Two Dimensional Geometries: Exercise pages

C. Herbert Clemens
Mathematics Department
Ohio State University
Columbus, OH 43210

EXERCISE 1. a) Write out the Theorem above in the case in which $n = m = p = 1$ to be sure that you recognize how it is just a several variable formulation of the most important theorems from your introductory course in calculus.

b) Write the full 2×2 matrix form of the Chain Rule when $n = m = p = 2$.

EXERCISE 2. a) Write out the above Theorem in the case in which $n = 1$ to be sure that you recognize how it is just a several variable formulation of the most important theorem from your introductory course in integal calculus.

b) Integrate $\sin\left(y_1^2 + y_2^2\right)$ over the region in the (y_1, y_2)-plane bounded by the unit circle.
Hint: Use the Substitution Rule to change (y_1, y_2) to polar coordinates.

Exercise 3. Given an angle $\angle BAC$ show by drawings the two regions into which it divides the plane. Show how the (signed) measure of the angle depends on which region you pick and on which is the initial ray and which is the final ray of the angle.

Exercise 4. Using a sketch on grid paper or an algebraic formulation in the Euclidean plane,

a) give a concrete example of a rigid motion that takes $(1, 2)$ to $(3, 5)$,

b) modify your answer in a) so that the tangent vector $(1, 0)$ emanating from $(1, 2)$ goes to the tangent vector $(0, 1)$ emanating from $(3, 5)$.

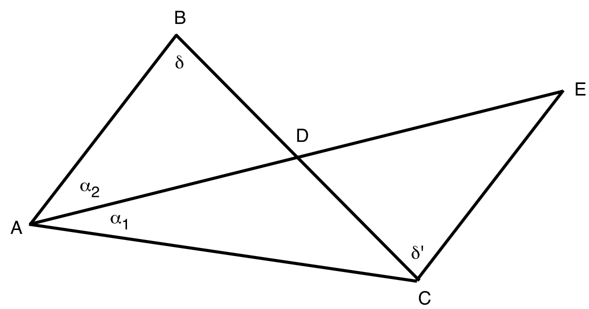
Exercise 5. (**NG**) Think back to high school days and write the triangle congruence rules SSS, SAS, and ASA, in other words "side-side-side," "side-angle-side," and "angle-side-angle." Be careful with your wording for each rule so that two triangles can be moved onto each other by a rigid motion if and only if they satisfy they satisfy that rule.

EXERCISE 6. Give a counterexample to show that there is no universal SSA law.

EXERCISE 7. a) Given an angle $\angle BAC$ and a ray R, use E1-E4 to produce an angle congruent to $\angle BAC$ but having R as one of its sides.

b) Use a) and E1-E4 to prove the SAS rule for congruent triangles.

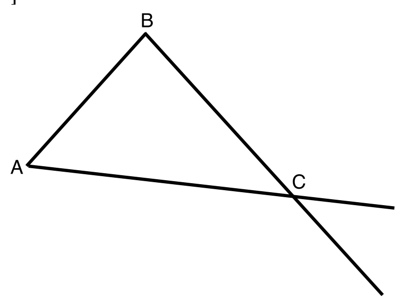
Exercise 8. (**NG**) Suppose, in the diagram below that $|BD| = |CD|$ and $|AD| = |ED|$.



a) Show that $\angle ADB = \angle CDE$ have the same measure and so are congruent. (A pair of opposite angles where two lines cross is called a *vertical angle pair.* Your argument will show that the two opposite angles in any vertical angle pair are congruent.)

b) Show that triangle $\triangle BDA$ and triangle $\triangle CDE$ are congruent. [MJG,138]

Exercise 9. In $\triangle ABC$ below, an exterior angle at vertex C is formed by a side of the triangle that contains C and the outward extension of the other side of the triangle containing C. Show in Neutral Geometry that the measure of either exterior angle of the triangle at C is greater than the measure of either remote interior angle. [MJG,119]



Hint: Compare this figure with the figure in Exercise 8.

EXERCISE 10. Use Exercise 9 to show that the sum of any two angles of a triangle is less than $180°$.
Hint: Begin with the fact that the sum of the measures of the interior angle at C and either exterior angle at C is $180°$.

Exercise 11. (**NG**) Show in **NG** that, if two lines cut by a transversal line have a pair of congruent alternate interior angles, then they are parallel. [MJG,117]

Hint: Suppose the assertion is false for some pair of lines. Find a triangle that violates the conclusion of Exercise 9.

EXERCISE 12. (**NG**) For the diagram in Exercise 8, show that the sum of the angles in $\triangle ACE$ is the same as the sum of the angles in $\triangle ACB$.

Exercise 13. (**NG**) Suppose that there is a triangle $\triangle ABC$ in **NG** for which the sum of the angles in a triangle $\triangle ABC$ is $(180 + x)$ with $x > 0$. For the $\triangle ABC$ in Exercise 8, show that one of the angles of $\triangle ACE$ is no more than half the size of $\angle BAC$. Yet by Exercise 12 the sum of the angles in a triangle $\triangle ABC$ is still $(180 + x)^\circ$.
Hint: In the diagram in Exercise 8, this new 'smaller' angle may or may not have vertex A. [MJG,125-127]

Exercise 14. (**NG**) Suppose that there is a triangle $\triangle ABC$ in **NG** for which the sum of the angles in a triangle $\triangle ABC$ is $(180 + x)^\circ$ with $x > 0$. Let α denote the measure of $\angle BAC$.

a) Rename the vertices of $\triangle ACE$ in Exercise 13 to get the same triangle denoted as $\triangle A_2B_2C_2$ but with A_2 denoting the vertex (either A or E) that has the smaller angle. Repeat the above constructions to construct a triangle for which the sum of the angles in a triangle $\triangle ABC$ is $(180 + x)^\circ$ but one of its angles is less than or equal to

$$\frac{1}{4}\alpha.$$

b) Repeat the construction in Exercise 13 over and over again n-times to construct a triangle with the sum of its angles still equal to $(180 + x)^\circ$ but such that one of its angles has size less than

$$\frac{1}{2^n}\alpha.$$

Hint: Use Exercise 13 over and over again.

EXERCISE 15. (**NG**) Suppose that there is a triangle $\triangle ABC$ in **NG** for which the sum of the angles in a triangle $\triangle ABC$ is $(180 + x)^\circ$ with $x > 0$. Show that there is a positive integer n_0 so that, if you repeat the construction in Exercise 13 n_0-times, the result will be a triangle $\triangle A_{n_0} B_{n_0} C_{n_0}$ with the sum of its angles still equal to $(180 + x)^\circ$ but with one of its angles having measure less than x. [MJG,125-127]
Hint: Remember that $x > 0$ is fixed once and for all at the beginning of the argument in Exercise 13 . Then use that $\lim_{n \to \infty} \frac{1}{2^n} \alpha = 0$.

EXERCISE 16. (**NG**) a) For $\triangle A_{n_0}B_{n_0}C_{n_0}$ in Exercise 15, explain why the sum of two of its interior angles would have to be greater than 180º.

b) Use Exercise 10 to conclude that you cannot have a triangle with two angles summing to more than 180°. [MJG,124]

Exercise 17. (**NG**) Show the following:

a) The sum of the interior angles in any quadrilateral is no greater than $360°$.

b) The sum of the interior angles of an n-gon is no greater than $(n-2)\cdot 180°$.

Hint: Divide the n-gon into a triangle and a $(n-1)$-gon and use induction.

Exercise 18. (**NG**) a) If we add the line segment $\overline{AD}$ to the figure above, show that $\angle BAD$ is congruent to $\angle ADC$.

b) Use Exercise 17 to show that the two congruent angles in a) must each have measure less that or equal to 90°.

EXERCISE 19. (**NG**) Show that the measure of $\angle BA'D$ is less than the measure of $\angle ADC$.
Hint: Use Exercise 18 and Exercise 9a).

EXERCISE 20. Use the three implications just above and pure logic to show that

$\angle BAD < \angle ADC$ implies that $|AB| > |CD|$,
$\angle BAD = \angle ADC$ implies that $|AB| = |CD|$,
$\angle BAD > \angle ADC$ implies that $|AB| < |CD|$.

Hint: If the first implication is false, then $\angle BAD < \angle ADC$ and either $|AB| < |CD|$ or $|AB| = |CD|$. If $|AB| < |CD|$, we have already shown that $\angle BAD > \angle ADC$, so we have a contradiction! Etc., etc.

Exercise 21. (**EG**) a) Show that, if two parallel lines L and M are cut by a transversal line, opposite interior angles are equal.
Hint: Concentrate on the words ‘only one’ in E5.

b) Show that, if two parallel lines L and M are cut by a transversal line, corresponding angles are equal.

Hint: Use a) and the fact that vertical angles are equal.

EXERCISE 22. a) Show that, if a line T crosses one of two parallel lines L and M, it must cross the other.

b) Show that, if two distinct lines L and M are both parallel to another line, then they are parallel to each other.

EXERCISE 23. a) Show that angles $\angle BAC$ and $\angle B'A'C'$ in the Euclidean plane are equal (i.e. have the same measure) if their corresponding rays are parallel. Hint: Construct a transversal through A and A'.

b) Show that $\angle BAC$ and $\angle B'A'C'$ are equal (i.e. have the same measure) if $\angle B'A'C'$ can be rotated around A' to obtain an angle $\angle B''A'C''$ with corresponding rays parallel to those of $\angle BAC$.

Exercise 24. (**EG**) Use the 'only one' assertion in E5 together with what we have established about Neutral Geometry to show that in **EG** the sum of the interior angles of any triangle is 180°.

Exercise 25. (**EG**) Show that in **EG** the sum of the interior angles of a quadrilateral is 360°.

EXERCISE 26. (**EG**) a) Show in **EG** that, given any positive real numbers a and b, there exists a rectangles with adjacent sides of lengths a and b.

