, that is,

$$\int_M p_t(x,y)\, d\mu(y) = 1$$

for all $t > 0$. Integrating in t from 0 to ∞ and using the definition (13.3) of the Green function, we obtain

$$\int_M g(x,y)\, d\mu(y) = \int_M \int_0^\infty p_t(x,y)\, d\mu(y) = \int_0^\infty \int_M p_t(x,y)\, d\mu(y) = \infty.$$

Since $g(x,\cdot)$ is locally integrable (see the proof of Theorem 13.17), we have

$$\int_K g(x,y)\, d\mu(y) < \infty$$

whence

$$\int_{M\setminus K} g(x,y)\, d\mu(y) = \infty,$$

which contradicts the hypothesis.

13.34. Fix some $R > 0$ and consider in B_R the function

$$h_R(x) = \int_{|x|}^R \frac{ds}{S(s)}. \tag{B.223}$$

By Exercise 13.2, this function is a fundamental solution in B_R of the Laplace operator at the pole o. Since h_R vanishes at ∂B_R, we conclude by Exercise 13.29 that $h(x) = g^{B_R}(x,o)$. Letting $R \to \infty$ and using that $g^{B_R} \to g$ as $R \to \infty$ we obtain from (B.223) that

$$g(x,o) = \int_r^\infty \frac{ds}{S(s)}, \tag{B.224}$$

which was to be proved.

To answer the last question, write

$$\int_M g(\cdot,o)\, d\mu = \int_0^\infty S(r) \int_r^\infty \frac{ds}{S(s)} = \int_0^\infty \frac{V(s)}{S(s)} ds$$

where $V(s) = \int_0^s S(r)\, dr$. Hence, the Green function $g(\cdot,o)$ belongs to $L^1(M)$ if and only if

$$\int_0^\infty \frac{V(s)}{S(s)} ds < \infty.$$

At 0 this integral is always convergent because $\frac{V(s)}{S(s)} = O(s)$ as $s \to \infty$. The convergence at ∞ occurs, for example, if, for large s, $V(s) = \exp\left(s^{2+\varepsilon}\right)$ with $\varepsilon > 0$.

Remark. Note that (B.224) remains valid also in the case when the integral in the right hand side diverges as in this case $g = \infty$.

13.35. Let $h(x)$ be the standard fundamental solution of the Laplace operator in $\mathbb{R}^n$, that is,

$$h(x) = \begin{cases} \frac{1}{\omega_n (n-2)} |x-y|^{2-n}, & n \neq 2, \\ \frac{1}{2\pi} \log \frac{1}{|x-y|}, & n = 2 \end{cases}$$

(cf. Exercise 13.2).

(a) Fix a point $y \in B$. Note that the right hand side of (13.44) coincides with the following function

$$f(x) = h(x-y) - \left(\frac{R}{|y|}\right)^{n-2} h(x-y^*).$$

Since y^* lies outside $\overline{B}$, the function $h(x-y^*)$ is harmonic in B. Therefore, we have $-\Delta_\mu f = \delta_y$, that is, f is a fundamental solution in B. Let us verify that $f = 0$ on ∂B. For any $x \in \partial B$, we have $|x| = R$, whence, by the definition of y^*,

$$\frac{|x|}{|y|} = \frac{|y^*|}{|x|}. \tag{B.225}$$

It follows that the triangles $0xy$ and $0y^*x$ are similar, whence

$$\frac{|x-y^*|}{|x-y|} = \frac{|x|}{|y|} = \frac{R}{|y|}. \tag{B.226}$$

Clearly, (B.226) implies that $f(x) = 0$ for $x \in \partial B$. By Exercise 13.29, we conclude that $g(x,y) = f(x)$ for all $x \in B$.

Solutions to (b) and (c) are similar to (a).

13.36. Since

$$g(x,y) = \int_0^\infty p_t(x,y)\,dt,$$

it suffices to prove that

$$J := \int_0^\infty \frac{1}{F(\sqrt{t})} \exp\left(-c\frac{r^2}{t}\right) dt \simeq \int_r^\infty \frac{sds}{F(s)}. \tag{B.227}$$

Split the integral into the sum

$$J = J_0 + J_\infty,$$

where

$$J_0 = \int_0^{r^2} \frac{1}{F(\sqrt{t})} \exp\left(-c\frac{r^2}{t}\right) dt, \tag{B.228}$$

$$J_\infty = \int_{r^2}^\infty \frac{1}{F(\sqrt{t})} \exp\left(-c\frac{r^2}{t}\right) dt. \tag{B.229}$$

Step 1. Let us show that

$$J_\infty \simeq \int_r^\infty \frac{sds}{F(s)}. \tag{B.230}$$

Indeed, observe that in the integral (B.229) $r^2/t \leq 1$, which implies

$$J_\infty \simeq \int_{r^2}^{\infty} \frac{1}{F\left(\sqrt{t}\right)} dt,$$

and the rest follows by the change $s = \sqrt{t}$.

Step 2. Let us show that

$$J_0 \leq C \frac{r^2}{F(r)}. \tag{B.231}$$

By (13.45) there are constants $\beta > 0$ and $b > 0$ such that

$$\frac{F(s)}{F(s')} \leq b \left(\frac{s}{s'}\right)^\beta, \quad \text{for all } s > s' > 0,$$

so that, for all $t \leq r^2$,

$$\frac{F(r)}{F\left(\sqrt{t}\right)} \leq b \left(\frac{r}{\sqrt{t}}\right)^\beta.$$

Substituting into (B.228) and changing in the integral $\tau = t/r^2$, we see that (B.231) amounts to

$$\int_0^1 \tau^\beta \exp\left(-\frac{1}{\tau}\right) d\tau \leq C,$$

which is obviously true.

Step 3. Let us show that

$$\frac{r^2}{F(r)} \leq A \int_r^\infty \frac{sds}{F(s)}. \tag{B.232}$$

Indeed, using (13.45), we have

$$\int_r^\infty \frac{sds}{F(s)} \geq \int_r^{2r} \frac{sds}{F(s)} \geq A^{-1} \int_r^{2r} \frac{sds}{F(r)} \geq A^{-1} \frac{r^2}{F(r)}.$$

Combining (B.230), (B.231), (B.232), we obtain (13.2).

To prove the second claim, let us show that, under the condition (13.47),

$$J \simeq \frac{r^2}{F(r)}.$$

The lower bound follows from (B.232). For the upper bound, it suffices to show that

$$J_\infty \leq C \frac{r^2}{F(r)}$$

since J_0 satisfies a similar estimate by (B.231). By (B.230), it suffices to prove that

$$\int_r^\infty \frac{sds}{F(s)} \leq C \frac{r^2}{F(r)},$$

which by (13.47) amounts to

$$\int_r^\infty \left(\frac{r}{s}\right)^\alpha sds \le Cr^2,$$

and which is true by $\alpha > 2$.

Solutions to Chapter 14

14.1. By Theorem 14.2, $\{P_t\}$ is $L^2 \to L^{p^*}$ ultracontractive with the rate function $\theta(t)$. Hence, for any $f \in L^p \cap L^2$, we have

$$\|P_t f\|_{p^*} = \|P_{t/2}\left(P_{t/2} f\right)\|_{p^*} \le \theta(\frac{t}{2})\|P_{t/2} f\|_2 \le \theta^2(\frac{t}{2})\|f\|_p,$$

whence the claim follows.

14.2. Let $\{\Omega_k\}$ be a compact exhaustion sequence of Ω. Then we have

$$\lambda_{\min}(\Omega_k) \ge \Lambda(\mu(\Omega_k)) \ge \Lambda(\mu(\Omega)).$$

Letting $k \to \infty$ and using Exercise 10.6, we obtain $\lambda_{\min}(\Omega) \ge \Lambda(\mu(\Omega))$.

14.3. If $u \in C_0^\infty(\Omega)$ then by the Cauchy-Schwarz inequality

$$\|u\|_1^2 \le \mu(\Omega)\|u\|_2^2$$

whence

$$\frac{\int_M |\nabla u|^2 d\mu}{\int_M u^2 d\mu} \ge \Lambda\left(\frac{\|u\|_1^2}{\|u\|_2^2}\right) \ge \Lambda(\mu(\Omega)).$$

The rest follows from the variational principle of Theorem 10.8.

14.4. Left to the reader

14.5. Let $V(r) = \mu(B(x,r))$. Then using the Lipschitz cutoff function φ of $B(x,r/2)$ in $B(x,r)$ (see Exercise 11.6) as a test function in the variational property of the first eigenvalue, we obtain

$$\begin{aligned} V(r/2) &\le \int_{B(x,r)} \varphi^2 d\mu \le \lambda_{\min}(B(x,r))^{-1} \int_{B(x,r)} |\nabla\varphi|^2 d\mu \\ &\le \frac{16}{r^2}\left(aV(r)^{-2/\nu}\right)^{-1} V(r) \end{aligned}$$

whence

$$V(r) \ge c\left(ar^2 V(r/2)\right)^\theta,$$

where $\theta = \frac{\nu}{\nu+2}$ and $c = c(\nu) > 0$. Iterating this, we obtain

$$\begin{aligned} V(r) &\ge ca^\theta r^{2\theta} V\left(\frac{r}{2}\right)^\theta \\ &\ge c^{1+\theta} a^{\theta+\theta^2} r^{2\theta}\left(\frac{r}{2}\right)^{2\theta^2} V\left(\frac{r}{4}\right)^{\theta^2} \\ &\ge c^{1+\theta+\theta^2} a^{\theta+\theta^2+\theta^3} r^{2\theta}\left(\frac{r}{2}\right)^{2\theta^2}\left(\frac{r}{4}\right)^{2\theta^3} V\left(\frac{r}{8}\right)^{\theta^3} \\ &\ldots \\ &\ge c^{1+\theta+\theta^2+\ldots} a^{\theta\left(1+\theta+\theta^2+\ldots\right)} r^{2\theta\left(1+\theta+\theta^2+\ldots\right)} 2^{-2\theta^2\left(1+\theta+\theta^2+\ldots\right)} V\left(\frac{r}{2^k}\right)^{\theta^k}. \end{aligned}$$

Letting $k \to \infty$ and noticing that

$$V\left(\frac{r}{2^k}\right) \sim c_\nu \left(\frac{r}{2^k}\right)^\nu$$

and, hence, $V\left(\frac{r}{2^k}\right)^{\theta^k} \to 1$, we obtain

$$V(r) \geq \operatorname{const} a^{\frac{\theta}{1-\theta}} r^{\frac{2\theta}{1-\theta}} = \operatorname{const} a^{\nu/2} r^\nu .$$

14.6. Let us first prove that the Sobolev inequality (14.19) implies the Faber-Krahn inequality

$$\lambda_{\min}(\Omega) \geq c\mu(\Omega)^{-2/\nu}, \tag{B.233}$$

for any relatively compact open set $\Omega \subset M$. Indeed, set

$$p = \frac{\nu}{\nu - 2}$$

and observe that, for any $u \in C_0^\infty(\Omega)$ we have by the Hölder inequality inequality and (14.19)

$$\int_\Omega u^2 d\mu \leq \left(\int_\Omega \left(u^2\right)^p\right)^{1/p} \mu(\Omega)^{1-1/p} \leq c^{-1}\mu(\Omega)^{\frac{2}{\nu}} \int_M |\nabla u|^2 \, d\mu,$$

whence (B.233) follows.

