An Introduction to Complex Analysis and Geometry

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Extensions, errata, and clarifications November 1, 2011.

The author thanks all those who have commented on the text. Robert Burckel and Jeff Johannes have made some particularly helpful comments. I organize the comments in order, by section title.

What is a natural number?

Page 9. Just after Definition 3.5, the clause "then n is an element of every inductive subset of $\mathbf{R}$" should include the words "containing 1" at the end. It might also be better to use "each" instead of "every". Thus the clause should be

"then n is an element of each inductive subset of $\mathbf{R}$ containing 1."

A glimpse at metric spaces

Page 22. After the paragraph on open and closed balls, it seems appropriate to mention the definition of continuous function between metric spaces. Suppose (X_1, ρ_1) and (X_2, ρ_2) are metric spaces and $f : X_1 \to X_2$ is a function. The analogue of the standard criterion in Exercise 1.20 works:

Definition. The function $f : X_1 \to X_2$ is continuous at a if and only if the following holds. For each $\epsilon > 0$, there is a $\delta > 0$ such that $\rho_1(x, a) < \delta$ implies $\rho_2(f(x), f(a)) < \epsilon$.

We use this concept several times when (X_1, ρ_1) is Euclidean space $\mathbf{R}^n$ and (X_2, ρ_2) is $\mathbf{R}$ or $\mathbf{C}$.

Page 23. The word "relatively" should be added in order that Definition 6.3 read as follows:

Definition. Let (X, δ) be a metric space. A subset S of X is called connected if the following holds: whenever $S = A \cup B$, where A and B are relatively open in S and $A \cap B = \phi$, then either $A = \phi$ or $B = \phi$.

Page 23. It should be remarked that the empty set is connected. Furthermore, the notion of *relatively open* should be elaborated. A subset of S is relatively open (in S) if it is open when regarded as a subset of the metric space (S, δ). For example, if X is the interval $[0, 1]$, and S is the interval $[0, 1)$, then S is relatively open in X, but S is not open as a subset of $\mathbf{R}$.

Uniform convergence and consequences

Page 40. Corollary 5.1. Only a sketch of the proof is given. Perhaps one sentence should be added to that sketch. Before the words "We may therefore invoke" add

"By the root test, each of the series $\sum nc_n z^{n-1}$ and $\sum \frac{c_n}{n+1} z^{n+1}$ converges for $|z| < R$ and each converges uniformly on any closed subdisk."

Logarithms

Page 47. The following exercise should be added.
Find all values of $\log(i^2)$ and all values of $2\log(i)$. Are these sets the same?

The Riemann sphere

Page 65. Exercise 3.31. Robert Burckel suggests the following reference for this exercise: Hua, Xia, A converse to a theorem on linear fractional transformations, Mathematics Magazine 81 (2008), 367-369.

A glimpse at harmonic functions

Pages 91-92. Theorem 5.1 is well-known; the proof in the text establishes only one implication: if u satisfies the Mean-Value Property, then u is harmonic. The converse assertion requires considerable additional discussion.

First, the "if and only if" just above (25) on page 92 should be changed to "implies". Second, after (25), the following should be added.

The easiest way to prove the converse assertion is to derive it from the Cauchy integral formula (Theorem 4.2 in Chapter 6). To do so, we first need to know that u is the real part of a complex analytic function f. Then we apply Theorem 4.2 to f, where γ is a circle of radius r about p. Put $\zeta = p + re^{i\theta}$. Then $d\zeta = rie^{i\theta}d\theta$ and we get

$$f(p) = \frac{1}{2\pi i}\int_0^{2\pi} f(p+re^{i\theta})i\ d\theta,$$

which gives the mean value property for f. Taking the real part of each side yields the mean-value property for u.

It is also possible to derive the result directly for harmonic functions without using complex variable theory; the analogous result holds in all dimensions. The proof uses ideas from calculus of several variables quite similar to those used in the proof of the Cauchy integral formula. First one establishes Green's Theorem (Theorem 2.1 of Chapter 6) and derives from it various formulas (called Green's identities) involving normal derivatives $\frac{\partial u}{\partial n}$ and surface measure $d\sigma$.