Hint: Construct a third side of length a *perpendicular* to the side of length b that is, the two sides meet in a right angles. Then use Exercises 18a) and 25.

b) If we define the area inside the rectangle with adjacent sides of lengths a and $b$as $a{\cdot}b$, show that the area of any right triangle is $\frac{1}{2}\,(base)\cdot(height)$.

c) Show that the area of (i.e. inside) any triangle is $\frac{1}{2}\,(base)\cdot(height)$.

Hint: Find two right triangles such that the area of the given rectangle is the sum or difference of the areas of the right triangles.

Exercise 27. (**EG**) Show that there is a cartesian coordinate system on **EG**, that is, the set of points of **EG** are in $1-1$ correspondence with the set of pairs of real numbers. Write your solution as if you are explaining this to an eighth-grader.

EXERCISE 28. (**EG**) Show that the slope of a line in a cartesian plane is obtained by taking any two points (x_1, y_1) and (x_2, y_2) and computing

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for any two distinct points (x_1, y_1) and (x_1, y_1) on the line.

Exercise 29. (**EG**) a) Show that two lines in a cartesian plane are parallel if and only if they have the same slopes.
Hint: Use Exercise 21.

b) Show that two lines L and M intersecting at a point P are perpendicular if and only if the product of their slopes is -1 (or one has slope 0 and the other has slope $\pm\infty$).
Hint: Suppose that the slopes are not 0 or $\pm\infty$. Draw the line N parallel to the x-axis through P. Show that the following two statements are equivalent:

i) the right triangle having one side along N and having hypoteneuse along L and of length 1 is congruent to the right triangle with side on N and length 1 hypotenuse along M,

ii) L and M are perpendicular.

EXERCISE 30. (**EG**) State the Pythagorean theorem in **EG** and use Exercise 26 to prove it.
Hint: For the give right triangle, draw a cartesian coordinate system with $(0,0)$ at the vertex of the right triangle and with the x-axis containing one side of the right angle and the y-axis containg the other side. In that cartesian plane, draw a square with vertices $(0,0), (a+b,0), (0,a+b)$ and $(a+b,a+b)$. Inside that square, draw the square with vertices $(a,0), (a+b,a), (b,a+b)$ and $(0,b)$. Show by rearranging the triangles in your drawing that the area of the big square is the area of the little square plus the area of 4 right triangles, each of area $\frac{ab}{2}$. (Such an argument is one of the ones that depends least on the particular choice of cartesian coordinate system.

EXERCISE 31. Use the Pythagorean theorem to justify the definition (3.2.1).

Exercise 32. *(Triangle Inequality)* Show that, for any three points (a_1, b_1), (a_2, b_2), and (a_3, b_3) in the cartesian plane,

$$d\left(\left(a_1, b_1\right), \left(a_3, b_3\right)\right) \leq d\left(\left(a_1, b_1\right), \left(a_2, b_2\right)\right) + d\left(\left(a_2, b_2\right), \left(a_3, b_3\right)\right).$$

Hint: Write what you need to show in terms of

$$\begin{aligned} (x_1, x_2) &:= (a_2 - a_1, b_2 - b_1) \\ (y_1, y_2) &:= (a_3 - a_2, b_3 - b_2) \\ (x_1 + y_1, x_2 + y_2) &= (a_3 - a_1, b_3 - b_1). \end{aligned}$$

Then use that

$$(x_1 y_2 - x_2 y_1)^2 \geq 0.$$

Exercise 33. The next Theorem is called the Law of Cosines. As you work through the proof, construct a diagram or picture for each step.

EXERCISE 34. Give a quick proof of the triangle inequality using the Law of Cosines.

Exercise 35. Do algebra on the left-hand-side of the last equation above to derive Heron's formula

$$Area = \sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)}$$

where $s = \frac{a+b+c}{2}$.

EXERCISE 36. (**EG**) Using cartesian coordinates for the plane, show that the mapping D defined in (3.4.1) satisfies all three conditions in the definition of a dilation with magnification factor r and center (x_0, y_0).
Hint: For the second condition, use a parametric representation for a line through (x_0, y_0).

Exercise 37. (**EG**) Show (using the Substitution Rule, Theorem 2, if you wish) that a dilation with magnification factor r multiplies all areas by a factor of r^2.

Exercise 38. (**EG**) a) Show that the inverse mapping of a dilation is again a dilation with the same center but with magnification factor r^{-1}.
Hint: Use (3.4.2) and solve for (x, y) in terms of $\big(\underline{x}, \underline{y}\big)$.

b) Show that a dilation D takes lines to lines.
Hint: Start with a line

$$\underline{a}\underline{x} + \underline{b}\underline{y} = \underline{c}$$

in the $\big(\underline{x}, \underline{y}\big)$-plane. Show that it comes from a line in the (x, y)-plane by substitution using (3.4.2). Or start with a line

$$ax + by = c$$

in the (x, y)-plane and use the formula for D^{-1} you derived in a) .

EXERCISE 39. (**EG**) Show that a dilation takes any line to a line parallel (or equal) to itself.
Hint: Compute slopes.

EXERCISE 40. (**EG**) Show that a dilation by a factor of r takes any vector to r times itself.
Hint: Realize the vector as the difference of two points.

EXERCISE 41. (**EG**) Show that a dilation of the plane preserves angles. Hint: Use Exercise 23a) and Exercise 39.

Exercise 42. (**EG**) a) Show that, if two triangles are similar, then corresponding angles are equal.
Hint: You have to start from the supposition that the two triangles satisfy the definition of similar triangles.

b) Show that, if corresponding angles of $\triangle ABC$ and $\triangle A'B'C'$ are equal, then $\triangle ABC$ and $\triangle A'B'C'$ are similar.
Hint: You have to start from the supposition that corresponding angles of the two triangles have equal measure, then dilate $\triangle ABC$ with a dilation for which $r = |A'B'|/|AB|$. Finally use ASA to show that the dilation of $\triangle ABC$ is congruent to $\triangle A'B'C'$.

Exercise 43. (**EG**) Show that two triangles are similar if corresponding sides are parallel.
Hint: Use Exercise 23a).

Exercise 44. Show that two triangles are similar if corresponding sides are perpendicular.
Hint: Extend one of the rays of the first triangle until it crosses the corresponding ray of the second triangle.

EXERCISE 45. (**EG**) Denote the measure or area of a triangle $\triangle ABC$ as $|\triangle ABC|$. Show that, in the diagram below,

$$\frac{|AF|}{|FB|} = \frac{|\triangle AFC|}{|\triangle CFB|} = \frac{|\triangle AFX|}{|\triangle XFB|}.$$

EXERCISE 46. (**EG**) Use Exercise 45 to show by pure algebra that

$$\frac{|AF|}{|FB|} = \frac{|\triangle AXC|}{|\triangle CXB|}. \tag{0.0.1}$$

Hint: $|\triangle AXC| = |\triangle AFC| - |\triangle AFX|$, etc.

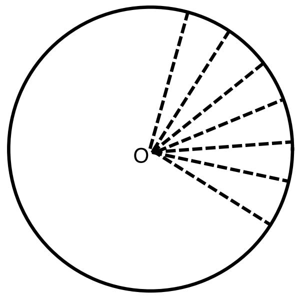
Exercise 47. (**EG**) For three concurrent segments $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ as given in Exercise 45, use Exercise 46 to show that

$$\frac{|AF|}{|FB|}\cdot\frac{|BD|}{|DC|}\cdot\frac{|CE|}{|EA|} = 1.$$

Hint: Use (0.0.1) with side $\overline{BC}$ replacing $\overline{AB}$ and (0.0.1) with side $\overline{CA}$ replacing $\overline{AB}$.

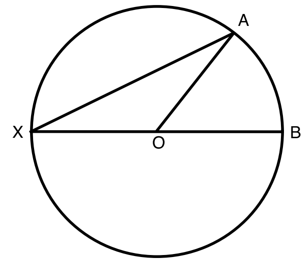
EXERCISE 48. (**EG**) A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side. Show that the medians of any triangle meet in a common point.
Hint: Use Ceva's Theorem.

Exercise 49. (**EG**) The circle of radius 1 has (interior) area that we denote as π. Use this to reason to the fact that the circle of radius 1 has circumference 2π.



Hint: Slice up the circle into equal size slices as illustrated in the above picture. Approximate a rectangle by rearranging those slices. Notice that the greater the number of slices, the better the approximation. Use that the area covered by the slices is independent of the number of slices to conclude that it must be the same as the area of the limiting rectangle.

EXERCISE 50. (**EG**) On the circle with center O below,

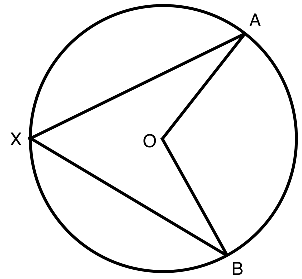


show that

$$\angle AXB = (1/2)(\angle AOB).$$

Hint: $\triangle OAX$ is isosceles.

EXERCISE 51. (**EG**) On the circle with center O below,

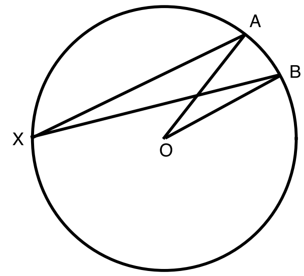


show that

$$\angle AXB = (1/2)(\angle AOB).$$

Hint: Draw the diameter through O and X, use Exercise 50 and add.

EXERCISE 52. (**EG**) On the circle with center O below,

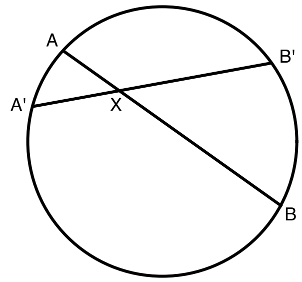


show that

$$\angle AXB = (1/2)(\angle AOB).$$

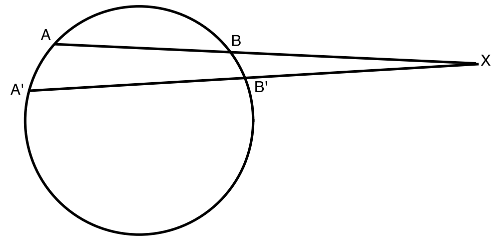
Hint: Draw the diameter through O and X, use Exercise 50 and subtract.