Next, let us deduce the Sobolev inequality from (B.233). Obviously, it suffices to assume that $u \geq 0$. Any non-negative function $u \in W_0^1(M)$ can be approximated by a sequence of non-negative functions $u_k \in C_0^\infty(M)$ such that

$$\|u_k - u\|_2 \longrightarrow 0 \quad \text{and} \quad \|\nabla u_k - \nabla u\|_2 \longrightarrow 0 \quad \text{as } k \to \infty$$

(see Lemma 5.4). Choose a subsequence $\{u_{k_i}\}$ which converges to u almost everywhere. If (14.19) holds for each u_{k_i} then we can pass to the limit and obtain (14.19) for u since by Fatou's lemma

$$\|u\|_{2p} \leq \liminf_{i\to\infty} \|u_{k_i}\|_{2p} .$$

Hence, we can assume in the sequel that u is a non-negative function from $C_0^\infty(M)$. Set for any $k \in \mathbb{Z}$

$$\Omega_k = \left\{x \in M : u(x) > 2^k\right\} \quad \text{and} \quad m_k = \mu(\overline{\Omega}_k).$$

Clearly, $\Omega_{k+1} \subset \Omega_k$, and the union of all sets Ω_k is $\{u > 0\}$. Hence, we have

$$\int_M u^{2p} d\mu = \sum_{k\in\mathbb{Z}} \int_{\Omega_k \setminus \Omega_{k+1}} u^{2p} d\mu \leq 4^p \sum_{k\in\mathbb{Z}} 4^{kp} m_k. \tag{B.234}$$

For any $k \in \mathbb{Z}$, consider the function

$$u_k(x) := \left(\min(u(x), 2^{k+1}) - 2^k\right)_+ = \begin{cases} 2^k, & x \in \Omega_{k+1}, \\ u(x) - 2^k, & x \in \Omega_k \setminus \Omega_{k+1}, \\ 0, & x \notin \Omega_k, \end{cases}$$

which belongs to $W_0^1(M)$ (cf. Exercise 5.3). Since $\nabla u_k = \nabla u$ on $\Omega_k \setminus \Omega_{k+1}$ and $\nabla u_k = 0$ outside $\Omega_k \setminus \Omega_{k+1}$, we obtain

$$\int_M |\nabla u|^2 \, d\mu = \sum_{k\in\mathbb{Z}} \int_M |\nabla u_k|^2 \, d\mu. \tag{B.235}$$

Let $\widetilde{\Omega}_k$ be an relatively open neighborhood of $\overline{\Omega}_k$, such that $\mu(\widetilde{\Omega}_k) \le 2\mu\left(\overline{\Omega}_k\right)$. Since $\operatorname{supp} u_k \subset \widetilde{\Omega}_k$, we conclude by Lemma 5.5 that $u_k \in W_0^1(\widetilde{\Omega}_k)$. Applying the Faber-Krahn inequality (B.233) for $\widetilde{\Omega}_k$, we obtain

$$\int_M |\nabla u_k|^2 \, d\mu \ge \frac{a}{\mu(\widetilde{\Omega}_k)^{2/\nu}} \int_{\Omega_k} u_k^2 d\mu \ge \frac{a}{(2m_k)^{2/\nu}} 2^{2k} m_{k+1} = \frac{a}{2^{2/\nu}} \frac{4^k m_{k+1}}{m_k^{2/\nu}},$$

whence

$$\int_M |\nabla u_k|^2 \, d\mu \ge \frac{a}{2^{2/\nu}} \sum_{k\in\mathbb{Z}} \frac{4^k m_{k+1}}{m_k^{2/\nu}}. \tag{B.236}$$

For arbitrary sequences of positive numbers $\{x_k\}$, $\{y_k\}$, and $r, s > 1$ such that $1/r + 1/s = 1$, we have by the Hölder inequality

$$\sum x_k^{1/r} = \sum \left(\frac{x_k}{y_k}\right)^{1/r} y_k^{1/r} \le \left(\sum \frac{x_k}{y_k}\right)^{1/r} \left(\sum y_k^{s/r}\right)^{1/s},$$

whence

$$\sum \frac{x_k}{y_k} \ge \frac{\left(\sum x_k^{1/r}\right)^r}{\left(\sum y_k^{s/r}\right)^{r/s}} \ge \frac{\sum x_k}{\left(\sum y_k^{\alpha}\right)^{1/\alpha}},$$

where $\alpha = s/r$ can take any positive value. Therefore, we obtain

$$\sum \frac{4^k m_{k+1}}{m_k^{2/\nu}} = \sum \frac{4^{kp} m_{k+1}}{4^{k(p-1)} m_k^{2/\nu}} \ge \frac{\sum 4^{kp} m_{k+1}}{\left(\sum 4^{k(p-1)\alpha} m_k^{2\alpha/\nu}\right)^{1/\alpha}}.$$

Choosing α so that $(p-1)\alpha = p$ that is, $\alpha = \frac{p}{p-1} = \nu/2$, we obtain

$$\sum \frac{4^k m_{k+1}}{m_k^{2/\nu}} \ge \frac{4^{-p} \sum 4^{kp} m_k}{\left(\sum 4^{kp} m_k\right)^{2/\nu}} = 4^{-p} \left(\sum 4^{kp} m_k\right)^{1-2/\nu}. \tag{B.237}$$

Combining together (B.234), (B.235), (B.236), (B.237), we obtain (14.19).

14.7. It suffices to prove that

$$\left(\int_M |u|^{\gamma}\right)^{2/\gamma} \ge \left(\int_M |u|^{\alpha} \, d\mu\right)^{-a} \left(\int_M |u|^{\beta} \, d\mu\right)^{b}$$

where $\gamma = \frac{2\nu}{\nu-2}$, that is,

$$\|u\|_{\beta}^{b\beta} \le \|u\|_{\alpha}^{a\alpha} \|u\|_{\gamma}^{2}.$$

Let us use the interpolation inequality

$$\|u\|_{\beta}^{y} \le \|u\|_{\alpha}^{x} \|u\|_{\gamma}^{z},$$

which is true whenever $\alpha < \beta < \gamma$ and x, y, z are positive numbers such that

$$\begin{cases} x + z = y \\ \frac{x}{a} + \frac{z}{\gamma} = \frac{y}{\beta} \end{cases}.$$

Substituting $x = a\alpha$, $y = b\beta$, and $z = 2$, we obtain the conditions (14.22).

14.8. Each of the functions $\Lambda = \Lambda_1 + \Lambda_2$ and $\Lambda = \max(\Lambda_1, \Lambda_2)$ is obviously non-negative, monotone decreasing, and right continuous. The condition (14.23) is satisfied because in the both cases $\Lambda \geq \Lambda_1$.

14.9. Let γ be the Γ-transform of Λ. For $t < t_1 := \int_0^{v_1} \frac{dv}{v\Lambda(v)}$ we have

$$\gamma(t) = \operatorname{const} t^{1/\alpha_1} \text{ and } \frac{\gamma'}{\gamma}(t) = \frac{1}{\alpha_1 t}.$$

Clearly, γ satisfies (14.36) for $t < t_1$ with $\delta = 1/2$. It suffices to check that γ satisfies (14.36) also for large enough t with some $\delta > 0$, which will imply by Lemma 14.15 that $\gamma \in \Gamma_{\delta'}$ for some $\delta' > 0$ and, hence, $\Lambda \in \mathbf{L}_{\delta'}$.

To treat large t, assume first that $c_2, \alpha_2 > 0$. Then, for $t > t_2 := \int_0^{v_2} \frac{dv}{v\Lambda(v)}$, we obtain

$$t - t_2 = \int_{v_2}^{\gamma(t)} \frac{dv}{v\Lambda(v)} = \frac{1}{c_2\alpha_2}\left(\gamma(t)^{\alpha_2} - v_2^{\alpha_2}\right),$$

whence, for some real c,

$$\gamma(t) = \operatorname{const}(t + c)^{1/\alpha_2} \text{ and } \frac{\gamma'}{\gamma}(t) = \frac{1}{\alpha_2(t + c)}.$$

Therefore, γ satisfies (14.36) for large enough t with $\delta < 1/2$.

Let $\alpha_2 = 0$ while $c_2 > 0$, that is, $\Lambda(v) = c_2$ for $v \geq v_2$. Then we have for $t > t_2$

$$t - t_2 = \int_{v_2}^{\gamma(t)} \frac{dv}{c_2 v} = \frac{1}{c_2}\log\frac{\gamma(t)}{v},$$

whence $\frac{\gamma'}{\gamma}(t) = c_2$. Hence, γ satisfies (14.36) for all $t > t_2$ with $\delta = 1$.

Finally, let $\alpha_2 = c_2 = 0$. In this case, $\Lambda(v) = 0$ for $v > v_2$, and by (14.25) $\gamma(t) = \text{const}$ for $t > t_2$. Hence, $\frac{\gamma'}{\gamma}(t) = 0$ for $t > t_2$, which again satisfies (14.36).

14.10. (*a*) Denote for simplicity $\gamma = \gamma_\Lambda$ and $\widetilde{\gamma} = b^{-1}\gamma(at)$. Then by (14.24)

$$\widetilde{\gamma}' = ab^{-1}\gamma'(at) = ab^{-1}\gamma(at)\Lambda(\gamma(at)) = a\widetilde{\gamma}\Lambda(b\widetilde{\gamma}) = \widetilde{\gamma}\widetilde{\Lambda}(\widetilde{\gamma}),$$

whence it follows by Lemma 14.10 that $\gamma_{\widetilde{\Lambda}} = \widetilde{\gamma}$.

(*b*) Set $\widetilde{\Lambda}(v) = b\Lambda_\gamma\left(a^{-1}v\right)$. Then by part (*a*) we have

$$\gamma_{\widetilde{\Lambda}}(t) = a\gamma_\Lambda(bt) = \widetilde{\gamma}(t)$$

whence by Lemma 14.10 $\Lambda_{\widetilde{\gamma}} = \widetilde{\Lambda}$.

(c) For the function $f=\gamma_{\Lambda_2}$, we have

$$\frac{f'}{f}=\Lambda_2(f)\geq\Lambda_1(f),$$

whence by Lemma 14.13 $f(t)\geq\gamma_{\Lambda_1}(t)$ for all $t\in(0,t_0)$ where

$$t_0=\int_0^{v_0}\frac{dv}{v\Lambda_1(v)}\quad\text{and}\quad v_0=\sup\{v>0:\Lambda_1(v)>0\}.$$

If $v_0=\infty$ then $t_0=\infty$ and $\gamma_{\Lambda_2}(t)\geq\gamma_{\Lambda_1}(t)$ for all $t>0$. If $v_0<\infty$ then by the monotonicity of γ_{Λ_2}, for $t\geq t_0$ we have

$$\gamma_{\Lambda_2}(t)\geq\gamma_{\Lambda_2}(t_0)\geq\gamma_{\Lambda_1}(t_0)=v_0=\gamma_{\Lambda_1}(t).$$

14.11. For the function $\gamma=\gamma_1\gamma_2$ we have

$$\frac{\gamma'}{\gamma}=\frac{\gamma_1'}{\gamma_1}+\frac{\gamma_2'}{\gamma_2}.$$

Therefore, γ_1 and γ_2 belong to $\widetilde{\Gamma}_\delta$ then

$$\frac{\gamma'}{\gamma}(2t)-\delta\frac{\gamma'}{\gamma}(t)=\left[\frac{\gamma_1'}{\gamma_1}(2t)-\delta\frac{\gamma_1'}{\gamma_1}(t)\right]+\left[\frac{\gamma_2'}{\gamma_2}(2t)-\delta\frac{\gamma_2'}{\gamma_2}(t)\right]\geq0$$

so that $\gamma\in\widetilde{\Gamma}_\delta$. If γ_1 and γ_2 belong to Γ_δ then similarly

$$\frac{\gamma'}{\gamma}(2t)-\delta\frac{\gamma'}{\gamma}(t)\geq-\frac{2\delta^{-1}}{(1+t)^{1+\delta}},$$

whence it follows that $\gamma\in\Gamma_{\delta/2}$.

