ADDITIONAL PROBLEMS FOR CONCEPTS IN ABSTRACT ALGEBRA

§0.1 For any sets A and B show that $A \subseteq B$ or $B \subseteq A$ exactly when $P(A \cup B) = P(A) \cup P(B)$.

§0.2 New exercise on the distributivity of $\times$ over $\cap$ and $\cup$:
Let $\{A_i\}_I$ and $\{B_j\}_J$ be collections of sets.
Show that $(\bigcup_i A_i) \times (\bigcup_j B_i) = \bigcup_i (\bigcup_j (A_i \times B_i))$ and that
$(\bigcap_i A_i) \times (\bigcap_j B_j) = \bigcap_i (\bigcap_j (A_i \times B_j))$.

§0.5 Exercise on restrictions of functions:
Find a nonconstant $h : \boldsymbol{Q} \to \boldsymbol{R}$ so that for no continuous $F : \boldsymbol{R} \to \boldsymbol{R}$ is $F|_{\boldsymbol{Q}} = h$.

§0.6 Let A be a set with $|A| > 1$ and let $P(A)$ be its power set. i) Describe some $f : A \to P(A)$ that is injective and find two different right inverses for f. ii) Describe some $g : P(A) \to A$ that is surjective and find two different left inverses for it.

Write any $n \in \boldsymbol{N}$ as $n = 2^s t$ for $s \geq 0$ and t odd. Define $h : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{Q}$ by: $h((2^s a, 2^t b)) = 2^s(a-1)/2^t b$ if $st \neq 0$ and $h((2^s a, 2^t b)) = -2^s(a-1)/2^t b$ if $st = 0$. Determine if h is injective and whether it is surjective.

§1.1. For $n \in \boldsymbol{N}$ and $b_1, b_2, \ldots, b_k \in \boldsymbol{Z}$, show the equivalence of:
i) $\{n\boldsymbol{Z} + b_j \mid 1 \leq j \leq k\}$ is a partition of $\boldsymbol{Z}$;
ii) for each $0 \leq i < n$ there is exactly one b_j with $n\boldsymbol{Z} + i = n\boldsymbol{Z} + b_j$;
iii) there are n different sets in $\{n\boldsymbol{Z} + b_j \mid 1 \leq j \leq k\}$.

For $n \in \boldsymbol{N}$ with $n \geq 2$ let c_n be the number of partitions of n using only 2 and 3: that is, the number of tuples $(a_1, a_2, \ldots, a_k)$ with $n = a_1 + \cdots + a_k$ and $3 \geq a_1 \geq \cdots \geq a_k \geq 2$. Show that for $n \geq 6$, c_n can be expressed in terms of c_{n-3}; this expression depends on whether n is even or odd. Compute c_2, c_3, c_4, c_5, and c_6 directly, then use the relation with c_{n-3} to write out all the c_j for $7 \leq j \leq 23$. Find an expression for c_n in terms of n and the remainder when n is divided by 6. Show this expression is correct by an induction argument.

§1.3. Let $n, m \in \boldsymbol{N}$ be relatively prime, $K = \{(i, j) \mid 0 \leq i < n \text{ and } 0 \leq j < m\}$, and for $(i, j) \in K$, set $S(i, j) = \{x \in \boldsymbol{Z} \mid n \mid (x - i) \text{ and } m \mid (x - j)\}$. i) Show that each $S(i, j) \neq \varnothing$. Hint: consider when $an + i$ and $bn + i$ have the same remainder on division by m and look at $a\boldsymbol{N} + i$.
ii) Show that $\{S(i, j)\}_K$ is a partition of $\boldsymbol{Z}$.

For $n \in \boldsymbol{N}$ show that $(n! + 1, (n + 1)! + 1) = 1$.

§1.4 The 2-adic representation of $n \in \boldsymbol{N}$ is the unique way if writing n as a sum of distinct powers of 2, as in Example 0.27. Let $2^{(n)}$ be the maximal power of 2 dividing $n!$
i) Show that if the 2-adic representation of n uses k powers of 2, then $2^{(n)} = n - k$.

ii) For $1 \le s \le n$, show that $2^{(n)} = 2^{(s)} + 2^{(n-s)} \Leftrightarrow$ the 2-adic representation of s uses a subset of the powers of 2 used to represent n.

iii) Determine when $\binom{n}{s}$ is odd.

§1.6 Let $S \neq \varnothing$ and set $A = \{f : S \to S\}$. Define a relation ~ on A by $g \sim h \Leftrightarrow h = f \circ g$ for some $f \in \mathrm{Bij}(S)$. a) Show that ~ is an equivalence relation. b) Show that $\mathrm{Bij}(S) = [I_S]_\sim$.
c) If $C_y \in A$ is the constant function $C_y(s) = y$ on S, show that $[C_y]_\sim = \{C_s \in A \mid s \in S\}$.