Exercise 53. (**EG**) Use similar triangles and the previous Exercises to show that $|AX|\cdot|XB| = |A'X|\cdot|XB'|$ in the figure below.



Hint: Draw $\overline{AB'}$ and $\overline{A'B}$.

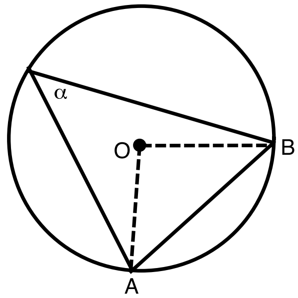
Exercise 54. (**EG**) Use similar triangles and the previous Exercises to show that $|AX|{\cdot}|XB| = |A'X|{\cdot}|XB'|$ in the figure below.



Hint: Draw $\overline{AB'}$ and $\overline{A'B}$.

Exercise 55. Show that, given any three non-collinear points in the cartesian plane, there is a unique circle passing through the three points.
Hint: The three non-collinear points are the three vertices of a unique triangle. Pick any two sides of that triangle and form their perpendicular bisectors. Show that the center of the circle must be the point at which those perpendicular bisectors intersect.

EXERCISE 56. (Extended Law of Sines) In the diagram below, O is the center of the circle and d is its diameter.



Show that

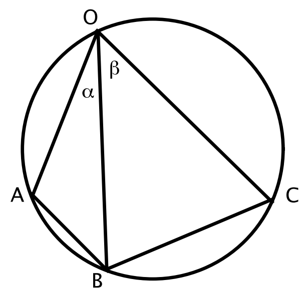
$$\frac{\sin\alpha}{|AB|} = \frac{1}{d}.$$

Hint: Bisect $\overline{AB}$ at a point C. Compare inscribed to central angle to show that

$$\frac{|AC|}{|OA|} = \sin\alpha = \frac{|BC|}{|OB|}.$$

Exercise 57. (**EG**) In the diagram below, show that

$$\frac{|AB|}{|CB|} = \frac{\sin\alpha}{\sin\beta} = \frac{\sin(\angle AOB)}{\sin(\angle COB)}.$$



Hint: Use the Extended Law of Sines. Alternatively notice that by Theorem 9

$$m(\angle BAO) + m(OCB) = 180^\circ$$

so that

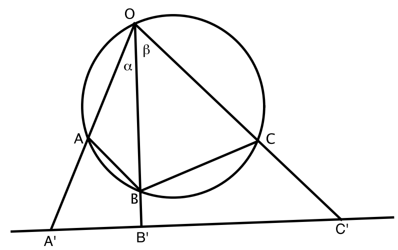
$$\sin(\angle BAO) = \sin(\angle OCB).$$

Now use the Law of Sines.

EXERCISE 58. (**EG**) Show that if, in the above figure, B moves along the circle to the other side of C, it is still true that

$$\frac{|AB|}{|CB|} = \frac{\sin(\angle AOB)}{\sin(\angle COB)}$$

EXERCISE 59. (**EG**) In the diagram



show that

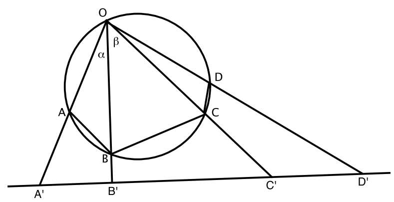
$$\frac{|A'B'|}{|C'B'|} = \frac{\sin(\angle A'OB')}{\sin(\angle C'OB')} \div \frac{\sin(\angle B'A'O)}{\sin(\angle B'C'O)}.$$

[MJG,266-267]

EXERCISE 60. (**EG**) Show that if, in the above figure, B' moves along the line to the other side of C', it is still true that

$$\frac{|A'B'|}{|C'B'|} = \frac{\sin(\angle A'OB')}{\sin(\angle C'OB')} \div \frac{\sin(\angle B'A'O)}{\sin(\angle B'C'O)}.$$

EXERCISE 61. a) Show that, in the figure



we have the equality

$$(A : B : C : D) = (A' : B' : C' : D')\,.$$

Hint: Use Exercises 57-60.

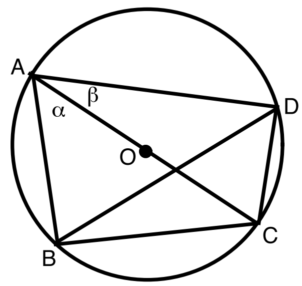
b) What happens in a) if we move B to the other side of C?

Exercise 62. a) Draw four points in the Euclidean plane, no 3 of which are collinear, that cannot lie on a single circle. With the help of a ruler, show that the distances between pairs of the four points violate the equality in Ptolemy's theorem.

b) Draw four points in the Euclidean plane that do lie on a single circle. With the help of a ruler, show that the distances between pairs of the four points satisfy the equality in Ptolemy's theorem.

EXERCISE 63. (Addition Law for Sines) The diameter of the circle below is 1. Use Ptolemy's Theorem and the Extended Law of Sines to show the Addition Law for Sines

$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta.$$



Hint: Examine the central angles associated to the chords $|AD|$ and $|BC|$.

EXERCISE 64. (**EG**) Show that an $r \times r \times r$ cube can be constructed from three equal pyramids with an $r \times r$ square base. Conclude that the volume of each pyramid is 1/3 the volume of the cube, namely

$$\frac{1}{3}r^3$$

Hint: Suppose the cube had a hollow interior and infinitely thin faces. Put your (infinitely tiny) eye at one vertex of the cube and look inside. How many faces of the cube can you see?

EXERCISE 65. Show that Cavalieri's Principle is true for the pyramid using several variable calculus.

Hint: Put the base of the pyramid P so that its vertices are $(0,0)$, $(r,0)$, $(0,r)$ and (r,r) in 3-dimensional Euclidean space. Consider the transformation

$$\left(\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}\right) = (\hat{x}, \hat{y}, \hat{z}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$$

and notice that

$$\int_{\underline{P}} d\underline{\hat{x}} d\underline{\hat{y}} d\underline{\hat{z}} = \int_P \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \right| d\hat{x} d\hat{y} d\hat{z}.$$

EXERCISE 66. Prove the magnification principle using several variable calculus. Hint: Use the transformation

$$\left(\underline{\hat{x}_1}, \ldots, \underline{\hat{x}_n}\right) = (\hat{x}_1, \ldots, \hat{x}_n) \begin{pmatrix} r_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & r_n \end{pmatrix}$$

and the Substitution Rule (Theorem 2) to compare $\int_{\underline{P}} d\underline{\hat{x}_1} \ldots d\underline{\hat{x}_n}$ with $\int_P d\hat{x}_1 \ldots d\hat{x}_n$.

Exercise 67. (**EG**) Use this magnification principle to justify the volume formula

$$(1/3)B{\cdot}h$$

for any pyramid with rectangular base of area B and vertical altitude h.

EXERCISE 68. (**EG**) a) Why is the area B of the base of the pyramid caught between $B - \varepsilon$ and $B + \varepsilon$?

b) Show that the volume V of the pyramid is caught between $(1/3){\cdot}t{\cdot}h$ and $(1/3){\cdot}T{\cdot}h$.

Exercise 69. (**EG**) a) Argue that, given any positive real number ε, however small, the volume V of the pyramid is caught between $(1/3)\cdot(B-\varepsilon)\cdot h$ and $(1/3)\cdot(B+\varepsilon)\cdot h$.

b) Show the formula

$$V = (1/3)\cdot B\cdot h$$

for the volume of a pyramid, where B is the area of the base and h is the vertical height.

Exercise 70. (**SG**) a) Why can you think of the disco-ball as being made up of pyramids, with each pyramid having base one of the tiny mirrors and vertex at the interior point O at the center of the disco-ball.

b) Argue that the volume of the disco-ball is (1/3) times the distance h from a mirror to O times the sum of the areas of all the mirrors.

EXERCISE 71. (**SG**) Argue that, as the mirrors are made to be smaller and smaller,

1) the sum of the areas of the mirrors approaches the surface area of a sphere,

2) the distance h approaches the radius R of that sphere,

3) the volume of the disco-ball approaches the volume of the region whose boundary is the sphere.

4) Use limits to conclude that, for a sphere of radius R in Euclidean 3-space, the relation between the volume V of the region bounded by the sphere and the surface area S of the sphere is given by the formula

$$V = \frac{R \cdot S}{3}.$$

EXERCISE 72. (**SG**) Explain why the area of the α-lune is $2\alpha \cdot R^2$.

Exercise 73. Give a formula for the area of any spherical n-gon.
Hint: Divide the spherical n-gon into spherical triangles.

Exercise 74. Give the formula for the length $\left|\hat{V}\right|$ of a vector $\hat{V} = \left(\hat{a}, \hat{b}, \hat{c}\right)$ in 3-dimensional Euclidean space in terms of dot product.

EXERCISE 75. Lemma 2 does not distinguish between the two angles that the vectors determine in the plane that they span. What can you say about the cosine of the two angles formed by two vectors $\hat{V}_1$ and $\hat{V}_2$, that is, what is the relation between the formula using ϑ and the formula using $(360° - \vartheta)$?