14.12. Choose function $f(t)=\frac{\gamma'(t)}{\gamma(t)}$ for large t to be a constant c_k on any interval $[2^k,2^{k+1})$. Then $\gamma\notin\Gamma_\delta$ provided for any $\delta>0$ there exists k such that

$$c_{k+1}+\frac{\delta^{-1}}{2^{k(1+\delta)}}<\delta c_k. \tag{B.238}$$

In fact, it suffices to ensure (B.238) for $\delta=\frac{1}{k}$, which will be the case if the following two inequalities hold:

$$c_{k+1}<\frac{1}{2}\frac{c_k}{k}\quad\text{and}\quad c_k>\frac{k^2}{2^k}. \tag{B.239}$$

A sequence $\{c_k\}$ that satisfies (B.239) for arbitrarily large k can be constructed inductively as follows. Set $c_1=1$. If c_l has been defined then define c_k to be equal to c_l for some values of $k=l+1,l+2,...$ until k is so big that $\frac{k^2}{2^k}<c_l$. For this k, set $c_k=c_l$ and $c_{k+1}=\frac{1}{3}\frac{c_k}{k}$.

14.13. Set

$$f(t)=\int_0^t\frac{ds}{F(s)}\quad\text{and}\quad g(t)=\frac{1}{F'(t)}.$$

Then

$$\frac{g'}{f'}=-\frac{F''F}{(F')^2}\to-c\ \text{ as }t\to\infty.$$

By l'Hospital rule,

$$\frac{g}{f} \to -c \text{ as } t \to \infty,$$

whence the claim follows.

14.14. We have for $t_0 = \gamma^{-1}(1)$

$$t - t_0 = \int_1^{\gamma(t)} \frac{dv}{v\Lambda(v)} = e \int_1^{\gamma(t)} \frac{dv}{F(v)},$$

where

$$F(v) = ev\Lambda(v) = v \exp\left(-v^\beta\right).$$

Set also $f = \log F$ so that

$$\frac{F''F}{(F')^2} = 1 + \frac{f''}{(f')^2}.$$

A simple computation shows that

$$f'(v) = \frac{1}{v} - \beta v^{\beta-1} \text{ and } f''(v) = -\frac{1}{v^2} - \beta(\beta-1)v^{\beta-2},$$

whence $\frac{f''}{(f')^2} \to 0$ and $\frac{F''F}{(F')^2} \to 1$ as $v \to \infty$. By Exercise 14.13, we conclude that

$$\int_1^s \frac{dv}{F(v)} \sim -\frac{1}{F'(s)} \sim \frac{\exp(s^\beta)}{\beta s^\beta} \quad \text{as } s \to \infty,$$

whence

$$t \sim \frac{\exp\left(\gamma(t)^\beta + 1\right)}{\beta\gamma(t)^\beta} \text{ as } t \to \infty.$$

Taking log, we obtain

$$\log t \sim \gamma(t)^\beta + 1 - \log\left(\beta\gamma(t)^\beta\right) \sim \gamma(t)^\beta,$$

whence

$$\gamma(t) \sim (\log t)^{1/\beta}.$$

14.15. We can assume that $f \geq 0$ since $|P_t f| \leq P_t |f|$. For any natural number k, consider the function

$$f_k = \min(f, k),$$

which is obviously in $L^1 \cap L^2$. By Theorem 14.19, we have

$$\|P_t f_k\|_2^2 \leq \frac{4}{\gamma(t)} \|f_k\|_1^2 \leq \frac{4}{\gamma(t)} \|f\|_1^2$$

Since $f_k \uparrow f$, we obtain by the monotone convergence theorem that

$$\begin{aligned}\|P_t f_k\|_2^2 &= (P_t f_k, P_t f_k) = (P_{2t} f_k, f_k) \\ &= \int_M \int_M p_{2t}(x,y) f_k(x) f_k(y) \, d\mu(x) \, d\mu(y) \\ &\to \int_M \int_M p_{2t}(x,y) f(x) f(y) \, d\mu(x) \, d\mu(y) \\ &= \|P_t f\|_2^2,\end{aligned}$$

whence the claim follows.

Solutions to Chapter 15

15.1. Fix $x \in B(x_0, r/2)$ and $t \in (0,T)$. Since the Faber-Krahn inequality (15.19) holds in $B(x, r/2)$, it follows from Theorem 15.1 that

$$u^2(t,x) \le \frac{Ca^{-n/2}}{\min\left(\sqrt{t}, r\right)^{n+2}} \int_{t/2}^{t} \int_{B(x,r/2)} u^2(s,y) \, d\mu(y) \, ds. \tag{B.240}$$

Applying inequality (12.39) of Theorem 12.9 with $A = B(x, r/2)$ and $B = B(x_0, R)$ and noticing that $d(A, B^c) \ge \delta$, we obtain

$$\int_{B(x,r/2)} u^2(s,y) \; d\mu(y) \le \mu(B(x_0,R)) \, \|u\|_{L^\infty}^2 \Phi\left(\frac{\delta^2}{2s}\right) \tag{B.241}$$

where

$$\Phi(\xi) = \max(\xi, 1) \exp(1 - \xi).$$

Observe that the function $\Phi(\xi)$ is decreasing in ξ. Consequently, $\Phi\left(\frac{\delta^2}{2s}\right)$ is decreasing in δ and increasing in s. Replacing in the right hand side of (B.241) s by t and substituting into (B.240), we obtain

$$u^2(t,x) \le \frac{Ca^{-n/2} t}{\min\left(\sqrt{t}, r\right)^{n+2}} \mu(B(x_0,R)) \, \|u\|_{L^\infty}^2 \Phi\left(\frac{\delta^2}{2t}\right),$$

whence (15.20) follows.

15.2. (a) Let $r(x)$ be the function from Theorem 15.4 and set $r = \min_{x \in M} r(x)$. Then in any ball $B(x,r)$ the Faber-Krahn inequality holds with function $av^{-2/n}$. Applying Theorem 15.11 to the family $\{B(x,r)\}_{x \in M}$, we obtain the claim.

It is obvious from the above argument that compact manifolds have bounded geometry so that the claim follows also from part (b).

(b) By definition of a manifold of bounded geometry, there is $r > 0$ such that all balls $B(x,r)$ are uniformly quasi-isometric to a Euclidean ball. It follows that the Faber-Krahn inequality holds in every ball $B(x,r)$ with function $av^{-2/n}$, and the rest follows from Theorem 15.11.

15.3. Setting in (15.43) $t = t_0$ and

$$F(x,s) = \min(s, r(x))^n$$

we obtain (15.51).

15.4. Setting in (15.43) $t = t_0$ and replacing $r(x)$ by $\inf r(x) > 0$ (cf. Remark 15.5), we obtain (15.52).

15.5. Integrating the heat kernel upper bound (15.49) in t and using Exercise 13.36, we obtain

$$g(x,y) \le Cd(x,y)^{2-n}. \tag{B.242}$$

Conversely, let us show that (B.242) implies the Faber-Krahn inequality with the function $\Lambda(v) = cv^{-2/n}$. Due to the estimate (13.21) of Exercise 13.13, it suffices to prove that, for any relatively compact open subset Ω of M and for all $x \in \Omega$,

$$\int_\Omega g^\Omega(x,y)\,d\mu(y) \le C\mu(\Omega)^{2/n}.$$

Since $g^\Omega \le g$ and g satisfies the upper bound of (B.242), it suffices to show that, for any fixed $x \in \Omega$,

$$\int_\Omega \rho^{2-n} d\mu \le C\mu(\Omega)^{2/n} \tag{B.243}$$

where $\rho = d(x,\cdot)$. Choose some $R > 0$ and estimate the integral in the left hand side as follows:

$$\int_\Omega \rho^{2-n} d\mu \le \int_{\Omega\cap\{\rho\ge R\}} \rho^{2-n}\,d\mu + \int_{\{\rho<R\}} \rho^{2-n}\,d\mu.$$

The first integral is bounded by

$$\int_{\Omega\cap\{\rho\ge R\}} \rho^{2-n}\,d\mu \le R^{2-n}\mu(\Omega),$$

whereas the second integral can be estimates by

$$\begin{aligned}
\int_{\{\rho<R\}} \rho^{2-n}\,d\mu &= \sum_{k-0}^{\infty} \int_{\{2^{-k-1}R\le\rho<2^{-k}R\}} \rho^{2-n}\,d\mu \\
&\le \sum_{k=0}^{\infty} \left(2^{-k-1}R\right)^{2-n} \mu\left(B\left(x,2^{-k}R\right)\right) \\
&\le C\sum_{k=0}^{\infty} \left(2^{-k}R\right)^{2-n} \left(2^{-k}R\right)^{n} \\
&= CR^2 \sum_{k=0}^{\infty} 4^{-k} \\
&\le C'R^2.
\end{aligned}$$

Hence, we obtain

$$\int_\Omega \rho^{2-n} d\mu \le R^{2-n}\mu(\Omega) + C'R^2,$$

whence (B.243) follows by setting $R = \mu(\Omega)^{1/n}$.

15.6. Using (15.50), rewrite the inequality (15.48) of Corollary 15.17 as follows:

$$p_t(x,y) \le \frac{C}{\min(t,T)^{n/2}} \exp\left(-\kappa \frac{\rho^2}{4t} - \lambda(t-T)_+\right) \tag{B.244}$$

where $t, T > 0$, and $\kappa \in (0,1)$ is arbitrary, while $C = C_{\kappa,n,c}$. Let us verify that (B.244) implies the following estimate:

$$p_t(x,y) \le \frac{C}{t^{n/2}} \exp\left(-\kappa \frac{\rho^2}{4t} - \kappa\lambda t\right), \tag{B.245}$$

where $C = C(n,\kappa,\lambda)$. Indeed, choose in (B.244) $T = 1$. If $t \le 1$ then (B.245) follows trivially from (B.244); if $t > 1$ then use the inequality

$$t^{n/2} \le C_{n,\delta} \exp(\delta t),$$

where $\delta = (1-\kappa)\lambda$.

Since $\frac{\rho^2}{4t} + \lambda t \ge \rho\sqrt{\lambda}$, it follows from (B.245) that, for any $\varepsilon \in (0,\kappa)$,

$$p_t(x,y) \le \frac{C}{t^{n/2}} \exp\left(-(\kappa-\varepsilon)\rho\sqrt{\lambda} - \varepsilon\frac{\rho^2}{4t} - \varepsilon\lambda t\right),$$

whence

$$\begin{aligned} g(x,y) &= \int_0^\infty p_t(x,y)\,dt \\ &\le C\exp\left(-(\kappa-\varepsilon)\rho\sqrt{\lambda}\right)\int_0^\infty t^{-n/2}\exp\left(-\varepsilon\frac{\rho^2}{4t} - \varepsilon\lambda t\right)dt \end{aligned} \tag{B.246}$$

If $n > 2$ then the integral in (B.246) is estimated from above by

$$\int_0^\infty t^{-n/2}\exp\left(-\varepsilon\frac{\rho^2}{4t}\right)dt = \text{const}\,\rho^{2-n},$$

where we have used (A.60). Together with (B.246), this implies (15.53) in the case $n > 2$.

