Partials, Gradients, and Lagrange Multipliers on a Pringles® Chip
Class Handout

Your group needs at least three chips, a tube of decorator's icing, paper, and a pencil.

1. Draw two xy-coordinate systems side by side. On one, place the Pringles chip so that it can be reasonably modeled by $f(x, y) = 2x^2 - y^2$. On the other coordinate system, sketch the contour lines for the surface of the chip.
   (a) Compute f_x, f_y, f_{xx}, f_{yy}, and f_{xy} at the point $(1, 1)$.
   (b) At $(1, 1)$ in the domain space, sketch vectors in the $\vec{i}$ and $\vec{j}$ direction.
   (c) Use icing to represent the vectors tangent to the surface and above the $\vec{i}$ and $\vec{j}$ vectors based at $(1, 1)$.
   (d) Determine if $f(x, y)$ is increasing or decreasing as you move in the $\vec{i}$ and $\vec{j}$ directions as well as the concavity of the surface there. Compare these to the signs of $f_x(1, 1)$, $f_y(1, 1)$, $f_{xx}(1, 1)$, and $f_{yy}(1, 1)$. What can you conclude?

2. Use the partial derivatives to find all critical points analytically, i.e., places where there is a horizontal tangent plane. Describe what the surface looks like at all critical point(s). Is there a local maximum, minimum, or saddle there?

3. Draw two new xy-coordinate systems. On one, place a new Pringles chip so that it can be reasonably modeled by $f(x, y) = 2x^2 - y^2$ as before. On the other, re-sketch the contour lines.
   (a) Compute the gradient vectors at $(1, 1)$, $(0, 1)$, $(-1, -1)$, and $(0, -1)$.
   (b) At each of these points, sketch the gradient vector on the contour diagram.
   (c) Use icing to represent the vectors tangent to the surface and above each of these gradient vectors.
   (d) Describe how the gradient vectors are situated with respect to the contour lines. Compare this to what is happening to the vectors along the surface. What conclusions can you make?

4. Draw two new xy-coordinate systems. On one, place a new Pringles chip so that it can be reasonably modeled by $f(x, y) = 2x^2 - y^2$ as before. On the other, re-sketch the contour lines.
   (a) Draw the curve $g(x, y) = x^2 + y^2 = 1$ on the contour diagram.
   (b) Use icing to represent the curve on the surface and above $g(x, y) = 1$. Locate the places along the icing curve where the surface has the highest elevation and lowest elevation, i.e., the maximum and minimum values of $f(x, y)$. Mark the corresponding points on the contour diagram.
   (c) At the points you just marked, sketch the gradient vectors for $f(x, y)$ and $g(x, y)$. Describe what you see. Does this property occur anywhere else along the constraint curve $g(x, y) = 1$?
   (d) Use the method of Lagrange multipliers to find where $f(x, y)$ reaches a maximum and minimum along the curve $g(x, y) = 1$. Compare this to your observations in the previous problems.