EXERCISE 76. Show that the area of the parallelogram determined by $\hat{V}_1$ and $\hat{V}_2$ emanating from the same point in Euclidean 3-space is given by

$$\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \sin\vartheta. \tag{0.0.2}$$

Exercise 77. Show that we have the following equality of matrices

$$\begin{pmatrix} \hat{V}_1 \bullet \hat{V}_1 & \hat{V}_2 \bullet \hat{V}_1 \\ \hat{V}_1 \bullet \hat{V}_2 & \hat{V}_2 \bullet \hat{V}_2 \end{pmatrix} = \begin{pmatrix} \left(\hat{V}_1\right) \\ \left(\hat{V}_2\right) \end{pmatrix} \cdot \begin{pmatrix} \left(\hat{V}_1\right)^t & \left(\hat{V}_2\right)^t \end{pmatrix}.$$

Exercise 78. a) Give two examples of smooth curves,

$$\hat{X}_1(s) = (\hat{x}_1(s), \hat{y}_1(s), \hat{z}_1(s))$$
$$\hat{X}_2(t) = (\hat{x}_2(t), \hat{y}_2(t), \hat{z}_2(t))$$

neither of which is a straight line, in 3-dimensional Euclidean space. Do this so that the two curves pass through a common point and go in distinct tangent directions at that point. Please choose curves so that none of the coordinate functions of s or t is a constant function. [DS,71ff]

b) Compute the tangent vectors of each of the two curves at each of their points.

c) For the two curves you defined in a), what are the coordinates of the point in Euclidean 3-space at which the two curves intersect?

d) Use the dot product formula to compute the angle ϑ between (the tangent vectors to) your two example curves in a) at the point at which the curves intersect. [DS,20-21]

EXERCISE 79. Compute the length of the tangent vector

$$l(t) = \sqrt{\frac{d\hat{X}}{dt} \bullet \frac{d\hat{X}}{dt}} = \sqrt{\begin{pmatrix} \frac{d\hat{x}}{dt} & \frac{d\hat{y}}{dt} & \frac{d\hat{z}}{dt} \end{pmatrix} \cdot \begin{pmatrix} \frac{d\hat{x}}{dt} \\ \frac{d\hat{y}}{dt} \\ \frac{d\hat{z}}{dt} \end{pmatrix}}$$

to each of your two example curves in Exercise 78 at each of their points.

Exercise 80. Write the formula for the tangent vector to the path (7.2.1) at each point using $(\hat{x}(t), \hat{y}(t), \hat{z}(t))$-coordinates. Show that the length of this path is $R\pi$.

EXERCISE 81. Compute the length of each of your two example curves in Exercise 78.

Exercise 82. Use spherical coordinates

$$\hat{x}(s,t) = R\cdot \sin s\cdot \cos t$$
$$\hat{y}(s,t) = R\cdot \sin s\cdot \sin t$$
$$\hat{z}(s,t) = R\cdot \cos s$$

and the definition just above to compute the area of an α-lune on the sphere of radius R.

EXERCISE 83. Suppose $\hat{M}$ is such that

$$\hat{M} \cdot \hat{M}^t = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find a vector $\hat{V} = \left(\hat{a}, \hat{b}, \hat{c}\right)$ such that

$$\hat{V} \cdot \hat{M} \cdot \hat{M}^t \cdot \hat{V}^t \neq \hat{V} \cdot \hat{V}^t.$$

EXERCISE 84. Show that the matrix

$$\hat{M} = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is orthogonal. Can you describe geometrically what this rigid motion is doing to the points in Euclidean 3-space?

EXERCISE 85. Show that the matrix

$$\hat{M} = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}$$

is orthogonal. Can you describe geometrically what this rigid motion is doing to the points in Euclidean 3-space?

Exercise 86. (**SG**) Show that the set of orthogonal matrices $\hat{M}$ form a group. That is, show that

a) the product of two orthogonal matrices is orthogonal,

b) recall from linear algebra that matrix multiplication (when defined) is associative,

c) the identity matrix is orthogonal,

d) the inverse matrix $\hat{M}^{-1}$ of an orthogonal matrix $\hat{M}$ is orthogonal.

Hint: Write

$$\hat{M} \cdot \hat{M}^{-1} = I = \hat{M} \cdot \hat{M}^{t}$$

and use matrix multiplication to reduce to showing that the transpose of an orthogonal matrix is orthogonal. [MJG,311]

.

EXERCISE 87. a) Show that the transformation

$$(x,y) \mapsto \left(\underline{x},\underline{y}\right) = (x,y) \cdot \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

is a $1-1$, onto (linear) transformation of $\mathbb{R}^2$ as long as

$$\begin{vmatrix} d & b \\ c & a \end{vmatrix} \neq 0. \tag{0.0.3}$$

b) For the transformation in a), show that every line through the origin in (x,y)-space is sent to a line through the origin in $\left(\underline{x},\underline{y}\right)$-space. The slope t of the line through $(0,0)$ and (x,y) is of course $t=\frac{y}{x}$. What is the slope $\underline{t}$ of the line through $(\underline{0},\underline{0})$ and $\left(\underline{x},\underline{y}\right)$? Show that

$$\underline{t} = \frac{at+b}{ct+d}. \tag{0.0.4}$$

EXERCISE 88. Show that a linear fractional transformation

$$\overline{\mathbb{R}} \to \overline{\mathbb{R}}$$

$$t \mapsto \underline{t} = \frac{at+b}{ct+d}$$

is $1-1$ and onto. What is its inverse function? (Your answer should show that the inverse function is also a linear fractional transformation.)
Hint: Algebraically solve for t in terms of $\underline{t}$. Then graph

$$\underline{t} = \frac{at+b}{ct+d}$$

in the $(t, \underline{t})$-plane. If $c = 0$ show that the graph is a straight line with non-zero slope and

$$\infty \mapsto \infty.$$

If $c \neq 0$, show that the graph has exactly one horizontal asymptote where $t \mapsto \infty$ and one vertical asymptote where $\underline{t} \mapsto \infty$.

Exercise 89. Show that the set of linear fractional transformations form a *group* under the operation of composition of functions:

a) Show that the composition of two linear fractional transformations is again a linear fractional transformation.

b) Since composition of functions is always associative, compositions of linear fractional transformations are automatically associative.

c) Show that the identity map from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ is a linear fractional transformation.

d) Show that the inverse function of a linear fractional transformation is a linear fractional transformation. That is, show that for any linear fractional transformation f there is an (inverse) linear fractional transformation g such that

$$f \circ g = \text{identity transformation}$$
$$g \circ f = \text{identity transformation.}$$

EXERCISE 90. a) Show that, for any three distinct fixed points t_2, t_3 and t_4, the function of t given by the formula

$$\underline{t} = \frac{t_3 - t_4}{t_3 - t_2} \frac{t - t_2}{t - t_4} = \frac{t - t_2}{t_3 - t_2} \div \frac{t - t_4}{t_3 - t_4}$$

is a linear fractional transformation. That is, show that it is a function of the form (0.0.4) for which the condition (0.0.3) holds.

b) Show that the linear fractional transformation in a) takes t_2 to 0, takes t_3 to 1 and takes t_4 to ∞.

EXERCISE 91. Show that any linear fractional transformation (0.0.4) that leaves 0, 1, and ∞ fixed is the identity map.

EXERCISE 92. If for two ordered sets of points $\{s_k\}$ and $\{t_k\}$,

$$(s_1 : s_2 : s_3 : s_4) = (t_1 : t_2 : t_3 : t_4)\,,$$

explicitly define the linear fractional transformation f such that, for $k = 1, \ldots, 4$,

$$f\,(s_k) = t_k.$$

Hint: Begin with

$$\frac{s - s_2}{s_3 - s_2} \div \frac{s - s_4}{s_3 - s_4} = \frac{t - t_2}{t_3 - t_2} \div \frac{t - t_4}{t_3 - t_4}$$

and solve for s.

Exercise 93. a) Sketch the solution set in (x, y, z)-coordinates representing the sphere

$$R^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 2^2$$

of radius 2 in Euclidean three-space.

b) Sketch the solution set in (x, y, z)-coordinates representing the sphere

$$R^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 10^2$$

of radius 10 in Euclidean three-space.

c) Sketch the solution set in (x, y, z)-coordinates representing the sphere

$$R^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 10^{-2}$$

of radius 10^{-1} in Euclidean three-space.

EXERCISE 94. a) Suppose we have functions

$$\left(\hat{x}\left(x,y,z\right),\hat{y}\left(x,y,z\right),\hat{z}\left(x,y,z\right)\right)$$

where

$$\begin{aligned} x &= f\left(t\right) \\ y &= g\left(t\right) \\ z &= h\left(t\right). \end{aligned}$$

State the Chain Rule (Theorem 1) for

$$\begin{aligned} \frac{d\hat{x}}{dt} &= \\ \frac{d\hat{y}}{dt} &= \\ \frac{d\hat{z}}{dt} &= . \end{aligned}$$

b) Rewrite the Chain Rule in matrix notation

$$\left(\begin{array}{ccc} \frac{d\hat{x}}{dt} & \frac{d\hat{y}}{dt} & \frac{d\hat{z}}{dt} \end{array}\right) = \left(\begin{array}{ccc} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{array}\right) \cdot \left(\quad \right).$$

EXERCISE 95. Recalling that R is a positive constant, use (9.1.2) and the Chain Rule to show that, for any path $\hat{X}(t) = (\hat{x}(t), \hat{y}(t), \hat{z}(t))$ in Euclidean 3-space,

$$\frac{d\hat{x}}{dt} = \frac{dx}{dt}$$
$$\frac{d\hat{y}}{dt} = \frac{dy}{dt}$$
$$\frac{d\hat{z}}{dt} = R\frac{dz}{dt}.$$

Exercise 96. Use matrix multiplication [DS,307] and Exercise 95 to show that

$$\frac{d\hat{X}(t)}{dt} = \left(\frac{dX(t)}{dt}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}$$

$$\frac{dX(t)}{dt} = \left(\frac{d\hat{X}(t)}{dt}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{pmatrix}.$$

Exercise 97. Use Lemma 3 to show that, if we have any two vectors in Euclidean three-space that are tangent to the R-sphere at some point on it, but the two vectors are given to us in (x, y, z)-coordinates as

$$\begin{aligned} V_1 &= (a_1, b_1, c_1) \\ V_2 &= (a_2, b_2, c_2)\,, \end{aligned}$$

then the area of the parallelogram spanned by those two vectors in Euclidean 3-space is

$$\sqrt{\begin{vmatrix} V_1 \bullet_K V_1 & V_2 \bullet_K V_1 \\ V_1 \bullet_K V_2 & V_2 \bullet_K V_2 \end{vmatrix}} = \sqrt{\left| \begin{pmatrix} (V_1) \\ (V_2) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \begin{pmatrix} (V_1)^t & (V_2)^t \end{pmatrix} \right|}.$$

Exercise 98. Starting from the equality (9.3.1) describing the transformation in Euclidean coordinates, explain why

$$\begin{pmatrix} \underline{x} & \underline{y} & \underline{z} \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} \cdot \hat{M} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{pmatrix}.$$

EXERCISE 99. Finish the matrix algebra computations just above to show that the condition that a transformation M in (x, y, z)-coordinates preserves distances in Euclidean 3-space is the condition that

$$M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot M^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix}. \tag{0.0.5}$$

EXERCISE 100. Check that (0.0.5) is the correct condition by showing that any 3×3 matrix M that satisfies (0.0.5) also satisfies

$$((V) \cdot M) \bullet_K ((V) \cdot M) = V \bullet_K V$$

where

$$V = X_2 - X_1.$$

That is, the transformation given in (x, y, z)-coordinates by a matrix M that satisfies (0.0.5) preserves the K-dot product.