Consider the case $n = 2$. Then we split the integral in (B.246) into two parts: from 0 to ρ and from ρ to ∞. Making change $s = s^2/t$, we estimate the first part as follows:

$$\begin{aligned} \int_0^\rho t^{-1}\exp\left(-\varepsilon\frac{\rho^2}{4t} - \varepsilon\lambda t\right)dt &\le \int_0^\rho t^{-1}\exp\left(-\varepsilon\frac{\rho^2}{4t}\right)dt \\ &= \int_\rho^\infty s^{-1}\exp\left(-\varepsilon\frac{s}{4}\right)ds, \end{aligned} \tag{B.247}$$

while for the second part we use the trivial estimate

$$\int_\rho^\infty t^{-1}\exp\left(-\varepsilon\frac{\rho^2}{4t} - \varepsilon\lambda t\right)dt \le \int_\rho^\infty t^{-1}\exp(-\varepsilon\lambda t)\,dt. \tag{B.248}$$

Since the integrals in the right hand sides of (B.247) and (B.248) are similar, it suffices to estimate one of them, say (B.247). If $\rho \geq 1$ then we have

$$\int_{\rho}^{\infty} s^{-1} \exp\left(-\varepsilon \frac{s}{4}\right) ds \leq \int_{1}^{\infty} s^{-1} \exp\left(-\varepsilon \frac{s}{4}\right) ds =: c,$$

while for $\rho < 1$ we have

$$\int_{\rho}^{\infty} s^{-1} \exp\left(-\varepsilon \frac{s}{4}\right) ds \leq c + \int_{\rho}^{1} s^{-1} ds = c + \log \frac{1}{\rho}.$$

Combining these estimates together, we obtain

$$\int_{0}^{\infty} t^{-1} \exp\left(-\varepsilon \frac{\rho^2}{4t} - \varepsilon \lambda t\right) dt \leq 2\left(c + \log_{+} \frac{1}{\rho}\right), \tag{B.249}$$

which together with (B.246) implies (15.53) in the case $n = 2$.

15.7. Fix $x \in M$ and let Ω be an open set containing x. Since the function

$$u(y) = g(x,y) - g^{\Omega}(x,y)$$

is harmonic in Ω and, hence, is bounded in a neighborhood of x, we have

$$g(x,y) \leq g^{\Omega}(x,y) + \text{const}$$

provided y is close enough to x. Since the right hand side of (15.54) goes to ∞ as $y \to x$, it suffices to prove (15.54) for g^{Ω} instead of g.

Consider the function $r(y)$ from Theorem 15.4 and set

$$r_0 := \frac{1}{2} \inf_{y \in B(x, r(x))} r(y) > 0.$$

Consider $\Omega = B(x, 2r_0)$ as a manifold itself and observe that, for any $y \in B(x, r_0)$, the ball $B(y, r_0)$ is contained in Ω and $r_0 < r(y)$ (indeed, we have $r_0 < r(x)$ whence $y \in B(x, r(x))$ and $r_0 < r(y)$). Applying Theorem 15.14 for the family of two balls $B(x, r_0)$, $B(y, r_0)$ in the manifold Ω, we obtain, for all $t \geq t_0 > 0$ and $y \in B(x, r_0)$, that

$$p_t^{\Omega}(x,y) \leq \frac{C}{\min\left(t_0, r_0^2\right)^{n/2}} \exp\left(-\frac{\rho^2}{5t} - \lambda(t - t_0)\right),$$

where $\rho = d(x,y)$, $\lambda = \lambda_{\min}(\Omega)$, and $C = C(n)$. Choosing $t_0 = t$ or $t_0 = r_0^2$, we obtain the following two estimates:

$$p_t^{\Omega}(x,y) \leq C \begin{cases} t^{-n/2} \exp\left(-\frac{\rho^2}{5t}\right), & t \leq r_0^2, \\ r_0^{-n} \exp\left(-\lambda\left(t - r_0^2\right)\right), & t > r_0^2. \end{cases}$$

It follows that

$$\begin{aligned} g^{\Omega}(x,y) &= \int_0^{r_0^2} p_t(x,y)\, dt + \int_{r_0^2}^{\infty} p_t(x,y)\, dt \\ &\leq C \int_0^{r_0^2} t^{-n/2} \exp\left(-\frac{\rho^2}{5t}\right) dt + C r_0^{-n} \int_{r_0^2}^{\infty} \exp\left(-\lambda\left(t - r_0^2\right)\right) \end{aligned} \tag{B.250}$$

The second term in (B.250) is equal to $C\lambda^{-1}r_0^{-n}$. Observing that

$$\lambda = \lambda_{\min}(B(x,2r_0)) \simeq r_0^{-2}$$

(cf. Exercise 11.25), we obtain

$$r_0^{-n}\int_{r_0^2}^{\infty}\exp\left(-\lambda\left(t-r_0^2\right)\right)dt \le Cr_0^{2-n} \le C\rho^{2-n}. \tag{B.251}$$

To estimate the first term in (B.250), assume first that $n>2$. Then by (A.60) (or by Exercise 13.36) we have

$$\int_0^{r_0^2} t^{-n/2}\exp\left(-\frac{\rho^2}{5t}\right)dt \le \int_0^{\infty} t^{-n/2}\exp\left(-\frac{\rho^2}{5t}\right)dt = C\rho^{2-n},$$

where $C=C(n)$. Combining the above estimates together, we obtain from (B.250) that

$$g^{\Omega}(x,y) \le C\rho^{2-n},$$

which proved (15.54) for the case $n>2$.

Consider now the case $n=2$. Assuming $r_0<1$ and making change $s=\rho^2/t$, we obtain as in (B.247)

$$\begin{aligned}\int_0^{r_0^2} t^{-1}\exp\left(-\frac{\rho^2}{5t}\right)dt &\le \int_{\rho^2}^{\infty} s^{-1}e^{-s/5}ds \\ &\le \int_{\rho^2}^{1}\frac{ds}{s}+\int_1^{\infty}e^{-s/5}d\tau \\ &= 2\log\frac{1}{\rho}+\text{const}.\end{aligned}$$

Combining with (B.250), (B.251) and choosing ρ small enough to absorb the constant, we obtain

$$g^{\Omega}(x,y) \le C\log\frac{1}{\rho},$$

which proves (15.54) for the case $n=2$.

15.8. If M satisfies the relative Faber-Krahn inequality then, by Theorem 15.21, the volume function $V(x,r)$ is doubling and

$$p_t(x,y) \le \frac{C}{V\left(x,\sqrt{t}\right)}\exp\left(-c\frac{d^2(x,y)}{t}\right).$$

By Exercise 13.36, we obtain, for all $x\neq y$,

$$g(x,y) \le C\int_{d(x,y)}^{\infty}\frac{rdr}{V(x,r)} < \infty,$$

which was to be proved.

If the Green function is finite then by Exercise 13.30 the manifold M is non-parabolic, and by Theorem 11.14, we obtain

$$\int^{\infty}\frac{rdr}{V(x,r)} < \infty$$

for all $x \in M$.

15.9. See [**154**, Theorem 5.9].

Solutions to Chapter 16

16.1. (a) Extend $\gamma(t)$ to $(0, +\infty)$ by setting $\gamma(t) = \gamma(T-)$ for all $t \geq T$. It is easy to see that the doubling property (16.24) holds for all $t > 0$ so that γ is regular on $(0, +\infty)$ in the sense of Definition 16.1. Since $p_t(x, x)$ decreases in t, we conclude that (16.23) holds for all $t > 0$.

Theorem 16.3 yields that, for all $D > 2$ and $t > 0$,

$$E_D(t, x) \leq \frac{C}{\gamma(ct)}.$$

Using again the doubling property of γ, we obtain $\gamma(ct) \geq \varepsilon\gamma(t)$ where $\varepsilon = \varepsilon(c) > 0$, which finishes the proof.

16.2. By Theorem 7.7, for any $x \in M$ there exists a finite constant $C(x)$ such that, for all $t > 0$,

$$p_t(x, x) = \|p_{t/2,x}\|^2 \leq C(x)\left(1 + t^{-\sigma}\right)^2. \tag{B.252}$$

Hence, by Exercise 16.1, $E_D(t, x)$ admits the upper bound

$$E_D(t, x) \leq C(x)\left(1 + t^{-\sigma}\right)^2, \tag{B.253}$$

whence the finiteness of $E_D(t, x)$ follows.

16.3. Set

$$F(t, x) = \sqrt{E_D(\tfrac{1}{2}t, x)}. \tag{B.254}$$

By Exercise 16.2, this function is finite. By Theorem 12.1, the function $F(t, x)$ is decreasing in t. By Lemma 15.13, we have the inequality

$$p_t(x, y) \leq F(t_0, x) F(t_0, y) \exp(\lambda_{\min} t_0) \exp\left(-\frac{d^2(x, y)}{2Dt} - \lambda_{\min} t\right), \tag{B.255}$$

for all $t \geq t_0 > 0$. Set

$$\Phi(t, x) = \exp\left(\frac{1}{2}\lambda_{\min}\right) \begin{cases} F(t, x), & t < 1, \\ F(1, x), & t \geq 1. \end{cases}$$

Obviously, this function is decreasing in t. We claim that the required inequality (16.26) holds with this function Φ. Indeed, if $t \geq 1$ then (16.26) follows from (B.255) with $t_0 = 1$. If $t < 1$ then (16.26) follows from (B.255) with $t_0 = t$.

16.4. By Theorem 14.19, we have, for all $x \in M$ and $t > 0$,

$$p_t(x, x) \leq \frac{4}{\gamma(t/2)}.$$

Then the claim follows from Corollary 16.4.

16.5. By Exercise 16.1, we have

$$E_D(t,x) \le \frac{C}{V\left(x,\sqrt{t}\right)},$$

for all $x \in M$ and $t \in (0,T)$, and by Lemma 15.13

$$p_t(x,y) \le \frac{C}{V\left(x,\sqrt{t}\right)^{1/2} V\left(y,\sqrt{t}\right)^{1/2}} \exp\left(-\frac{d^2(x,y)}{2Dt}\right).$$

Finally, using the estimate

$$\frac{V\left(x,\sqrt{t}\right)}{V\left(y,\sqrt{t}\right)} \le C\left(1+\frac{d^2(x,y)}{t}\right)^c$$

which follows from the doubling property of $V(x,r)$, we finish the proof.

Solutions to Appendix A

A.1. Using the bilinearity of the inner product, we obtain

$$\begin{aligned}(x_k,y_k)-(x,y) &= (x_k,y_k)-(x_k,y)+(x_k,y)-(x,y)\\ &= (x_k,y_k-y)+(x_k-x,y)\end{aligned}$$

whence, by the triangle inequality and the Cauchy-Schwarz inequality,

$$|(x_k,y_k)-(x,y)| \le \|x_k\|\,\|y_k-y\| + \|x_k-x\|\,\|y\|.$$

Since $\|y_k-y\|$ and $\|x_k-x\|$ tend to 0 and $\|x_k\|$ remains bounded, the claim follows.

A.2. Passing to a subsequence, we can assume that $\lim\|x_k\|$ exists. Using the definition of the weak convergence and the Cauchy-Schwarz inequality, we obtain

$$\|x\|^2 = (x,x) = \lim_{k\to\infty}(x,x_k) \le \lim_{k\to\infty}\|x\|\|x_k\|,$$

whence the claim follows.