Let Ω be a domain in $\mathbf{R}^2$ with smooth boundary S. Assume u, v are twice differentiable on the closure of Ω. Then one of Green's identities states that

$$\int_S (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})\ d\sigma = \int_\Omega (v\nabla u - u\nabla v)\ dxdy. \tag{G}$$

First take u to be harmonic and put $v = 1$ in (G); we conclude that

$$\int_S \frac{\partial u}{\partial n} d\sigma = 0. \tag{H1}$$

Then take u to be harmonic and put $v = -\log(|z-p|)$. In a deleted neighborhood of p, v is also harmonic. Apply (G) when Ω is an annular region centered at p with inner radius ϵ and outer radius r. Let $S_r(p)$ be the circle of radius r about p. Then $d\sigma$ is a multiple of $d\theta$. The right-hand side of (G) is 0, and we conclude that the integral

$$\int_{S_r(p)} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})d\theta \tag{H2}$$

is independent of the radius r. Since v is constant on the circle, the first term in (H2) vanishes by (H1). Also $\frac{\partial v}{\partial n}$ is constant on the circle. It follows that the integral of u over the circle is independent of the radius. Since u is continuous at the center point p, the integral over the circle equals $2\pi u(p)$, and the mean-value property follows.

Pages 92-93. This proof can be made completely rigorous, but doing so is not appropriate for the intended audience. Nonetheless, rewriting the last sentence might make the point more clearly. After (27), write:

Formula (27) exhibits f as a differentiable function that is independent of $\overline{z}$. Hence f is complex analytic. Formula (27) also shows that u is the real part of f.

What is a differential form?

This section, as stated on page 96, is intended to be informal. It is not possible to include precise definitions of exterior algebras, cotangent bundles, and so on here, while being true to the intended audience of the book.

The Cauchy Integral Formula

Page 112. In the statement of Theorem 4.2, the $z \in \Omega$ should of course be replaced by $z \in \mathrm{int}(\gamma)$.

Fourier transforms

Page 129. Definition 3.1. To clarify that ξ is assumed to be real here, add the words "for $\xi \in \mathbf{R}$" to read.

Let $f : \mathbf{R} \to \mathbf{C}$ be an integrable function. Its Fourier transform, denoted by $\hat{f}$, is defined for $\xi \in \mathbf{R}$ by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx. \tag{21}$$

Winding numbers, zeroes, and poles

Page 146. Here is a simple proof of Remark 3.1.

Proof. Suppose $|f-g| < |f|+|g|$ holds on γ. Then neither f nor g can vanish on γ. Furthermore, consider the family of meromorphic functions F_t defined for $0 \le t < \infty$ by

$$F_t(z) = t + \frac{f(z)}{g(z)}.$$

Suppose $F_t(z) = 0$ for some $z \in \gamma$. Then $f(z) = -tg(z)$. Using $|f-g| < |f| + |g|$ yields

$$(t+1)|g(z)| = |-tg(z) - g(z)| = |f(z) - g(z)| < |f(z)| + |g(z)| = (t+1)|g(t)|,$$

which is a contradiction. Therefore $F_t(z)$ is also not zero on γ. We compute the number of zeroes minus the number of poles to get

$$Z(F_t) - P(F_t) = \frac{1}{2\pi i} \int_\gamma \frac{F_t'(z)}{F_t(z)} dz.$$

This expression depends continuously on t but takes integer values. Hence, because $[0, \infty)$ is connected, it is a constant. Letting t tend to infinity in the integral shows that this constant is 0. Setting $t = 0$ then gives $Z(f) - Z(g) = 0$. Hence $Z(f) = Z(g)$. □

Page 146. Just before Corollary 3.1, it should be remarked that mathematicians often write "roots" instead of "zeroes" when working with polynomials. These words are synonymous in the next several exercises.