EXERCISE 101. For $K \neq 0$, show that the condition (10.1.2) on V is exactly the same condition as

$$(x(t), y(t), z(t)) \bullet_K V = 0.$$

EXERCISE 102. a) Show that this last equality is always true if

$$M\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&K^{-1}\end{pmatrix}\cdot M^{t}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&K^{-1}\end{pmatrix}. \tag{0.0.6}$$

b) Show that, if M satisfies the identity 0.0.6, then the transformation $\left(\underline{x},\underline{y},\underline{z}\right)=(x,y,z)\cdot M$ takes the set of points (x,y,z) such that

$$1=K\left(x^{2}+y^{2}\right)+z^{2},$$

to the set of points $\left(\underline{x},\underline{y},\underline{z}\right)$ such that

$$1=K\left(\underline{x}^{2}+\underline{y}^{2}\right)+\underline{z}^{2}.$$

That is, M gives a $1-1$, onto mapping of K-geometry to itself.
Hint: For $K\neq 0$, write the equation $1=K\left(\underline{x}^{2}+\underline{y}^{2}\right)+\underline{z}^{2}$ in matrix notation as

$$\begin{pmatrix}\underline{x}&\underline{y}&\underline{z}\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&K^{-1}\end{pmatrix}\cdot\begin{pmatrix}\underline{x}\\\underline{y}\\\underline{z}\end{pmatrix}=\frac{1}{K}.$$

EXERCISE 103. For $K \neq 0$, show that the set of K-orthogonal matrices M form a group. That is, show that

a) the product of two K-orthogonal matrices is K-orthogonal,

b) again cite the fact that matrix mltiplication is associative,

c) the identity matrix is K-orthogonal,

d) the inverse matrix M^{-1} of a K-orthogonal matrix M is K-orthogonal

Hint: Write

$$M \cdot M^{-1} = I = M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot M^t \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{pmatrix}$$

and use matrix multiplication to reduce to showing that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot M^t \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{pmatrix}$$

is K-orthogonal.

EXERCISE 104. a) For the projection of the set (10.2.1) onto the $z = 1$ plane with center of projection O, write (x_c, y_c) as a function of (x, y, z).

b) For the projection of the set (10.2.1) onto the $z = 1$ plane with center of projection O, write (x, y, z) as a function of (x_c, y_c).

EXERCISE 105. Show that,

a) when $K > 0$, the entire (x_c, y_c)-plane parametrizes the upper hemisphere,

b) when $K = 0$, the (x_c, y_c)-plane coincides with the plane $z = 1$,

c) when $K < 0$ the region is the interior of a circle of radius $\frac{1}{\sqrt{|K|}}$ in the (x_c, y_c)-plane parametrizes the entire K-geometry.

Exercise 106. For the K-geometry coordinates

$$X(x_c, y_c) = (x(x_c, y_c), y(x_c, y_c), z(x_c, y_c))$$

use the formulas you derived in Exercise 104b) to calculate

$$dX = \left(\frac{\partial X}{\partial x_c}\right) dx_c + \left(\frac{\partial X}{\partial y_c}\right) dy_c.$$

That is, calculate the 2×3 matrix

$$D_c = \begin{pmatrix} \frac{\partial x}{\partial x_c} & \frac{\partial y}{\partial x_c} & \frac{\partial z}{\partial x_c} \\ \frac{\partial x}{\partial y_c} & \frac{\partial y}{\partial y_c} & \frac{\partial z}{\partial y_c} \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial X}{\partial x_c}\right) \\ \left(\frac{\partial X}{\partial y_c}\right) \end{pmatrix}.$$

Hint: Use logarithmic differentiation:

$$\begin{aligned} dx &= d(rx_c) = x_c dr + r dx_c \\ r^{-1} dx &= x_c d\ln(r) + dx_c \end{aligned}$$

and similarly for y and z since it is easier to compute $r^{-1}\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ than $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$. Next use that

$$\begin{aligned} 2d\ln(r) &= d\ln\left(r^2\right) = -d\ln\left(K\left(x_c^2 + y_c^2\right) + 1\right) \\ &= -\frac{1}{K\left(x_c^2 + y_c^2\right) + 1} d\left(K\left(x_c^2 + y_c^2\right) + 1\right) \\ &= -r^2 K\left(2x_c dx_c + 2y_c dy_c\right). \end{aligned}$$

EXERCISE 107. Now suppose we have a path,

$$(x_c(t), y_c(t)),\ a \le t \le b$$

in the (x_c, y_c)-plane, that is, in the central projection plane

$$(x_c, y_c, 1).$$

Use the formula you derived in Exercise 104b) to write the corresponding path

$$x(x_c(t), y_c(t)),\ y(x_c(t), y_c(t)),\ z(x_c(t), y_c(t))$$

in the K-geometry space of (x, y, z) such that $K(x^2 + y^2) + z^2 = 1$.

Exercise 108. For the path $(x(t), y(t), z(t))$ in Exercise 107 lying on the set (10.2.1), use the Chain Rule (Theorem 1) from calculus of several variables to compute

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(\frac{dx_c(t)}{dt}, \frac{dy_c(t)}{dt}\right) \cdot D_c.$$

EXERCISE 109. Compute the 2×2 matrix

$$P_c = D_c \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot D_c^t,$$

that that gives the K-dot product in (x_c, y_c)-coordinates. That is, use matrix multiplication to show that

$$P_c = \begin{pmatrix} r^2\left(1 - r^2 K x_c^2\right) & -r^4 K x_c y_c \\ -r^4 K x_c y_c & r^2\left(1 - r^2 K y_c^2\right) \end{pmatrix}.$$

Hint: For example

$$\begin{aligned}
\frac{\partial x}{\partial x_c} &= r\left(x_c \frac{\partial ln\,(r)}{\partial x_c} + 1\right) = -r^3 K x_c^2 + r \\
\frac{\partial y}{\partial x_c} &= r\left(y_c \frac{\partial ln\,(r)}{\partial x_c}\right) = -r^3 K x_c y_c \\
\frac{\partial z}{\partial x_c} &= r\left(\frac{\partial ln\,(r)}{\partial x_c}\right) = -r^3 K x_c
\end{aligned}$$

so that

$$\begin{aligned}
&\left(\frac{\partial x}{\partial x_c}, \frac{\partial y}{\partial x_c}, \frac{\partial z}{\partial x_c}\right) \bullet_K \left(\frac{\partial x}{\partial x_c}, \frac{\partial y}{\partial x_c}, \frac{\partial z}{\partial x_c}\right) \\
&= r^6 K^2 x_c^4 - 2r^4 K x_c^2 + r^2 + r^6 K^2 x_c^2 y_c^2 + r^6 K x_c^2 \\
&= \left(r^6 K^2 x_c^4 + r^6 K^2 x_c^2 y_c^2 + r^6 K x_c^2\right) - 2r^4 K x_c^2 + r^2 \\
&= r^4 K x_c^2 - 2r^4 K x_c^2 + r^2 = r^2\left(1 - r^2 K x_c^2\right).
\end{aligned}$$

EXERCISE 110. Show that, when $K = 0$, the formulas for the dot product $\bullet_c$ are exactly the ordinary formulas for (flat) plane geometry.

EXERCISE 111. Explain why a curve in K-geometry cut out by a plane through $(0, 0, 0)$ in (x, y, z)-coordinates corresponds to a line in the (x_c, y_c)-coordinate plane. Hint: See (10.2.2)) relating (x, y, z)-coordinates with (x_c, y_c)-coordinates.

EXERCISE 112. Using Lemma 3 and Exercise 97 show that

$$\begin{aligned}\hat{a}\left(\frac{d\hat{X}}{dx_c},\frac{d\hat{X}}{dy_c}\right)^2 &= \left|\begin{matrix}\frac{d\hat{X}}{dx_c}\bullet\frac{d\hat{X}}{dx_c} & \frac{d\hat{X}}{dy_c}\bullet\frac{d\hat{X}}{dx_c}\\ \frac{d\hat{X}}{dx_c}\bullet\frac{d\hat{X}}{dy_c} & \frac{d\hat{X}}{dy_c}\bullet\frac{d\hat{X}}{dy_c}\end{matrix}\right| \\ &= \left|\begin{matrix}\frac{dX}{dx_c}\bullet_K\frac{dX}{dx_c} & \frac{dX}{dy_c}\bullet_K\frac{dX}{dx_c}\\ \frac{dX}{dx_c}\bullet_K\frac{dX}{dy_c} & \frac{dX}{dy_c}\bullet_K\frac{dX}{dy_c}\end{matrix}\right| \\ &= \left|\begin{pmatrix}\left(\frac{dX}{dx_c}\right)\\ \left(\frac{dX}{dy_c}\right)\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & K^{-1}\end{pmatrix}\begin{pmatrix}\left(\frac{dX}{dx_c}\right)^t & \left(\frac{dX}{dy_c}\right)^t\end{pmatrix}\right| \\ &= \left|P_c\right|.\end{aligned}$$

EXERCISE 113. Use Exercise 109 to show that

$$\hat{a}\left(\frac{d\hat{X}}{dx_c},\frac{d\hat{X}}{dy_c}\right)^2 = r^6 = \frac{1}{\left(K\left(x_c^2+y_c^2\right)+1\right)^3}$$

as a function of (x_c, y_c).
Hint: Notice that the matrix D_c in Exercise 106 is simply the 2×3 matix whose rows are the vectors $\frac{dX}{dx_c}$ and $\frac{dX}{dy_c}$. So referring to Exercise 109, we know that

$$\begin{pmatrix} \frac{d\hat{X}}{dx_c} \bullet \frac{d\hat{X}}{dx_c} & \frac{d\hat{X}}{dy_c} \bullet \frac{d\hat{X}}{dx_c} \\ \frac{d\hat{X}}{dx_c} \bullet \frac{d\hat{X}}{dy_c} & \frac{d\hat{X}}{dy_c} \bullet \frac{d\hat{X}}{dy_c} \end{pmatrix} = \begin{pmatrix} r^2\left(1-r^2Kx_c^2\right) & -r^4Kx_cy_c \\ -r^4Kx_cy_c & r^2\left(1-r^2Ky_c^2\right) \end{pmatrix}.$$

Exercise 114. a) For the projection of the set (10.2.1) onto the $z = 1$ plane with center of projection S, write (x_s, y_s) as a function of (x, y, z).

b) For the projection of the set (10.2.1) onto the $z = 1$ plane with center of projection S, write (x, y, z) as a function of (x_s, y_s).