A.3. (a) The fact, that the strong convergence implies the weak one and the convergence of the norms, is obvious. Conversely, if $x_k \rightharpoonup x$ and $\|x_k\| \to \|x\|$ then

$$\lim_{k\to\infty}\|x-x_k\|^2 = \lim_{k\to\infty}\left(\|x\|^2+\|x_k\|^2-2(x,x_k)\right) = \|x\|^2+\|x\|^2-2(x,x)=0,$$

that is, $x_k \to x$.

(b) That the weak convergence implies the boundedness of the norms follows from the principle of uniform boundedness; the convergence "in distribution" is obvious. Conversely, in order to prove that $x_k \rightharpoonup x$, we must show that

$$(x_k,y)\to(x,y) \text{ for all } y\in\mathcal{H}.$$

Set $C=\sup_k\|x_k\|$ and let $\{y_i\}$ be a sequence from $\mathcal{D}$ such that $y_i\to y$ as $i\to\infty$. Then we have

$$(x_k,y) = (x_k,y_i)+(x_k,y-y_i)$$

whence, for any index i,

$$\limsup_{k\to\infty}(x_k,y)\le\lim_{k\to\infty}(x_k,y_i)+C\|y-y_i\|=(x,y_i)+C\|y-y_i\|.$$

Letting now $i\to\infty$, we obtain

$$\limsup_{k\to\infty}(x_k,y)\le(x,y).$$

In the same way one proves that

$$\liminf_{k\to\infty}(x_k,y)\ge(x,y),$$

whence the claim follows.

A.4. (a) We need to prove that, for any $y\in\mathcal{H}$,

$$c_k:=(v_k,y)\to 0\text{ as }k\to\infty.$$

Indeed, by Bessel's inequality

$$\sum_k|c_k|^2\le\|y\|^2<\infty$$

whence it follows that $c_k\to 0$.

(b) By the definition of weak convergence, we have, for any $y\in\mathcal{H}$,

$$(x,y)=\sum_k c_k(v_k,y).$$

In particular, setting $y=v_i$, we obtain

$$(x,v_i)=c_i.$$

Setting $y=x$, we obtain

$$\|x\|^2=\sum|c_k|^2,$$

that is, x satisfies Parseval's identity. It follows that the identity

$$x=\sum c_kv_k$$

holds also in the strong sense.

A.5. Let S be a closed subspace of $\mathcal{H}$. and let $S^\perp$ be the orthogonal complement of S. Since $S=\left(S^\perp\right)^\perp$, the condition $x\in S$ is equivalent to $x\perp S^\perp$, that is,

$$(x,y)=0\text{ for all }y\in S^\perp. \tag{B.256}$$

Clearly, the condition (B.256) is stable under weak convergence; that is, if every x_k satisfies it and $x_k\rightharpoonup x$ then also x satisfies it. Therefore, S contains all weak limits of its sequences, which means that S is weakly closed.

A.6. It suffices to prove (A.9) since (A.10) follows from (A.9) by changing f to $-f$. Denote by S the set of indicator functions of subsets of M with finite measures. Then we have, for any measurable function f,

$$\operatorname{esup} f=\sup_{\substack{A\subset M,\\ 0<\mu(A)<\infty}}\frac{1}{\mu(A)}\int_A f\,d\mu=\sup_{\varphi\in S\setminus\{0\}}\frac{(f,\varphi)}{\|\varphi\|_{L^1}}.$$

Since $(f_k, \varphi) \to (f, \varphi)$ as $k \to \infty$, we obtain

$$\begin{aligned} \operatorname{esup} f &= \sup_{\varphi \in S \setminus \{0\}} \lim_{k\to\infty} \frac{(f_k, \varphi)}{\|\varphi\|_{L^1}} \\ &\le \limsup_{k\to\infty} \left(\sup_{\varphi \in S \setminus \{0\}} \frac{(f_k, \varphi)}{\|\varphi\|_{L^1}} \right) \\ &= \limsup_{k\to\infty} \left(\operatorname{esup} f_k \right). \end{aligned}$$

Passing to a subsequence of $\{f_k\}$, one can replace lim sup by lim inf.

A.7. We have

$$\begin{aligned} \|f_k^2 - f^2\|_{L^1} &= \int_M \left|f_k^2 - f^2\right| d\mu = \int_M |f_k - f|\, |f_k + f|\, d\mu \\ &\le \left(\int_M |f_k - f|^2\, d\mu \right)^{1/2} \left(\int_M |f_k + f|^2\, d\mu \right)^{1/2} \\ &= \|f_k - f\|_{L^2} \|f_k + f\|_{L^2}. \end{aligned}$$

Since

$$\|f_k + f\|_{L^2} \le \|f_k\|_{L^2} + \|f\|_{L^2}$$

remains bounded as $k \to \infty$ and $\|f_k - f\|_{L^2} \to 0$, we obtain that $\|f_k^2 - f^2\|_{L^1} \to 0$, which was to be proved.

The second claim follows from

$$\int_M \left| f^2 g - f_k^2 g \right| d\mu \le \|f^2 - f_k^2\|_{L^1} \|g\|_{L^\infty}.$$

A.8. (*a*) This is obvious. Note that $A^{-1}A$ is not necessarily equal to id because $\operatorname{dom}\left(A^{-1}A\right) \subset \operatorname{dom} A$ where as $\operatorname{dom} \mathrm{id} = \mathcal{H}$.

(*b*) For any $x \in \mathcal{H}$, we have $A(Bx) = x$. In particular, this implies $\operatorname{ran} A = \mathcal{H}$. For any $x \in \operatorname{dom} A$, we have $B(Ax) = x$. It follows that $\ker A = \{0\}$ because $Ax = 0$ implies $x = B0 = 0$. Hence, A^{-1} exists. The equation $Ay = x$ has solution $y = Bx$ for any $x \in \mathcal{H}$ whence it follows that $A^{-1}x = Bx$ and $A^{-1} = B$.

A.9. Set

$$a := \sup_{x \in \operatorname{dom} A, \|x\| \le 1, \|y\| \le 1} (Ax, y)$$

and note that $a \ge 0$. By the Cauchy-Schwarz inequality, if $\|x\| \le 1$ and $\|y\| \le 1$ then

$$(Ax, y) \le \|Ax\| \|y\| \le \|A\|,$$

whence $a \le \|A\|$. Assume now that $a < \|A\|$. Then there exists $x \in \operatorname{dom} A$ with $\|x\| = 1$ and such that $\|Ax\| > a$. Setting $y = \frac{Ax}{\|Ax\|}$, we obtain

$$(Ax, y) = (Ax, \frac{Ax}{\|Ax\|}) = \|Ax\| > a,$$

which contradicts the definition of a.

A.10. By definition, $\operatorname{dom} A^*$ consists of those $y \in \mathcal{H}$ for which the linear functional $x \mapsto (Ax, y)$ is bounded. Since

$$|(Ax, y)| \leq \|Ax\| \|y\| \leq \|A\| \|y\| \|x\|,$$

we see that this functional is bounded for any y. Hence, $\operatorname{dom} A^* = \mathcal{H}$.

It follows from (A.18) and $(Ax, y) = (x, A^*y)$ that

$$\|A\| = \sup_{\|x\| \leq 1, \|y\| \leq 1} (Ax, y) = \sup_{\|x\| \leq 1, \|y\| \leq 1} (x, A^*y) = \sup_{\|x\| \leq 1, \|y\| \leq 1} (A^*y, x) = \|A^*\|.$$

In particular, A^* is bounded. The identity $\|A\| = \|A^*\|$ implies

$$\|A^*A\| \leq \|A^*\| \|A\| = \|A\|^2.$$

On the other hand, taking $x = y$ in (A.18) we obtain

$$\|A^*A\| \geq \sup_{\|x\| \leq 1} \|Ax\|^2 = \|A\|^2.$$

Hence, we conclude $\|A^*A\| = \|A\|^2$.

A.11. (a) Since A is non-negative definite, we have, for all $x, y \in \operatorname{dom} A$ and for any real t,

$$(A(x + ty), x + ty) \geq 0.$$

Using the linearity of A and the symmetry, we obtain

$$(A(x + ty), x + ty) = (Ax, x) + 2t(Ax, y) + t^2(Ay, y),$$

whence

$$(Ax, x) + 2t(Ax, y) + t^2(Ay, y) \geq 0. \tag{B.257}$$

If $(Ay, y) > 0$ then the left hand side of (B.257) is a quadratic function of t that is non-negative for all real t, whence we conclude

$$(Ax, y)^2 \leq (Ax, x)\,(Ay, y)\,. \tag{B.258}$$

If $(Ay, y) = 0$ then (B.257) becomes

$$(Ax, x) + 2t(Ax, y) \geq 0$$

which can be true for all real t only if $(Ax, y) = 0$ whence (B.258) follows again.

(b) It follows from (A.18) and (A.19) that

$$\begin{aligned} \|A\| &= \sup_{x,y \in \operatorname{dom} A, \|x\| \leq 1, \|y\| \leq 1} (Ax, y) \\ &\leq \sup_{x,y \in \operatorname{dom} A, \|x\| \leq 1, \|y\| \leq 1} \sqrt{(Ax, x)\,(Ay, y)} = \sup_{x \in \operatorname{dom} A, \|x\| \leq 1} (Ax, x)\,. \end{aligned}$$

The opposite inequality trivially follows from (A.18) by setting $y = x$.

A.12. (a) If $x \in (\operatorname{ran} A)^\perp$ then $x \perp \operatorname{ran} A$, that is, $(x, Ay) = 0$ for $y \in \operatorname{dom} A$. Hence, we have

$$(Ax, y) = (x, Ay) = 0$$

so that $Ax \perp \operatorname{dom} A$. Since $\operatorname{dom} A$ is dense in $\mathcal{H}$, we conclude $Ax = 0$ and hence $x \in \ker A$, which proves that $(\operatorname{ran} A)^{\perp} \subset \ker A$. For the opposite inclusion, for any $x \in \ker A$ and $y \in \operatorname{dom} A$, we have

$$(x, Ay) = (Ax, y) = 0$$

whence $x \perp Ay$ and hence $x \in (\operatorname{ran} A)^{\perp}$. Thus, we have $(\operatorname{ran} A)^{\perp} = \ker A$. Taking the orthogonal complements to both parts, we obtain $\overline{\operatorname{ran} A} = (\ker A)^{\perp}$.

(b) If A^{-1} exists and is bounded then there exists $C > 0$ such that, for all $y \in \mathcal{H}$,

$$\|A^{-1}y\| \leq C\|y\|.$$

Setting $y = Ax$, we obtain (A.20) with $c = C^{-1}$.

Let us prove the converse. The hypothesis (A.20) implies $\ker A = \{0\}$, and by part (a) we obtain

$$\overline{\operatorname{ran} A} = (\ker A)^{\perp} = \mathcal{H},$$

so that $\operatorname{ran} A$ is dense in $\mathcal{H}$. Let us show that in fact $\operatorname{ran} A = \mathcal{H}$. For any $y \in \mathcal{H}$, there is a sequence $\{y_k\} \subset \operatorname{ran} A$ such that $y_k \to y$. For some $x_k \in \operatorname{dom} A$, we have $y_k = Ax_k$. By hypothesis, we obtain that

$$\|Ax_k - Ax_l\| \geq c\|x_k - x_l\|$$

whence it follows that the sequence $\{x_k\}$ is Cauchy and, hence, converges to a vector $x \in \mathcal{H}$. Let us show that $x \in \operatorname{dom} A$ and $y = Ax$, which would imply $y \in \operatorname{ran} A$ and $\operatorname{ran} A = \mathcal{H}$. For any $z \in \operatorname{dom} A$, we have

$$(Ax_k, z) = (x_k, Az) \to (x, Az)\,.$$

Since $Ax_k \to y$, we conclude that

$$(Az, x) = (z, y)\,.$$

In particular, we see that the linear functional $z \mapsto (Az, x)$ is bounded, which implies, by the definition of the adjoint operator, that $x \in \operatorname{dom} A^*$ and $A^*x = y$. Since $A = A^*$, we conclude $x \in \operatorname{dom} A$ and $Ax = y$, which was claimed.

