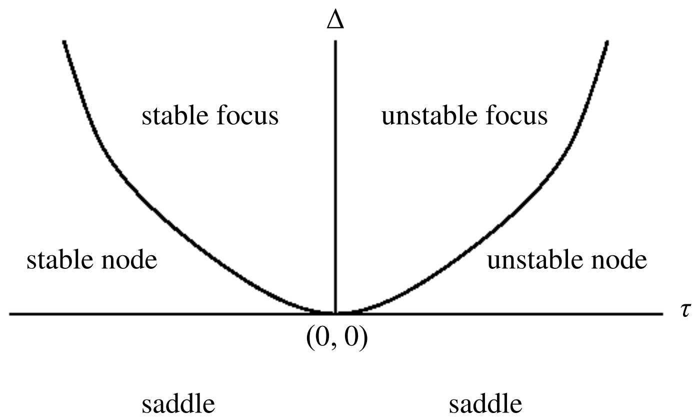
Two-dimensional Constant Coefficient Linear Systems

The eigenvalues of a two dimensional linear system can be determined from the trace τ and determinant Δ as given in Theorem 4.3. See the following figure.



Summary of Drawing the Phase Portraits

The first step is to find the eigenvalues r_1 and r_2 with corresponding eigenvectors $\mathbf{v}^1$ and $\mathbf{v}^2$.

1. If $r_1 < 0 < r_2$ (real), then $\mathbf{0}$ is a saddle. See Table 1 on page 25 and Figure 2 on page 24. Solutions along the line $\text{span}(\mathbf{v}^1)$ are attracted toward $\mathbf{0}$ and solutions along the line $\text{span}(\mathbf{v}^2)$ are repelled away from $\mathbf{0}$.
2. If $r_2 < r_1 < 0$ (real), then $\mathbf{0}$ is a stable node. See Table 2 on page 27 and Figure 3 on page 26. Trajectories off $\text{span}(\mathbf{v}^2)$ approach $\mathbf{0}$ tangent to $\text{span}(\mathbf{v}^1)$.
3. If $0 < r_1 < r_2$ (real), then $\mathbf{0}$ is an unstable node. See Figure 4 on page 28. Letting time decrease, trajectories off $\text{span}(\mathbf{v}^2)$ approach $\mathbf{0}$ tangent to $\text{span}(\mathbf{v}^1)$. As t goes to infinity, the growth rate of the component along $\mathbf{v}^2$ is faster than the one along $\mathbf{v}^1$.
4. If $r_1 = r_2 < 0$ with only one independent eigenvector, then $\mathbf{0}$ is a degenerate stable node. See Table 4 on page 41 and Figure 11 on page 39.
5. If $r_1 = r_2 > 0$ with only one independent eigenvector, then $\mathbf{0}$ is a degenerate unstable node. The phase plane is similar to case 4 with the directions reversed.
6. If the eigenvalues are complex $\alpha \pm i\,\beta$ with $\beta \neq 0$, then we have the following cases. See Table 3 on page 33. In the phase plane, it is important to get the direction correct, i.e., clockwise or counterclockwise.
   - **a.** If $\alpha = 0$, then $\mathbf{0}$ is an elliptic center. See Figure 6 on page 30.
   - **b.** If $\alpha < 0$, then $\mathbf{0}$ is a stable focus. See Figure 8 on page 32.
   - **c.** If $\alpha = 0$, then $\mathbf{0}$ is an unstable focus. Reverse the directions of the trajectories on Figure 8 on page 32.