Exercise 115. Suppose we have a path

$$X\left(x_s\left(t\right),y_s\left(t\right)\right)=\left(x\left(x_s\left(t\right),y_s\left(t\right)\right),y\left(x_s\left(t\right),y_s\left(t\right)\right),z\left(x_s\left(t\right),y_s\left(t\right)\right)\right)$$

lying on the set (10.2.1) given in terms of its projection $\left(x_s\left(t\right),y_s\left(t\right)\right)$ in the plane $z=1$. Use the formula you derived in Exercise 114b) and the Chain Rule (Theorem 1) to find the 2×3 matrix

$$D_s=\begin{pmatrix}\left(\frac{\partial X}{\partial x_s}\right)\\ \left(\frac{\partial X}{\partial y_s}\right)\end{pmatrix}$$

such that

$$\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=\left(\frac{dx_s\left(t\right)}{dt},\frac{dy_s\left(t\right)}{dt}\right)\cdot D_s.$$

Hint: Use logarithmic differentiation

$$dx=d\left(\rho x_s\right)=x_s d\rho+\rho dx_s$$
$$\rho^{-1}dx=x_s d\ln\left(\rho\right)+dx_s$$

and similarly for y. Also

$$\begin{aligned}d\ln\left(\rho\right)&=-d\ln\left(\frac{K}{4}\left(x_s^2+y_s^2\right)+1\right)\\&=-\frac{1}{\frac{K}{4}\left(x_s^2+y_s^2\right)+1}d\left(\frac{K}{4}\left(x_s^2+y_s^2\right)+1\right)\\&=-\rho\frac{K}{4}\left(2x_s dx_s+2y_s dy_s\right).\end{aligned}$$

EXERCISE 116. Use matrix multiplication to compute the 2×2 matrix

$$P_s = D_s \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot D_s^t,$$

that is, to compute the K-dot product in (x_s, y_s)-coordinates. (You may be surprised at the answer! It is quite simple and only involves the quantity ρ.)

Exercise 117. Write the formula for the K-dot product (x_s, y_s)-coordinates when $K = 0$. Does it look familiar?

EXERCISE 118. Suppose we intersect K-geometry (10.4.1) with a plane

$$ax + by + z = 0.$$

a) Show that the equation for the resulting path in stereographic projection coordinates is

$$\left(x_s - \frac{2a}{K}\right)^2 + \left(y_s - \frac{2b}{K}\right)^2 = \frac{4\left(K + a^2 + b^2\right)}{K^2}.$$

Hint: Use formula (10.3.1) to relate (x, y, z)-coordinates with (x_s, y_s)-coordinates.

b) What is the equation for the resulting path in stereographic projection coordinates if we intersect the K-geometry with a plane given by

$$ax + by = 0,$$

that is, a plane containing the z-axis?

Exercise 119. As in Lemma 3 show that

$$\hat{a}\left(\frac{d\hat{X}}{dx_s},\frac{d\hat{X}}{dy_s}\right)^2 = \begin{vmatrix} \frac{d\hat{X}}{dx_s}\bullet\frac{d\hat{X}}{dx_s} & \frac{d\hat{X}}{dy_s}\bullet\frac{d\hat{X}}{dx_s} \\ \frac{d\hat{X}}{dx_s}\bullet\frac{d\hat{X}}{dy_s} & \frac{d\hat{X}}{dy_s}\bullet\frac{d\hat{X}}{dy_s} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{dX}{dx_s}\bullet_K\frac{dX}{dx_s} & \frac{dX}{dy_s}\bullet_K\frac{dX}{dx_s} \\ \frac{dX}{dx_s}\bullet_K\frac{dX}{dy_s} & \frac{dX}{dy_s}\bullet_K\frac{dX}{dy_s} \end{vmatrix}.$$

EXERCISE 120. Use Exercise 116 to show that

$$\hat{a}\left(\frac{d\hat{X}}{dx_s}, \frac{d\hat{X}}{dy_s}\right)^2 = \rho^4 = \frac{1}{\left(\frac{K}{4}\left(x_s^2 + y_s^2\right) + 1\right)^4}.$$

Exercise 121. Do the algebra to write the explicit formulas for

$$(x_s(x_c, y_c), y_s(x_c, y_c))$$

and

$$(x_c(x_s, y_s), y_c(x_s, y_s)).$$

Exercise 122. a) Show that the slopes of the asymptotes to the hyperbola (10.4.4) are the limits as x goes to $\pm\infty$ of the slopes of the lines through $(0, 0)$ and the point (x, z) on the hyperbola.

b) Show that the slopes of the asymptotes to the hyperbola (10.4.4) are the limits as x goes to $\pm\infty$ of the slopes of the lines through $(0, -1)$ and the point (x, z) on the hyperbola.

c) Use b) to compute the radius of the disk around $(0, 0)$ in the (x_s, y_s)-plane that captures all the points (x, y, z) of K-geometry under stereographic projection.

EXERCISE 123. (**SG**) Show that the transformation (11.1.3) takes the R-sphere to itself if

$$\hat{M} \cdot \hat{M}^t = I$$

where I is the 3×3 identity matrix. (Recall that a matrix $\hat{M}$ satisfying this condition is called an orthogonal matrix.)

EXERCISE 124. (**SG**) Referring to Definition 19, show that the transformations

$$\left(\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}\right) = (\hat{x}, \hat{y}, \hat{z}) \cdot \hat{M}$$

and

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot M$$

give the same rigid motion of the R-sphere if

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} \cdot \hat{M} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{pmatrix}.$$

Exercise 125. (**SG**) Show that the matrix

$$\begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is both orthogonal and K-orthogonal and gives the same transformation of the Euclidean R-sphere. Describe geometrically what this transformation is doing to the R-sphere.

EXERCISE 126. (**SG**) Show that the matrix

$$\hat{M} = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

is orthogonal and that the matrix

$$M = \begin{pmatrix} \cos\varphi & 0 & R^{-1}\cdot\sin\varphi \\ 0 & 1 & 0 \\ -R\cdot\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

is the K-orthogonal matrix describing the same transformation of the R-sphere. Describe geometrically what this transformation is doing to the R-sphere.

EXERCISE 127. (**SG**) Write an explicit K-rigid motion of the type

$$M_1 = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes the point X_0 to a point $X_1 = (x_1, 0, z_0)$.
Hint: Start from the identity

$$\frac{-y_0}{\sqrt{x_0^2 + y_0^2}} \cdot x_0 + \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \cdot y_0 = 0$$

and the desired identity

$$\sin\vartheta \cdot x_0 + \cos\vartheta \cdot y_0 = 0$$

and conclude that there is a ϑ so that

$$\cos\vartheta = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}$$

$$\sin\vartheta = \frac{-y_0}{\sqrt{x_0^2 + y_0^2}}.$$

EXERCISE 128. (**SG**) Write an explicit K-rigid motion of the type

$$M_2 = \begin{pmatrix} \cos\varphi & 0 & R^{-1}\cdot\sin\varphi \\ 0 & 1 & 0 \\ -R\cdot\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

that takes the point $X_1 = (x_1, 0, z_0)$ to $N = (0, 0, 1)$.

Exercise 129. (**SG**) Write an explicit K-rigid motion of the type

$$M_3 = \begin{pmatrix} \cos\vartheta' & \sin\vartheta' & 0 \\ -\sin\vartheta' & \cos\vartheta' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes V_2 to the vector

$$\left(\sqrt{a_2^2 + b_2^2}, 0, 0\right) = \left(\sqrt{V_2 \bullet_K V_2}, 0, 0\right).$$

Why does the transformation given by M_3 leave the North Pole N fixed?

Exercise 130. (**SG**) Explain in words why the K-rigid motion

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot (M_1 \cdot M_2 \cdot M_3)$$

constructed over the last couple of sections takes the point X_0 to N and the tangent vector V_0 to $\left(\sqrt{V_0 \bullet_K V_0}, 0, 0\right)$

EXERCISE 131. (**SG**) Explain in words why the K-rigid motion given by

$$M \cdot (M')^{-1}$$

takes (X_0, V_0) to (X_0', V_0').

Exercise 132. (**SG**) Show that these spherical coordinates do actually parametrize the R-sphere, that is, that

$$K\left(x(\sigma,\tau)^2 + y(\sigma,\tau)^2\right) + z(\sigma,\tau)^2 \equiv 1$$

for all (σ,τ).