Hence, the operator A is injective and surjective and hence, the inverse A^{-1} exists with domain $\mathcal{H}$. The boundedness of A^{-1} immediately follows from (A.20).

A.13. (a) Let us show that (Az, x) is a bounded linear functional in $z \in \operatorname{dom} A$, which will implies that $x \in \operatorname{dom} A^* = \operatorname{dom} A$. Indeed, we have, for any $z \in \operatorname{dom} A$.

$$(Az, x) = \lim_{k\to\infty} (Az, x_k) = \lim_{k\to\infty} (z, Ax_k) = (z, y)\,.$$

Indeed, (z, y) is a bounded linear functional in z, whence it follows that $x \in \operatorname{dom} A$. Since

$$(Az, x) = (z, Ax)\,,$$

comparing the above two lines, we conclude $Ax = y$.

(b) The symmetry and the non-negative definiteness of A easily imply that the bilinear form

$$(x, y) + (Ax, Ay)$$

in $\operatorname{dom} A$ satisfies the axioms of an inner product. Let $\{x_k\}$ be a Cauchy sequence in $\operatorname{dom} A$. Then $\{x_k\}$ and $\{Ax_k\}$ are Cauchy sequences in $\mathcal{H}$ so that $x_k \to x$ and $Ax_k \to y$ for some $x, y \in \mathcal{H}$. By part (a), we obtain $x \in \operatorname{dom} A$ and $y = Ax$, which means that the sequence $\{x_k\}$ converges to x in the norm of the space $\operatorname{dom} A$, whence the completeness of $\operatorname{dom} A$ follows.

A.14. The fact that F_U is σ-additive implies that, for any increasing sequence $\{U_k\}$ of Borel sets,

$$F_{\cup_k U_k} = \lim_{k\to\infty} F_{U_k},$$

and, for any decreasing sequence $\{U_k\}$,

$$F_{\cap_k U_k} = \lim_{k\to\infty} F_{U_k}.$$

Hence, the required results follow from $F_{[a,b)} = F(b) - F(a)$ and the following observations:

$$\begin{aligned}(a, b) &= \bigcup_{\lambda\to a+} [\lambda, b), \\ [a, b] &= \bigcap_{\lambda\to b+} [a, \lambda), \\ \{a\} &= \bigcap_{\lambda\to a+} [a, \lambda),\end{aligned}$$

and $(a, b] = [a, b] \setminus \{a\}$.

A.15. (a) The fact that F is increasing is obvious from (A.25). If $\{\lambda_n\}$ is a monotone increasing sequence that converges to λ from the left then the sequence of sets $\{k : s_k < \lambda_n\}$ increases and the union of all these sets is $\{k : s_k < \lambda\}$. Therefore, we obtain that

$$F(\lambda_n) \to F(\lambda)$$

and hence F is left-continuous. If $\lambda_n \to +\infty$ then the union of the sets $\{k : s_k < \lambda_n\}$ is the set of all integers so that

$$F(+\infty) = \lim_{n\to\infty} F(\lambda_n) = \sum_{k\in\mathbb{Z}} t_k < \infty.$$

Finally, if $\lambda_n \to -\infty$ then the intersection of all the sets $\{k : s_k < \lambda_n\}$ is empty and we obtain $F(-\infty) = 0$.

(b) Denote by S_U the right hand side of (A.26). Clearly, S_U is a measure of the σ-ring of all Borel sets. For any interval $[a, b)$ we have

$$S_{[a,b)} = \sum_{\{k:a\le s_k<b\}} t_k = \sum_{\{k:s_k<b\}} t_k - \sum_{\{k:s_k<a\}} t_k = F(b) - F(a).$$

We see that $S_{[a,b)} = F_{[a,b)}$ and, hence, by the uniqueness of the extension, $S_U = F_U$ for all Borel sets U.

(c) If $\varphi = 1_U$ then (A.27) follows from (A.26). The the identity (A.27) extends by linearity to finite linear combinations of indicator functions, that is, to functions φ of the form

$$\varphi = \sum_{k=1}^{n} \alpha_k 1_{U_k}. \tag{B.259}$$

Observe that the both parts of (A.27) survive when taking monotone limit of a sequence of functions φ. Since any non-negative Borel function is a monotone limit of functions like (B.259), we conclude that (A.27) holds for all non-negative Borel functions φ.

(d) This immediately follows from (c).

A.16. If $\varphi = 1_{[a,b)}$ then the left hand side of (A.28) is equal to

$$\int 1_{[a,b)} dF(\lambda) = F_{[a,b)} = F(b) - F(a),$$

and the right hand side of (A.28) is equal to

$$\int_a^b F'(\lambda)\, d\lambda = F(b) - F(a).$$

Hence, (A.28) holds for all indicator functions. By linearity, (A.28) extends to all finite linear combinations of indicator functions. Finally, by taking monotone limits of such functions, we extend (A.28) to all non-negative Borel functions.

A.17. (a) If F is a monotone function then it is obvious that the right hand side of (A.29) does not depend on $\{\lambda_k\}$ and is equal to $|F(+\infty) - F(-\infty)|$. Hence, a bounded monotone function has a finite total variation. It also follows from (A.29) that $\operatorname{var} F = \operatorname{var}(-F)$ and

$$\operatorname{var}(F + G) \leq \operatorname{var}(F + G).$$

Hence, the sum and the difference of two functions with finite total variation is again a function of this class. Hence, the difference of two bounded monotone functions has a finite total variation.

Now, let $\operatorname{var} F < \infty$ and let us prove that F is the difference of two bounded increasing monotone functions. Define the total variation of F on $(-\infty, a]$ by

$$\operatorname*{var}_{(-\infty,a]} F = \sup_{\{\lambda_k\}} \sum_{k=0}^{+\infty} |F(\lambda_{k+1}) - F(\lambda_k)|$$

where $\{\lambda_k\}_{k\geq 0}$ is a decreasing sequence such that

$$\lambda_0 = a \ \text{ and } \ \lambda_k \to -\infty \text{ as } k \to +\infty. \tag{B.260}$$

Clearly, the function

$$G(a) := \operatorname*{var}_{(-\infty,a]} F \tag{B.261}$$

is non-negative and is bounded by $\operatorname{var} F$. Let us show that the functions G and $G-F$ are increasing, which will finish the proof because both these functions are bounded and

$$F = G - (G - F)\,.$$

Indeed, for any pair $a < b$ and for any decreasing sequence $\{\lambda_k\}_{k\geq 0}$ satisfying (B.260), we obtain by the definition of $\operatorname{var}_{(-\infty,b]} F$ that

$$G(b) = \operatorname*{var}_{(-\infty,b]} F \geq \sum_{k=0}^{+\infty} |F(\lambda_{k+1}) - F(\lambda_k)| + |F(b) - F(a)|$$

whence

$$G(b) \geq G(a) + |F(b) - F(a)|\,.$$

This implies that $G(b) \geq G(a)$ and

$$G(b) \geq G(a) + F(b) - F(a)\,,$$

that is,

$$G(b) - F(b) \geq G(a) - F(a)\,.$$

(b) If F is the difference of two functions satisfying (A.21) then F is left-continuous and, by part (a), $\operatorname{var} F < \infty$. To prove the converse, consider the function G defined by (B.261) and $H := G - F$. As was shown above, both G and H are bounded monotone increasing functions. Let $\widetilde{G}$ be the left-continuous modification of G, that is,

$$\widetilde{G}(\lambda) = G(\lambda-)\,.$$

Note that $\widetilde{G}$ and G coincide everywhere except for the set of points of jumps of G, which is at most a countable set. Defining the same way $\widetilde{H}$, we conclude that $\widetilde{G} - \widetilde{H}$ is a left-continuous function on $\mathbb{R}$, which coincides with $F = G - H$ outside a countable set. Since function F is also left-continuous, we conclude that $F = \widetilde{G} - \widetilde{H}$ at all points, which finishes the proof.

(c) and (d) Left to the reader

A.18. Let us first show that, for any two functions $F^{(1)}$ and $F^{(2)}$ satisfying (A.21) and for any non-negative Borel function φ,

$$\int_{-\infty}^{+\infty} \varphi dF^{(1)} + \int_{-\infty}^{+\infty} \varphi dF^{(2)} = \int_{-\infty}^{+\infty} \varphi d\left(F^{(1)} + F^{(2)}\right). \tag{B.262}$$

It suffices to show that, for any Borel set $U \subset \mathbb{R}$,

$$F_U^{(1)} + F_U^{(2)} = \left(F^{(1)} + F^{(2)}\right)_U.$$

By uniqueness of the extension in the Carathéodory Extension Theorem, it suffices to prove this for $U = [a, b)$, which is obvious because

$$F_U^{(1)} + F_U^{(2)} = F^{(1)}(b) - F^{(1)}(a) + F^{(2)}(b) - F^{(2)}(a) = \left(F^{(1)} + F^{(2)}\right)(b) - \left(F^{(1)} + F^{(2)}\right)(a)\,.$$

Let now $G^{(1)}$ and $G^{(2)}$ be another pair of functions satisfying (A.21) and such that $F = G^{(1)} - G^{(2)}$ and φ is integrable against $G^{(1)}$ and $G^{(2)}$. Let us prove that

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF^{(1)}(\lambda) - \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF^{(2)}(\lambda) = \int_{-\infty}^{+\infty} \varphi(\lambda)\, dG^{(1)}(\lambda) - \int_{-\infty}^{+\infty} \varphi(\lambda)\, dG^{(2)}(\lambda). \tag{B.263}$$

Since the integral of φ is defined as the difference of the integrals of φ_+ and φ_-, it suffices to prove the same identity separately for φ_+ and φ_-. Hence, we can assume without loss of generality that $\varphi \geq 0$. Then (B.263) follows from

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF^{(1)}(\lambda) + \int_{-\infty}^{+\infty} \varphi(\lambda)\, dG^{(2)}(\lambda) = \int_{-\infty}^{+\infty} \varphi(\lambda)\, dG^{(1)}(\lambda) + \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF^{(2)}(\lambda), \tag{B.264}$$

while (B.264) holds by (B.262) because $F^{(1)} + G^{(2)} = G^{(1)} + F^{(2)}$.

A.19. (*a*) Let $S = \operatorname{ran} P$ so that Px is determined by the conditions

$$Px \in S \quad \text{and} \quad x - Px \perp S. \tag{B.265}$$

If $x \in S$ then $Px = x$. Therefore, for any $x \in \mathcal{H}$,

$$P(Px) = Px$$

and hence $P^2 = P$.

If $x - Px \perp S$ and $y - Py \perp S$ then also

$$(x - Px) + (y - Py) \perp S,$$

that is

$$(x + y) - (Px + Py) \perp S,$$

whence it follows that $Px + Py = P(x + y)$. Similarly, one proves $P(\lambda x) = \lambda Px$ whence the linearity follows.

Let us prove that P is symmetric, that is,

$$(Px, y) = (x, Py). \tag{B.266}$$

Since $Py \in S$ and $x - Px \perp S$, we have

$$(x - Px, Py) = 0,$$

whence

$$(x, Py) = (Px, Py).$$

By switching x and y, we obtain

$$(y, Px) = (Py, Px),$$

which together with the previous line implies (B.266).