Elementary mappings

Page 151. In the first sentence after Theorem 5.1, "is not zero" means "has no zeroes".

Page 151. The Riemann mapping theorem is stated in the book. Its proof is considerably more advanced than other material in this book. We provide it here for interested readers. The author would appreciate feedback about whether this material should be included and elaborated in a possible second edition.

The Riemann mapping theorem

One crucial point in the proof is the existence of an explicit complex analytic function for which a certain supremum is attained. We therefore begin with ideas generalizing notions of compactness to spaces of functions.

Let Ω be an open, connected subset of $\mathbf{C}$. We write $\mathcal{O}(\Omega)$ to denote the space of complex analytic functions $f : \Omega \to \mathbf{C}$. Since the sum of analytic functions is analytic and a constant times an analytic function is analytic, $\mathcal{O}(\Omega)$ is a complex vector space. We naturally wish to introduce a notion of convergence in this space.

To do so, we regard $\mathcal{O}(\Omega)$ as a subspace of $C(\Omega)$, the continuous, complex-valued functions on Ω. Recall from Lemma 5.1 of Chapter 2 that the uniform limit of continuous functions is continuous. Based on our experience with power series, where we encountered uniform convergence on closed subdisks of a given disk, we make the following definition.

Definition 0.1. Let f_n be a sequence in $\mathcal{O}(\Omega)$ (or $C(\Omega)$). We say that f_n converges in $\mathcal{O}(\Omega)$ (or $C(\Omega)$) if, for each compact subset $K \subset \Omega$, f_n converges uniformly on K.

Lemma 0.1. *Suppose $f_n \in \mathcal{O}(\Omega)$ and that f_n converges in $\mathcal{O}(\Omega)$ to some function f. Then $f \in \mathcal{O}(\Omega)$. Thus $\mathcal{O}(\Omega)$ is a closed subset of $C(\Omega)$.*

Proof. Each f_n is continuously differentiable and satisfies $\overline{\partial}(f_n) = 0$. Hence the barred derivatives converge uniformly. We will show, for each compact set $K \subset \Omega$, that $\partial(f_n)$ converges uniformly on K. It then follows, by Theorem 5.1 of Chapter 2, that the limit function f is continuously differentiable and satisfies $\overline{\partial}(f_n) = 0$. Therefore $f \in \mathcal{O}(\Omega)$. It remains to establish the uniform convergence of the derivatives. This standard result follows from the Cauchy estimates, Corollary 4.2 of Chapter 6. Here are the details: Choose a compact subset K containing a closed disk of radius R about z. For each n, $|f_n|$ achieves its maximum $M = M(R, n)$ on this closed disk, and the Cauchy estimates yield

$$|f_n'(z)| \leq \frac{M(R,n)}{R}. \qquad (1)$$

It follows from the compactness of K that there is a constant C_K, depending on K, such that

$$\sup_K |f_n'(z) - f_m'(z)| \le C_K \sup_K |f_n(z) - f_m(z)|. \tag{2}$$

It follows from (2) and the uniform convergence of f_n that the sequence of derivatives also converges uniformly on K. □

Next we need to understand *compactness* in the space $\mathcal{O}(\Omega)$. Our treatment of this advanced material is not complete, because we omit the proof of the Arzela-Ascoli theorem characterizing compact subsets of $C(\Omega)$. Recall that the Bolzano-Weierstrass theorem in $\mathbf{R}$ or $\mathbf{C}$ states that a bounded sequence has a convergent subsequence. See Lemma 1.1 of this Chapter. The next result provides an *infinite-dimensional* analogue. Let $S \subset C(\Omega)$; thus S is a collection of continuous complex-valued functions on Ω.

DEFINITION 0.2. Let Ω be a domain in $\mathbf{C}$ and let S be a subset of $C(\Omega)$.