EXERCISE 133. (**SG**) a) Referring to (11.3.1) compute the 2×3 matrix

$$D_{sph} = \begin{pmatrix} \frac{dx}{d\sigma} & \frac{dy}{d\sigma} & \frac{dz}{d\sigma} \\ \frac{dx}{d\tau} & \frac{dy}{d\tau} & \frac{dz}{d\tau} \end{pmatrix}.$$

b) Show that, if a path in K-geometry is given by a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane,

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \cdot D_{sph}.$$

c) For two paths in K-geometry given by paths $(\sigma_1(t), \tau_1(t))$ and $(\sigma_2(t), \tau_2(t))$ in the (σ, τ)-plane, use a) and b) to show that

$$\left(\frac{d\hat{x}_1}{dt}, \frac{d\hat{y}_1}{dt}, \frac{d\hat{z}_1}{dt}\right) \cdot \left(\frac{d\hat{x}_2}{dt}, \frac{d\hat{y}_2}{dt}, \frac{d\hat{z}_2}{dt}\right)^t =$$

$$\left(\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}, \frac{dz_2}{dt}\right)^t =$$

$$\left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \cdot \begin{pmatrix} K^{-1} & 0 \\ 0 & K^{-1}\sin^2\sigma \end{pmatrix} \cdot \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right)^t$$

d) Explain why the formula

$$\left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \bullet_{sph} \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right) = \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \cdot \begin{pmatrix} K^{-1} & 0 \\ 0 & K^{-1}\sin^2\sigma \end{pmatrix} \cdot \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right)^t$$

allows us to compute the ordinary Euclidean dot product of two tangent vectors to the R-sphere in Euclidean space if we just know the values of the vectors in the (σ, τ)-plane that correspond to the two tangent vectors.

EXERCISE 134. (**SG**) Show that the length L of any path on the R-sphere given by

$$(\sigma, \tau(\sigma)), \ 0 \leq \sigma \leq \varepsilon$$

with

$$\tau(0) = 0.$$

and

$$\tau(\varepsilon) = 0$$

is given by the formula

$$L = \int_0^{\varepsilon} \sqrt{\left(1, \frac{d\tau}{d\sigma}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \sigma \end{pmatrix} \cdot \left(1, \frac{d\tau}{d\sigma}\right)^t} d\sigma.$$

Hint: Notice that the Exercise makes the first coordinate σ the parameter for the curve, that is $\sigma(t) = t$, $\frac{d\sigma}{dt} \equiv 1$.

EXERCISE 135. Show that central projection of a point on the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$-space to the plane $\hat{z} = R$ is the same as central projection of the corresponding point in (x, y, z)-coordinates to the plane $z = 1$.
Hint: Recall (10.2.2) and write the corresponding relation $\hat{r}(\hat{x}_c, \hat{y}_c, R) = (\hat{x}, \hat{y}, \hat{z})$ in $(\hat{x}, \hat{y}, \hat{z})$-coordinates. Conclude that $\hat{r} = r$. (Why?)

Exercise 136. (**SG**) Show that lines (i.e. shortest paths in **SG**) correspond under central projection to straight lines in the (x_c, y_c)-coordinates.
Hint: See Exercise 111 and Theorem 15. Or just write the equation for a line in (x_c, y_c)-coordinates and substitute (10.4.2). Then reverse the process.

Exercise 137. (**SG**) Explain why we know from an Exercise in Part 5 that, if a region $\hat{G}$ on the R-sphere is parametrized by a region G_c in (x_c, y_c)-coordinates, then the area $\hat{A}$ of $\hat{G}$ is given by the formula

$$\hat{A} = \int_{G_c} \left(K\left(x_c^2 + y_c^2\right) + 1\right)^{-3/2} dx_c dy_c.$$

Exercise 138. (**SG**) a) Compute the stereographic projection of a point on the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$-space to the plane $\hat{z} = R$.

b) Show that the coordinates $(\hat{x}_s, \hat{y}_s)$ of the stereographic projection of a point on the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$-space to the plane $\hat{z} = R$ are the same as the coordinates (x_s, y_s) of the stereographic projection of the corresponding point in (x, y, z)-coordinates to the plane $z = 1$.
Hint: Reduce to showing that

$$R\left(\frac{2\hat{x}}{\hat{z}+R}, \frac{2\hat{y}}{\hat{z}+R}\right) = \left(\frac{2x}{z+1}, \frac{2y}{z+1}\right).$$

EXERCISE 139. (**SG**) a) Show that stereographic projection is conformal, that is, that the angle between two paths through a point on the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$-space is the same as the usual (Euclidean) angle between the corresponding two paths through the corresponding point in the (x_s, y_s)-plane.
Hint: See Subsection 10.3.3.

b) Draw a picture of an angle between two paths through a point on the Euclidean R-sphere and the stereographic projection of that angle onto the plane $\hat{z} = R$. Try to give an intuitive geometric explanation for why it should have the same measure as the original angle.
Hint: Let L denote the line through $(\hat{x}, \hat{y}, \hat{z})$ and (x_s, y_s, R). Compare the angle between L and the tangent plane to the R-sphere at $(\hat{x}, \hat{y}, \hat{z})$ to the angle between L and the plane $\hat{z} = R$. (In fact, by rotational symmetry, you can assume that $\hat{y} = y_s = 0$.)

Exercise 140. a) Use Exercise 118b) to compute the equations for the other two sides of T_s.

b) In the (x_s, y_s)-plane, draw T_s as accurately as you can when $K = 4$, then when $K = \frac{1}{4}$, then finally when $K = 0$, that is, for the sphere of infinite radius.

c) Use Subsection 6.5.2 and the fact that stereographic projection is conformal to compute the area of T in all cases in b).

Hint: Your job will be easier if you notice that the y_s-axis divides T_s into two congruent isosceles triangles. Then the only calculation you will need to do is to calculate the radian measure of the angle at the vertex $(x_s, y_s) = (2, 0)$ of T_s. (Why?) To calculate the interior angle, calculate $\frac{dy_s}{dx_s}$ by implicit differentiation of the first quadrant equation in a), then take $\pi - \arctan\left(\frac{dy_s}{dx_s}\right)$ radians.

Exercise 141. Explain why we know from an Exercise in Part 5 that in all of the cases in Exercise 140 the area of the spherical triangle T is also given by the formula

$$\int_{T_s} \frac{1}{\left(1+\frac{K}{4}\left(x_s^2+y_s^2\right)\right)^2} dx_s dy_s.$$

Exercise 142. Suppose that a vector V emanates from $(0,0,0)$ in (x,y,z)-space.

a) Show that $V \bullet_K V = 0$ if and only if V points in a direction of the light cone.

b) Show that $V \bullet_K V < 0$ if and only if V points in a direction inside the light cone.

c) Show that $V \bullet_K V > 0$ if and only if V points in a direction outside the light cone.

Hint: Use that the (Euclidean) angle ϑ that the light cone makes with the plane $z = 0$ is given by taking any point (x,y,z) on the light cone with $z > 0$ and computing

$$\tan \vartheta = \frac{z}{\sqrt{x^2 + y^2}} = |K|^{1/2}.$$

EXERCISE 143. (**HG**) Show that the matrix

$$\begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is K-orthogonal. Describe geometrically what this transformation is doing to the K-geometry.

Exercise 144. (**HG**) Show that the matrix

$$\begin{pmatrix} \cosh\varphi & 0 & |K|^{1/2}\cdot\sinh\varphi \\ 0 & 1 & 0 \\ |K|^{-1/2}\cdot\sinh\varphi & 0 & \cosh\varphi \end{pmatrix}$$

is K-orthogonal. Describe geometrically what this transformation is doing to the the K-geometry.

EXERCISE 145. (**HG**) Write an explicit K-rigid motion

$$M_1 = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes the point X_0 to a point $X_1 = (x_1, 0, z_0)$.

Exercise 146. (**HG**) Write an explicit K-rigid motion

$$M_2 = \begin{pmatrix} \cosh\varphi & 0 & |K|^{1/2}\cdot\sinh\varphi \\ 0 & 1 & 0 \\ |K|^{-1/2}\cdot\sinh\varphi & 0 & \cosh\varphi \end{pmatrix}$$

that takes the point $X_1 = (x_1, 0, z_0)$ to $N = (0, 0, 1)$.
Hint: Notice that

$$Kx_1^2 + z_0^2 = 1 = -\left(-|K|^{1/2}\cdot x_1\right)^2 + z_0^2.$$

So there is a φ with

$$\cosh\varphi = z_0$$

and

$$\sinh\varphi = -|K|^{1/2}\cdot x_1.$$

Try that φ in M_2.

EXERCISE 147. (**HG**) Write an explicit K-rigid motion

$$M_3 = \begin{pmatrix} \cos\vartheta' & \sin\vartheta' & 0 \\ -\sin\vartheta' & \cos\vartheta' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes V_2 to the vector

$$\left(\sqrt{a_2^2 + b_2^2}, 0, 0\right) = \left(\sqrt{V_2 \bullet_K V_2}, 0, 0\right).$$

Why does the transformation given by M_3 leave the North Pole N fixed?

EXERCISE 148. (**HG**) Explain why the K-rigid motion

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot (M_1 \cdot M_2 \cdot M_2)$$

constructed over the last couple of sections takes the point X_0 to N and the tangent vector V_0 to $\left(\sqrt{V_0 \bullet_K V_0}, 0, 0\right)$

EXERCISE 149. (**HG**) Explain why the K-rigid motion given by

$$M \cdot (M')^{-1}$$

takes (X_0, V_0) to (X_0', V_0').

EXERCISE 150. (**HG**) Show that these hyperbolic coordinates do actually parametrize the K-geometry, that is, that

$$K\left(x\left(\sigma,\tau\right)^{2}+y\left(\sigma,\tau\right)^{2}\right)+z\left(\sigma,\tau\right)^{2}\equiv 1$$

for all (σ,τ).