(*b*) Let us first verify that $\operatorname{ran} A$ is closed. For any $x \in \operatorname{ran} A$ we have $Ax = x$ because, for some $y \in \mathcal{H}$, $x = Ay$ and hence $Ax = A^2 y = Ay = x$. Let $\{x_k\}$ be a sequence from $\operatorname{ran} A$ that converges to $x \in \mathcal{H}$. Then we have $Ax_k = x_k$, and $x_k \to x$ implies by the boundedness of A than $Ax_k \to Ax$. Hence, $Ax = x$ and $x \in \operatorname{ran} A$, which shows that $\operatorname{ran} A$ is closed.

Observe that $\ker A \perp \operatorname{ran} A$ because for any $x \in \ker A$ and $y \in \operatorname{ran} A$, we have $y = Az$ for some z and hence

$$(x, y) = (x, Az) = (Ax, z) = 0.$$

On the other hand, for any $x \in \mathcal{H}$, we have $x - Ax \in \ker A$ because

$$A(x - Ax) = Ax - A^2 x = Ax - Ax = 0.$$

Therefore, $x - Ax$ is orthogonal to $\operatorname{ran} A$. Since $Ax \in \operatorname{ran} A$, we obtain that A is the projection onto $\operatorname{ran} A$.

(c) By (B.265), we have

$$(x - Px, Px) = 0$$

whence

$$(x, Px) = (Px, Px) \geq 0.$$

Hence, P is non-negative definite. The above identity also yields

$$\|Px\|^2 = (x, Px) \leq \|x\| \|Px\|,$$

whence $\|Px\| \leq \|x\|$ and hence

$$\|P\| \leq 1.$$

If $P \neq 0$ then $\operatorname{ran} P$ contains a non-zero vector, say x. For this vector, we have $Px = x$ and $\|Px\| = \|x\|$ whence $\|P\| = 1$.

A.20. Let $\{u_i\}$ an orthonormal basis in $\operatorname{ran} P$. Then

$$Pv_k = \sum_i (v_k, u_i) u_i$$

and, applying twice the Parseval Identity, we obtain

$$\|Pv_k\|^2 = \sum_i (v_k, u_i)^2$$

and

$$\sum_k \|Pv_k\|^2 = \sum_k \sum_i (v_k, u_i)^2 = \sum_i \sum_k (v_k, u_i)^2 = \sum_i \|u_i\|^2 = \sum_i 1 = \dim \operatorname{ran} P.$$

A.21. (a) By definition, E_a and E_b are a projector, and $\operatorname{ran} E_a \subset \operatorname{ran} E_b$. Therefore, for any vector x,

$$x - E_b x \perp \operatorname{ran} E_a$$

so that $E_a (x - E_b x) = 0$ and hence $E_a x = E_a E_b x$. Since $E_a x \in \operatorname{ran} E_b$, we conclude also that $E_a x = E_b (E_a x) = E_b E_a x$.

(b) The operator $E_b - E_a$ is bounded and self-adjoint, so to prove that it is a projector it suffices to verify that $(E_b - E_a)^2 = E_b - E_a$, which immediately follows from (A.39):

$$(E_b - E_a)^2 = E_b^2 - E_b E_a - E_a E_b + E_a^2 = E_b - 2E_a + E_a = E_b - E_a.$$

In fact, one can show that $\operatorname{ran}(E_b - E_a)$ is the orthogonal complement of $\operatorname{ran} E_a$ in $\operatorname{ran} E_b$.

Let us prove that $\|E_a x\| \leq \|E_b x\|$. We have:

$$\|E_b x\|^2 - \|E_a x\|^2 = (E_b x, x) - (E_a x, x) = ((E_b - E_a)\, x, x) \geq 0,$$

because $E_b - E_a$ is a projector and, hence, is non-negative definite.

(c) By the definition of the numerical Lebesgue-Stieltjes integration, we have, for any $x \in \mathcal{H}$,

$$\int_{[a,b)} d\|E_\lambda x\|^2 = \int_{-\infty}^{+\infty} 1_{[a,b)} d\|E_\lambda x\|^2 = \|E_b x\|^2 - \|E_a x\|^2 = \|\,(E_b - E_a)\, x\|^2,$$

where we have also used (A.39). Therefore, for all $x, y \in \mathcal{H}$,

$$\begin{aligned}
\int_{[a,b)} d\,(E_\lambda x, y) &= \frac{1}{4}\int_{[a,b)} d\|E_\lambda\,(x+y)\,\|^2 - \frac{1}{4}\int_{[a,b)} d\|E_\lambda\,(x-y)\,\|^2 \\
&= \frac{1}{4}\left(\|\,(E_b - E_a)\,(x+y)\|^2 - \|\,(E_b - E_a)\,(x-y)\|^2\right) \\
&= ((E_b - E_a)\, x, (E_b - E_a)\, y) \\
&= ((E_b - E_a)\, x, y)\,,
\end{aligned}$$

whence it follows that

$$\int_{[a,b)} dE_\lambda = E_b - E_a.$$

(d) For any two projectors P and Q such that $PQ = 0$, we have $P\,(\operatorname{ran} Q) = \{0\}$ which implies $\operatorname{ran} P \perp \operatorname{ran} Q$. Hence, to prove that $\operatorname{ran} E_{[a_1,b_1)}$ and $\operatorname{ran} E_{[a_2,b_2)}$ are orthogonal, it suffices to verify that $E_{[a_1,b_1)} E_{[a_2,b_2)} = 0$. Assuming for simplicity that $a_1 < b_1 \leq a_2 < b_2$ and using (A.39) and (A.40), we obtain

$$\begin{aligned}
(E_{b_1} - E_{a_1})\,(E_{b_2} - E_{a_2}) &= E_{b_1}E_{b_2} - E_{b_1}E_{a_2} - E_{a_1}E_{b_2} + E_{a_1}E_{a_2} \\
&= E_{b_1} - E_{b_1} - E_{a_1} + E_{a_1} = 0.
\end{aligned}$$

A.22. Note that $P_i^2 = P_i$ and $P_i P_j = 0$ for $i \neq j$. Therefore, when expanding the expression

$$A^m = \left(\sum_{i=1}^{k} \lambda_i P_i\right)^m,$$

all the terms coming from the products of projectors P_i with different indices i will vanish, and the term P_i^m will amount to P_i. Hence, we obtain

$$A^m = \sum_{i=1}^{k} \lambda_i^m P_i.$$

By linearity, we obtain

$$\varphi\,(A) = \sum_{i=1}^{k} \varphi\,(\lambda_i)\, P_i.$$

Note that operators A and $\varphi\,(A)$ are bounded.

Finally, for any $x \in \mathcal{H}$,

$$\begin{aligned} \|\varphi(A)\|^2 &= (\varphi(A)x, \varphi(A)x) = \left(\sum_{i=1}^{k} \varphi(\lambda_i) P_i x, \sum_{j=1}^{k} \varphi(\lambda_j) P_j x\right) \\ &= \sum_{i.j=1}^{k} \varphi(\lambda_i)\varphi(\lambda_j)(P_i x, P_j x). \end{aligned}$$

If $i \neq j$ then

$$(P_i x, P_j x) = (x, P_i P_j x) = 0,$$

whence it follows that

$$\|\varphi(A)\|^2 = \sum_{i.j=1}^{k} \varphi(\lambda_i)^2 (P_i x, P_i x).$$

The identity (A.42) obviously follows from (A.41). The identity (A.43) is also implied by (A.41) as follows:

$$\begin{aligned} \varphi(A)\psi(A) &= \sum_{i=1}^{k} \varphi(\lambda_i) P_i \sum_{j=1}^{k} \psi(\lambda_j) P_j \\ &= \sum_{i,j=1}^{k} \varphi(\lambda_i)\psi(\lambda_j) P_i P_j = \sum_{i=1}^{k} \varphi(\lambda_i)\psi(\lambda_i) P_i = (\varphi\psi)(A), \end{aligned}$$

where we have used the following properties of the projectors: $P_i P_j = 0$ if $i \neq j$ and $P_t^2 = P_i$.

A.23. (a) As it follows from (A.55), it suffices to verify that

$$\operatorname{dom}\varphi(A) \cap \operatorname{dom}\psi(A) \supset \operatorname{dom}(\varphi+\psi)(A). \tag{B.267}$$

By (A.48), we have

$$\operatorname{dom}\varphi(A) = \left\{x : \int |\varphi(\lambda)|^2 \, d\|E_\lambda x\|^2 < \infty\right\},$$

and the similar expressions for $\operatorname{dom}\psi(A)$ and $\operatorname{dom}(\varphi+\psi)(A)$. Hence, (B.267) will follow if we prove that

$$\max\left(\varphi^2, \psi^2\right) \leq C_1 + C_2 (\varphi+\psi)^2, \tag{B.268}$$

for some positive constants C_1, C_2. If functions φ and ψ are non-negative then

$$\varphi^2 + \psi^2 \leq (\varphi+\psi)^2,$$

and we are done. If one of them is bounded, say, $|\psi| \leq 1$, then

$$(\varphi+\psi)^2 \geq \varphi^2 + 2\varphi\psi \geq \varphi^2 - \left(\frac{1}{2}\varphi^2 + 2\psi^2\right) \geq \frac{1}{2}\varphi^2 - 2$$

and

$$2(\varphi+\psi)^2 + 4 \geq \varphi^2,$$

whence (B.268) follows.

(*b*) By (A.56), it suffices to prove that

$$\operatorname{dom}(\varphi\psi)(A) \cap \operatorname{dom}\psi(A) \supset \operatorname{dom}(\varphi\psi)(A),$$

which is obviously true because ψ is bounded and hence $\operatorname{dom}\psi(A) = \mathcal{H}$.

A.24. (*a*) Since A^{-1} exists and is bounded, 0 is a regular value of A and hence $0 \notin \operatorname{spec} A$. Since $\operatorname{spec} A$ is closed, this means that a small neighbourhood of 0 is also outside $\operatorname{spec} A$, which implies that function $\psi(\lambda) = \frac{1}{\lambda}$ is continuous and bounded on $\operatorname{spec} A$. Hence, $\psi(A) = \frac{1}{A}$ is a bounded self-adjoint operator. Let $\varphi(\lambda) = \lambda$. Since $\varphi\psi \equiv 1$, (A.56) implies

$$\varphi(A)\psi(A) = \mathrm{id} \quad \text{and} \quad \psi(A)\varphi(A) \subset \mathrm{id}.$$

Therefore, $\varphi(A) = A$ has the inverse $\psi(A)$, which was to be proved. Consequently, we see that A^{-1} is a bounded self-adjoint operator.