- S is *closed* if, whenever $f_n \in S$ and f_n converges to f in $C(\Omega)$, then $f \in S$.
- S is *compact* if every sequence in S has a convergent subsequence.
- S is *uniformly bounded* if the set of all values of all functions in S is a bounded set in $\mathbf{C}$.
- S is *equicontinuous* if the following strong form of continuity holds: For all $\epsilon > 0$, there is a $\delta > 0$ such that $|z - w| < \delta$ implies $|f(z) - f(w)| < \epsilon$ for all $f \in S$.

THEOREM 0.1 (Arzela-Ascoli). *Let S be a subset of $C(\Omega)$. Then S is compact if and only if S satisfies the following three conditions.*

- *S is closed.*
- *S is uniformly bounded.*
- *S is equicontinuous.*

This theorem works well in complex analysis because of the Cauchy estimates. Let S be a subset of $\mathcal{O}(\Omega)$, and suppose that S is uniformly bounded. The Cauchy estimates then imply that S is equicontinuous. The reason is simple. When f is complex analytic, the following inequality holds, and the constants are independent of f.

$$|f(z) - f(w)| \le C_1 \sup_\zeta (|f'(\zeta)|)\ |z - w| \le C_2 \sup_\zeta (|f(\zeta)|)\ |z - w|. \tag{$*$}$$

The first step in (*) is elementary, as $f(z) - f(w)$ is the integral of f' from z to w. The second step in (*) follows from the Cauchy estimates. Consider a uniformly bounded set S of analytic functions. The inequality in (*) becomes, for a new constant C,

$$|f(z) - f(w)| \le C|z - w|. \tag{$**$}$$

But (**) obviously implies equicontinuity.

In the proof of the Riemann mapping theorem we will first reduce to the case when the domain is a subset of the unit disk $\mathbf{B}$ containing 0. After doing so, we will consider a collection $\mathcal{F}$ of analytic functions on a domain Ω. The domain Ω will be a subset of the unit disk, and $f(0) = 0$ for each $f \in \mathcal{F}$. The image of each f in $\mathcal{F}$ will be a subset of $\mathbf{B}$ and hence $\mathcal{F}$ is uniformly bounded. Furthermore each function in the class $\mathcal{F}$ will be injective. We then wish to choose that f for which $|f'(0)|$ is maximal. The crucial point is that this f will be in $\mathcal{F}$.

Given a sequence $f_n \in \mathcal{F}$ with $|f_n'(0)|$ converging to $\sup_{\mathcal{F}}|f_n'(0)|$, the Arzela-Ascoli theorem guarantees that some subsequence of f_n converges to an f in $\mathcal{F}$. By continuity, $f(0) = 0$. A theorem of Hurwitz, which can be derived easily from Rouche's theorem, states that the uniform limit of injective complex analytic mappings is either itself injective or a constant mapping. In our situation we have $f'(0) \neq 0$, and therefore f is not a constant. Hence f is injective. We will then need to prove that this f is surjective in order for it to be the conformal mapping from Ω to the unit disk.

Proof. Let Ω be a simply connected domain in $\mathbf{C}$, and suppose that Ω is not all of $\mathbf{C}$. We will construct a conformal map from Ω to the unit disk $\mathbf{B}$.

The first step is to reduce to the case when Ω is contained in $\mathbf{B}$. Choose z_0 not in Ω. Because Ω is simply connected, we can choose a branch $\log(z - z_0)$ of the logarithm that is analytic on Ω. We then define an analytic square root:

$$B(z) = \sqrt{z - z_0} = e^{\frac{1}{2}\log(z-z_0)}.$$

Note that B is injective. Find w in the image of B. A simple computation shows that $-w$ is *not* in the image of B. By the open mapping theorem, the image of B is an open set. Hence there is a neighborhood of $-w$ that is also not in the image of B. Therefore the function $z \to \sqrt{z - z_0} + w$ omits a neighborhood of 0. Next put $G(z) = \frac{1}{\sqrt{z-z_0}+w}$. This function is complex analytic, injective, and bounded on Ω. Thus the image of G is contained in a ball. After composing with a translation, and then rescaling, we may assume that there is a complex analytic, injective mapping $H : \Omega \to \mathbf{B}$. We may also assume, for some point $c \in \Omega$, that $H(c) = 0$. What have we done so far? We have shown that there is a complex analytic, injective, H with $H(c) = 0$ and $H(\Omega) \subset \mathbf{B}$.