EXERCISE 151. (**HG**) a) Compute the 2×3 matrix D_{hyp} such that

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \cdot D_{hyp}$$

when a path in K-geometry is given by a path in the (σ, τ)-plane.
Hint: By the Chain Rule from several variable calculus

$$D_{hyp} = \begin{pmatrix} \frac{dx}{d\sigma} & \frac{dy}{d\sigma} & \frac{dz}{d\sigma} \\ \frac{dx}{d\tau} & \frac{dy}{d\tau} & \frac{dz}{d\tau} \end{pmatrix}.$$

b) Use a) to compute the K-dot product in (σ, τ)-coordinates, namely

$$\begin{aligned}
\left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \bullet_{hyp} \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right) &= \left(\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}\right) \bullet_K \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}, \frac{dz_2}{dt}\right) \\
&= \left(\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}, \frac{dz_2}{dt}\right)^t \\
&= \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \cdot D_{hyp} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot D_{hyp}^t \cdot \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right)^t.
\end{aligned}$$

EXERCISE 152. (**HG**) Show that the length L of any path in our K-geometry is given by

$$(\sigma, \tau(\sigma)), \ 0 \leq \sigma \leq \varepsilon$$

with

$$\tau(0) = 0.$$

and

$$\tau(\varepsilon) = 0$$

is given by the formula

$$L = |K|^{-1/2} \int_0^{\varepsilon} \sqrt{\left(1, \frac{d\tau}{d\sigma}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 \sigma \end{pmatrix} \cdot \left(1, \frac{d\tau}{d\sigma}\right)^t} \, d\sigma.$$

Exercise 153. (**HG**) Explain why lines in **HG** extend infinitely in each direction.
Hint: There is a K-rigid motion that takes any two points to $(0,0,1)$ and $\left(|K|^{-1/2}\cdot\sinh\varepsilon, 0, \cosh\varepsilon\right)$ for some $\varepsilon > 0$. Why does that mean that the K-distance between the two points is equal to $|K|^{-1/2}\cdot\varepsilon$?

Exercise 154. a) Use the formula (10.2.4) for K-rigid motions to check that for K-rigid motions

$$\begin{pmatrix} \underline{x} & \underline{y} & \underline{z} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & K^{-1} \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{pmatrix} = 0$$

whenever

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & K^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

and *vice versa.*

b) Recall that $m_{13}x + m_{23}y + m_{33}z = \underline{z}$ for the K-rigid motion given by the matrix M in (x, y, z)-coordinates. If $K < 0$, use a) to explain why (x, y, z) lies on the light cone if and only if $(\underline{x}, \underline{y}, \underline{z})$ lies on the light cone.

c) Use

$$\underline{z} = m_{13}x + m_{23}y + m_{33}z$$

to conclude that the plane given by

$$m_{13}x + m_{23}y + m_{33}z = 0$$

intersects the light cone only at $(0, 0, 0)$.

d) Show that the line given by

$$m_{13}x_c + m_{23}y_c + m_{33} = 0$$

never intersects the edge of the universe in central projection coordinates (nor in stereographic projection coordinates), no matter the (negative) value of K.

EXERCISE 155. (**HG**) a) Explain why the K-line $y = 0$ is given by the x_c-axis and the North Pole N is given by $(x_c, y_c) = (0, 0)$.

b) Explain why the point $\left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$ in the K-geometry is given by the point

$$(x_c, y_c) = \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}}, 0\right).$$

c) Explain why the K-distance between $(x_c, y_c) = (0, 0)$ and $(x_c, y_c) = \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}}, 0\right)$ is $|K|^{-1/2} \cdot \varepsilon$.

Hint: Convert the statement to a statement about the distance between two points in (x, y, z)-coordinates, a distance that we have computed previously.

EXERCISE 156. (**HG**) Use (x_c, y_c)-coordinates to show that **HG** satisfies the four Euclidean postulates E1, E2, E3, and E4. Thus hyperbolic geometry is a Neutral Geometry (**NG**).

EXERCISE 157. Explain why there is a K-rigid motion M_c that takes any three points P', R' and Q' in order along the edge of the universe to any other three points P'', R'' and Q'' in order along the edge of the universe.
Hint: Use that the set of K-rigid motions form a group under the composition operation.

Exercise 158. (**HG**) Explain why the above discussion implies that the angles $\angle P'A'R'$ and $\angle Q'A'R'$ must both be K-right angles, that is, their K-measures must each be 90°. So the line segments $\overline{P'Q'}$ and $\overline{A'R'}$ are K-perpendicular. [MJG,238-239]

Hint: You may need to use the fact that, since there is a K-rigid motion that interchanges $\left(-|K|^{-1/2},0\right)$ and $\left(|K|^{-1/2},0\right)$ and leaves $(0,0)$ fixed, the x_c-axis and the y_c-axis are K-perpendicular.

EXERCISE 159. (**HG**) Use the previous Exercise and the fact that A' can be any point along the chord $\overline{P'Q'}$ in the figure above to explain why the Klein model is not conformal, that is, it does not faithfully represent the measure of angles in **HG**.

EXERCISE 160. Use (x_c, y_c)-coordinates to show that **HG** does not satisfy Euclid's postulate E5. That is, through a point not on a line, it is not true that there passes a unique parallel (i.e. non-intersecting) line.

EXERCISE 161. (**HG**) a) Draw a picture of the Klein K-disk, the edge of the universe, and the four points on the x_c-axis.

b) Show that

$$\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}} : |K|^{-1/2}\right) = \left(0 : -1 : \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}} : 1\right).$$

In particular, notice that the computation doesn't depend on K.

c) Show that

$$\left(0 : -1 : \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}} : 1\right) = e^{-2\varepsilon}.$$

EXERCISE 162. For $K = -1$, calculate the K-distance between the two points given in (x_c, y_c)-coordinates by $(0, 0)$ and $(1/2, 0)$.

EXERCISE 163. Show that

$$A_K(\alpha) = |K|^{-1}(\pi - \alpha).$$

Hint: Use the substitution

$$\underline{x_c} = |K|^{1/2} x_c$$
$$\underline{y_c} = |K|^{1/2} y_c$$

to reduce the computation to the computation in the case that $|K| = 1$. Then use polar coordinates to get

$$A_{-1}(\alpha) = \int_{\vartheta=-\beta}^{\vartheta=\beta} \left(\int_{r=0}^{r=\frac{\cos\beta}{\cos\vartheta}} \frac{1}{(1-r^2)^{3/2}} r \cdot dr \right) d\vartheta.$$

Then do the substitution

$$u = 1 - r^2$$
$$du = -2r dr$$

to compute $\int_{r=0}^{r=\frac{\cos\beta}{\cos\vartheta}} \frac{1}{(1-r^2)^{3/2}} r \cdot dr$. In the final step use the substitution

$$t = \sin\vartheta$$

to reduce to an integral of the form

$$\int \frac{a}{\sqrt{1-\left(\frac{t}{a}\right)^2}} dt.$$

EXERCISE 164. (**HG**) a) Show that stereographic projection is conformal, that is, that the measure of K-angles between K-lines on K-geometry is just the ordinary Euclidean measure of angles formed by their (usually circular) stereographic projections.
Hint: See Subsection 10.3.3.

b) For $K = -1$, construct the K-line in (x_s, y_s)-coordinates that meets the K-line

$$(x_s - 2)^2 + (y_s - 2)^2 = 4$$

perpendicularly in the point $\left(2 - \sqrt{2}, 2 - \sqrt{2}\right)$.

Exercise 165. Show that in K-geometry for any K, the between two tangent vectors at the North Pole is the same as the ordinary Euclidean angles between the two corresponding tangent vectors in the (x_c, y_c)-plane and that angle is also the same as the ordinary Euclidean angles between the two corresponding tangent vectors in the (x_s, y_s)-plane.

Exercise 166. (**HG**) a) Use Exercise 118b) to compute the equations for the other two sides of T_s.

b) In the (x_s, y_s)-plane, draw T_s as accurately as you can when $K = -\frac{1}{4}$, then when $K = -1$.

EXERCISE 167. Draw the α-lune in Exercise 163 in (x_s, y_s)-coordinates.

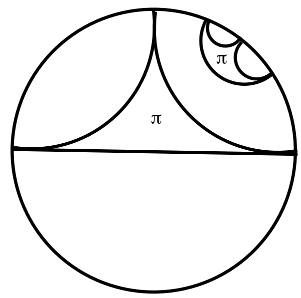
Exercise 168. (**HG**) Use Exercise 163 to show that in the above picture the K-area that lies in the union of the α-lune and the β-lune but does not lie in the $(\alpha+\beta)$- lune is $|K|^{-1}\pi$.

Exercise 169. Move the vertex of the α-lune in Exercise 167 to any other point of K-geometry by a rigid motion in (x_s, y_s)-coordinates. Draw the resulting figure (that we will continue to call an α-lune).

Exercise 170. a) (**HG**) Use Exercise 157 to show that the area of (the interior of) any infinte triangle has K-area

$$|K|^{-1} \cdot \pi.$$

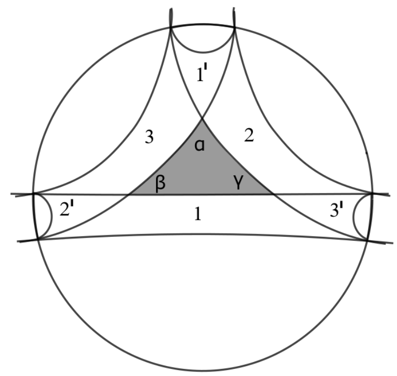
For example, if $K = -1$ we have



b) Use a) to give a formula for the K-area of any infinite n-gon in **HG**, that is, a figure described by a set of n disjoint K-lines that is the limit of a family of finite n-gons, all of whose vertices have gone to infinity. In particular, what is the area of any infinite hexagon?

Hint: Divide the infinite n-gon into infinite triangles.

EXERCISE 171. **HG**) a) Use the picture



(and remarks just above to explain why the K-area of the hyperbolic triangle is

$$|K|^{-1} \cdot (\pi - (\alpha + \beta + \gamma)) \, .$$

Hint: Locate α-lunes, two β-lunes and two γ-lunes in the picture and notice that they cover the hyperbolic triangle three times.

b) Use a) to give a formula for the K-area of a hyperbolic n-gon.