§1.7 Find the remainder when: i) $(6!)^{100001}$ is divided by 7; ii) $(10!)^{30605}$ is divided by 13;
iii) $(10!)^{30605}$ is divided by 17.

§1.8 4c) If $m \ge 3$ show that $x^2 \equiv 1 \pmod{2^m}$ has exactly four solutions in $\boldsymbol{Z}_{2^m}$.
(Hint: consider $[2^{m-1} \pm 1]$.)

§2.4 If $g \in G$, a group, and $o(g) = n \in \boldsymbol{N}$, show that $o(g)$ is odd $\Leftrightarrow o(g) = o(g^2)$.

§2.6 Show that the following groups are cyclic: i) U_{17}; ii) U_{23} (try $[5]_{23}$); iii) U_{29}; iv) U_{31}; v) U_{37}.

In each case show that the group is not cyclic: i) $(\boldsymbol{Q}, +)$; ii) $(\boldsymbol{Q}^+, \cdot)$; iii) $UT(2, \boldsymbol{Z})$;
iv) $Diag(2, \boldsymbol{Z}_5)$; v) $(\boldsymbol{Z}^2, \oplus)$ where $(n, m) \oplus (x, y) = (n + x, m + y)$.

If $G = \langle g \rangle$ is infinite and $H \le G$, show that $H = \langle g^m \rangle$ for some $m \ge 0$.

§3.1 Let G be a cyclic group of order n. If p is a prime dividing n show that G contains exactly $p - 1$ elements of order p.

§3.2 How many elements of prime order are contained in: i) U_{21}; ii) U_{23}; iii) U_{49}; iv) U_{67}.

Define ~ on $\boldsymbol{Z}_n$ for $n \ge 3$ by $[a]_n = [b]_n$ when $[b]_n = [a]_n[c]_n$ for $[c]_n \in U_n$. i) Show that ~ is an equivalence relation on $\boldsymbol{Z}_n$. ii) Find the equivalence classes of ~ explicitly when $n = 15$ and when $n = 27$. iii) In general show that a transversal for the equivalence classes is $\{[0]_n\} \cup \{[d]_n \mid 0 < d \text{ is a divisor of } n\}$.

§3.3 Define $F : I(10) \to I(10)$ by $F(i) = j$ when $i^3 + 1 \equiv j \pmod{10}$ and $1 \le j \le 10$. Show that F is a bijection and find its orbits.

§3.3 Find all n so that S_n has an element of order $n + 1$.

§3.4 Find a bijection $g : D_3 \to S_3$ so that if $x, y \in D_3$ then $g(xy) = g(x)g(y)$ in S_3. Thus the product of elements in D_3 corresponds to the product of the corresponding elements in S_3.

§4.2 If $H \le G$, a group, and if $G = Hg_1 \cup Hg_2 \cup \cdots \cup Hg_k$, show that $[G : H]$ is finite.

§4.3 Let G be a group with $|G| = p^2q^2$ for $p < q$ primes. If $H, K \leq G$ with $[G : H] = p$ and $[G : K] = q$, show that $|H \cap K| = pq$.

§5.3 Let G be a finite Abelian group with $|G| = p^m s$ for p a prime and $(p, s) = 1$.
Set $G_p = \{x \in G \mid o(x) = p^m \text{ for some } m \geq 0\}$.
i) Show that $G_p \leq G$ and that $|G_p| = p^t$ for $t \leq m$.
ii) Show that $|G_p| = p^m$ (consider G/G_p and its order, then use Cauchy's theorem).

§6.1 Show that there is an injective homomorphism $\varphi : S_n \to A_{n+2}$.

§7.3 How many elements of prime order are contained in: i) U_{210}; ii) U_{225}; iii) U_{900}.

Describe how many elements of prime order are contained in U_n in terms of the prime factorization of n.

§14.4 Let $p(x) \in F[x]$ be irreducible and let K be a splitting field over F for $p(x)$. If $\alpha \in K$ satisfies $p(\alpha) = 0$ show that the number of roots of $p(x)$ in $F(\alpha) \subseteq K$ divides $deg\ p(x)$.

§15.4 Let $f(x) = x^{15} - 15 \in \boldsymbol{Q}[x]$. Show that $f(x)$ is irreducible. Let L be a splitting field for $f(x)$ over $\boldsymbol{Q}$ with $G = Gal(L/\boldsymbol{Q})$. Find $|G|$ and justify your answer. Which Sylow subgroups of G are normal? What is the structure of each Sylow subgroup of G?