In the second step we assume that Ω is contained in $\mathbf{B}$ and that $0 \in \Omega$. If we can find a conformal map in this case, then we compose it with the function H from the first step to solve the problem in general.

Consider the set $\mathcal{F}$ of all complex analytic maps $g : \Omega \to \mathbf{B}$ such that

- $g(0) = 0$
- g is injective
- $g'(0) > 0$.

This set is non-empty, as it contains the identity map. The set of values of functions in $\mathcal{F}$ is bounded by 1. Hence $\mathcal{F}$ is uniformly bounded. It also follows from the Cauchy estimates that the set of values of $g'(0)$ for $g \in \mathcal{F}$ is a bounded set. Let α be the supremum of $g'(0)$ for $g \in \mathcal{F}$. We can find a sequence $f_n \in \mathcal{F}$ such that

$$\lim_{n\to\infty} f_n'(0) = \alpha.$$

By the Arzela-Ascoli theorem, there is a subsequence of f_n which converges to an $f \in \mathcal{F}$ with $f'(0) = \alpha$. We claim that this particular f does the job. That f is injective follows from the Hurwitz result mentioned earlier. To prove that f is surjective is harder. We suppose otherwise and seek a contradiction.

First we recall the linear fractional transformations defined in Section 5.1 and compute their derivatives. These functions are conformal mappings from the unit disk to itself. We have

$$\phi_a(z) = \frac{z-a}{1-\overline{a}z}. \tag{3}$$

We compute ϕ_a' by the quotient rule, obtaining after simplifying the result

$$\phi_a'(z) = \frac{1-|a|^2}{(1-\overline{a}z)^2}. \tag{4}$$

Suppose that b is not in the image of f. We put $w_1 = \sqrt{\phi_b(f)}$. Thus we have

$$w_1(z) = \sqrt{\frac{f(z)-b}{1-\overline{b}f(z)}}. \tag{5}$$

Let $a = w_1(0)$. We then put $w_2 = \phi_a(w_1)$. Thus

$$w_2(z) = \frac{w_1(z) - w_1(0)}{1-\overline{w}_1(0)w_1(z)} \tag{6}$$

and $w_2(c) = 0$. It is evident that w_1 and w_2 are injective. After multiplication by a factor of modulus 1, we obtain an element of the class $\mathcal{F}$. The contradiction results from showing that $|w_2'(0)| > f'(0)$.

We compute $w_2'(0)$ using the chain rule. Since $w_2 = \phi_a(w_1)$, we get

$$w_2'(0) = \frac{1-|a|^2}{(1-\overline{a}w_1(0))^2} w_1'(0) = \frac{1}{1-|a|^2} w_1'(0). \tag{7}$$

We compute $w_1'(0)$ using the chain rule on $w_1(z) = \sqrt{\phi_b(f(z))}$. We get

$$w_1'(z) = \frac{1}{2w_1(z)} \frac{1-|b|^2}{(1-\overline{b}(f(z))^2} f'(z).$$

Using $f(0) = 0$ we get

$$w_1'(0) = \frac{1}{2a}(1-|b|^2)f'(0). \tag{8}$$

Combining (7) and (8) and using $|b| = |a|^2$ finally gives

$$|w_2'(0)| = \frac{1}{1-|a|^2}\frac{1}{2|a|}(1-|a|^4)f'(0) = \frac{1+|a|^2}{2|a|}f'(0). \tag{9}$$

The factor in front of $f'(0)$ on the right-hand side of (9) exceeds 1, since $|a| < 1$. Multiply by a factor of modulus 1 to ensure that $w_2'(0)$ is positive. We obtain a function in $\mathcal{F}$ whose derivative at 0 exceeds that of f. Our assumption that f is not surjective must then have been false, and hence f is the needed conformal map. □