(*b*) Since $\operatorname{spec} A \subset [0, a]$, the function $\psi(\lambda) = \sqrt{\lambda}$ is defined on $\operatorname{spec} A$. Setting $X = \psi(A) = \sqrt{A}$, we obtain by (A.56)

$$X^2 \subset A \text{ and } \operatorname{dom} X^2 = \operatorname{dom} A \cap \operatorname{dom} X.$$

Let us show that $\operatorname{dom} A \subset \operatorname{dom} X$, which will imply $\operatorname{dom} X^2 = \operatorname{dom} A$ and hence $X^2 = A$. Indeed, by (A.48),

$$\operatorname{dom} A = \left\{ x \in \mathcal{H} : \int_{\operatorname{spec} A} \lambda^2 d\|E_\lambda x\|^2 < \infty \right\}$$

and

$$\operatorname{dom} X = \left\{ x \in \mathcal{H} : \int_{\operatorname{spec} A} \lambda d\|E_\lambda x\|^2 < \infty \right\}.$$

Then $\operatorname{dom} A \subset \operatorname{dom} X$ follows from $\lambda \le \lambda^2 + 1$. The operator X is non-negative definite because, by the Spectral Mapping Theorem,

$$\operatorname{spec} X = \overline{\psi(\operatorname{spec} A)} \subset [0, +\infty).$$

(*c*) Set $\varphi(\lambda) = \lambda$ and $\psi(\lambda) = e^{-\lambda}$ so that $\varphi(A) = A$ and $\psi(A) = e^{-A}$. The function ψ is bounded on $\operatorname{spec} A$ whence by Exercise A.23

$$\varphi(A)\psi(A) = (\varphi\psi)(A).$$

The function $\varphi\psi(\lambda) = \lambda e^{-\lambda}$ is also bounded on $\operatorname{spec} A$ whence we obtain $\operatorname{dom}(\varphi\psi)(A) = \mathcal{H}$. This implies that $\operatorname{dom}(\varphi(A)\psi(A)) = \mathcal{H}$ and hence $\operatorname{ran}\psi(A) \subset \operatorname{dom}\varphi(A)$, which was to be proved.

A.25. Obviously, all eigenvalues λ_k belong to $\operatorname{spec} A$. Let us show that λ is a real number that is not equal to any λ_k or 0 then λ is a regular value of A.

For any $x \in \mathcal{H}$, denote by x^0 the projection of x onto $\ker A$, and by $x^\perp$ the projection of x onto $(\ker A)^\perp$, so that $x = x^0 + x^\perp$. Assuming that $\{v_k\}$ is an orthonormal basis in $(\ker A)^\perp$, we obtain the expansion

$$x^\perp = \sum_k x_k v_k,$$

where

$$x_k = \left(x^{\perp}, v_k\right) = (x, v_k).$$

Therefore,

$$x = x^0 + \sum_k x_k v_k,$$

which implies that

$$Ax = \sum_k \lambda_k x_k v_k, \tag{B.269}$$

where we have used $Ax^0 = 0$ and $Av_k = \lambda_k v_k$.

The inverse operator $(A - \lambda \,\mathrm{id})^{-1}$ can be explicitly constructed as follows. Define operator B in $\mathcal{H}$ by

$$Bx = -\frac{1}{\lambda} x^0 + \sum_k \frac{1}{\lambda_k - \lambda} x_k v_k. \tag{B.270}$$

The sequence $\{\lambda_k\}$ is either finite or tends to 0; therefore, the differences $|\lambda_k - \lambda|$ are separated from 0. Set

$$\alpha := \min\left(|\lambda|, \min_k |\lambda_k - \lambda|\right) > 0.$$

Then the coefficients $\frac{1}{\lambda}$ and $\frac{1}{\lambda_k - \lambda}$ in (B.270) are all bounded by α^{-1}, which implies that the series in (B.270) converges for all $x \in \mathcal{H}$ and

$$\|Bx\| \leq \alpha^{-1} \|x\|.$$

Using (B.269) and (B.270), we obtain

$$\begin{aligned} (A - \lambda \,\mathrm{id})\, Bx &= \sum_k \frac{\lambda_k}{\lambda_k - \lambda} x_k v_k - \lambda \left(-\frac{1}{\lambda} x^0 + \sum_k \frac{1}{\lambda_k - \lambda} x_k v_k\right) \\ &= x^0 + \sum_k x_k v_k = x, \end{aligned}$$

and, similarly, $B\,(A - \lambda \,\mathrm{id})\, x = x$. Hence, B is the inverse of $A - \lambda \,\mathrm{id}$, which finishes the proof.

A.26. (a) Let us show that if A is non-negative definite then $\operatorname{spec} A \subset [0, +\infty)$. It suffices to show that $B = A + \lambda \,\mathrm{id}$ is invertible for any $\lambda > 0$ and the inverse B^{-1} is bounded. Since B is self-adjoint, it suffice to prove B satisfies the condition (A.20) of Exercise A.12, that is, for some $c > 0$,

$$\|Bx\| \geq c\|x\| \text{ for all } x \in \operatorname{dom} B = \operatorname{dom} A.$$

Indeed, we have

$$(Bx, Bx) = (Ax, Ax) + 2\lambda\,(Ax, x) + \lambda^2\,(x, x),$$

and the hypothesis $(Ax, x) \geq 0$ implies

$$\|Bx\| \geq \lambda \|x\|.$$

Assume now that $\operatorname{spec} A \subset [0, +\infty)$ and show that A is non-negative definite. Indeed, by (A.46), we have

$$(Ax, x) = \int_{\operatorname{spec} A} \lambda d\left(E_\lambda x, x\right) \geq 0,$$

because the function $\lambda \mapsto (E_\lambda x, x)$ is monotone increasing and $\lambda \geq 0$ in the domain of integration $\operatorname{spec} A$.

(b) By the definition of a, the operator $A - a\,\mathrm{id}$ is non-negative definite. By part (a), its spectrum is contained in $[0, +\infty)$, which implies that $\operatorname{spec} A \subset [a, +\infty)$ and, hence,

$$a' := \inf \operatorname{spec} A \geq a. \tag{B.271}$$

By the definition of a', the spectrum of the operator $A - a'\,\mathrm{id}$ is contained in $[0, +\infty)$ and, hence, this operator is non-negative definite, which implies that

$$(Ax, x) \geq a' \|x\|^2,$$

for all $x \in \operatorname{dom} A$. Therefore,

$$a = \inf_{\substack{x \in \operatorname{dom} A \\ \|x\|=1}} (Ax, x) \geq a',$$

which together with (B.271) proves that $a = a'$. The supremum of the spectrum is handled in the same way.

A.27. (a) Since function $\varphi\left(l\right) = 1_U\left(\lambda\right)$ is bounded, E_U is a bounded self-adjoint operator. To prove that it is a projector, it suffices to verify that $E_U^2 = E_U$, which follows from $\varphi^2 = \varphi$. If $U = [a, b)$ then, for all $x, y \in \mathcal{H}$,

$$\left(E_U x, y\right) = \int_{-\infty}^{\infty} 1_U\left(\lambda\right) d\left(E_\lambda x, y\right) = \int_{[a,b)} d\left(E_\lambda x, y\right) = \left(E_\lambda x, y\right),$$

whence $E_U = E_\lambda$.

(b) Since $1_{U_2} 1_{U_1} = 1_{U_1}$, we obtain $E_{U_2} E_{U_1} = E_{U_1}$ whence

$$E_{U_2}\left(\operatorname{ran} E_{U_1}\right) = \operatorname{ran} E_{U_1},$$

whence the claim follows.

(c) Since $1_{U_1} 1_{U_2} = 0$, we obtain $E_{U_1} E_{U_2} = 0$, whence the claim follows.

(d) In the both cases, we have the pointwise convergence $1_{U_i} \to 1_U$ and the sequence $\{1_{U_i}\}$ is uniformly bounded. Hence, Lemma 4.8 yields that, for all $x \in \mathcal{H}$,

$$\int_{-\infty}^{+\infty} 1_{U_i} dE_\lambda x \to \int_{-\infty}^{+\infty} 1_U dE_\lambda x,$$

whence $E_{U_i} x \to E_U x$, which was to be proved.

A.28. (a) Let $S = \ker\left(A - a\,\mathrm{id}\right)$ be the eigenspace of a. Let us show that $x \in S$ if and only if $E_{\{a\}} x = x$, which will settle the claim. If $x \in S$ then $\|Ax - ax\| = 0$. On the other hand, we have

$$Ax - ax = \int_{-\infty}^{+\infty} \left(\lambda - a\right) dE_\lambda x$$

and

$$\|Ax - ax\|^2 = \int_{-\infty}^{+\infty} (\lambda - a)^2 \, d\|E_\lambda x\|^2.$$

The function $(\lambda - a)^2$ is positive everywhere except for $\lambda = a$. Since function $\lambda \to \|E_\lambda x\|^2$ is increasing, the only way the above integral can be equal to 0 is that the function $\lambda \to \|E_\lambda x\|^2$ is identical constant for $\lambda < a$ and for $\lambda > a$. Letting $\lambda \to -\infty$ and $\lambda \to +\infty$ we conclude that

$$\|E_\lambda x\| = \begin{cases} 0, & \lambda < a, \\ \|x\|, & \lambda > a. \end{cases}$$

This implies, that

$$E_\lambda x = \begin{cases} 0, & \lambda < a, \\ x, & \lambda > a, \end{cases}$$

whence

$$E_{\{a\}} x = \lim_{\varepsilon \to 0+} E_{(a-\varepsilon, a+\varepsilon)} x = \lim_{\varepsilon \to 0+} (E_{a+\varepsilon} x - E_{a-\varepsilon} x) = x - 0 = x.$$

To prove the converse, consider first functions $\varphi(\lambda) = \lambda$ and $\psi(\lambda) = 1_{\{a\}}(\lambda)$. We have $\varphi\psi = a\psi$ whence

$$\varphi(A)\, \psi(A) = a\psi(A),$$

that is

$$AE_{\{a\}} = aE_{\{a\}}.$$

Therefore, if $E_{\{a\}} x = x$ then $AE_{\{a\}} x = Ax$ and

$$AE_{\{a\}} x = aE_{\{a\}} x = ax,$$

that is, $Ax = ax$ and $x \in S$.

(*b*) By part (*a*), $x \in \operatorname{ran} E_{\{\alpha\}}$. By Exercise A.27, this implies that $E_U x = 0$ for any Borel set $U \subset \mathbb{R}$ that does not contain α. It follows that

$$\varphi(A)\, x = \int_{\mathbb{R}} \varphi(\lambda)\, dE_\lambda x = \int_{\{\alpha\}} \varphi(\lambda)\, dE_\lambda x = \varphi(\alpha)\, E_{\{\alpha\}} x = \varphi(\alpha)\, x,$$

which was to be proved.

A.29. It follows from (A.46) that

$$A = \int_{-\infty}^{+\infty} \lambda 1_{\operatorname{spec} A} dE_\lambda = \sum_{i=1}^{k} \int_{-\infty}^{+\infty} \lambda 1_{\{\lambda_i\}} dE_\lambda = \sum_{i=1}^{k} \lambda_i E_{\{\lambda_i\}}.$$

By Exercise A.28, $E_{\{\lambda_i\}} = P_i$, which finishes the proof.

A.30. Since $|\varphi(\lambda)| \le \Phi(\lambda)$ and

$$\operatorname{dom} \varphi(A) = \left\{ x \in \mathcal{H} : \int_0^\infty |\varphi(\lambda)|^2 \, d\|E_\lambda x\|^2 < \infty \right\},$$

we obtain that $x \in \operatorname{dom} \varphi(A)$. In the same way, we have also $x \in \operatorname{dom} \varphi_n(A)$. Since

$$|\varphi(\lambda) - \varphi_n(\lambda)|^2 \le 4\Phi^2(\lambda)$$

where the function $\Phi^2(\lambda)$ is integrable against the measure $d\|E_\lambda x\|^2$, the Lebesgue dominated convergence theorem implies that

$$\|\varphi(A)\,x - \varphi_n(A)\,x\|^2 = \int_0^\infty |\varphi(\lambda) - \varphi_n(\lambda)|^2\, d\|E_\lambda x\|^2 \longrightarrow 0 \ \text{ as } n \to \infty,$$

which was to be proved.