J-holomorphic Curves and Symplectic Topology

Second Edition

erratum

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22 October 2016

page 33: Here is a cleaner argument. The map $w' : U_0 \to \mathbb{C}$ defined on page 32 by

$$w'(z) := \prod_{\zeta \in U_0,\, \zeta \sim z} w(z) \tag{1}$$

is holomorphic and nonconstant and satisfies $w'(z_0) = 0$. Hence a theorem in complex analysis asserts that there exists a positive integer $\ell \in \mathbb{N}$, a neighbourhood $U_1 \subset U_0$ of z_0, and a biholomorphic map $\phi : U_1 \to V$ onto an open neighbourhood $V \subset \mathbb{C}$ of zero such that $\phi(z_0) = 0$ and

$$w'(z) = \phi(z)^\ell \qquad \text{for all } z \in U_1$$

(see [1, pp131–133, Thm 11] or [2, Satz 3.61]). Define $U := w(U_1)$ and $f := \phi \circ w^{-1} : U \to V$. Then $U \subset \mathbb{C}$ is an open neighbourhood of zero, $f : U \to V$ is a biholomorphic map, $f(0) = 0$, and

$$w'(z) = \bigl(f(w(z))\bigr)^\ell \qquad \text{for all } z \in U_1. \tag{2}$$

Choose $\delta > 0$ such that

(a) $\delta|w| \le |f(w)| \le \delta^{-1}|w|$ for all $w \in U$ (shrinking U if necessary).

Then the following holds.
(b) $\delta^\ell |w(z)|^\ell \le |w'(z)| \le \delta^{-\ell}|w(z)|^\ell$ for all $z \in U_1$, by (2) and (a).
(c) If $z, \zeta \in U_1$ and $z \sim \zeta$ then $w'(z) = w'(\zeta)$ and hence, by (b),

$$\delta^2 |w(z)| \le |w(\zeta)| \le \delta^{-2}|w(z)|.$$

(d) $\delta^{2m_0}|w(z)|^{m_0} \le |w'(z)| \le \delta^{-2m_0}|w(z)|^{m_0}$ for all $z \in U_1$ by (1) and (c). Here $m_0 := m(z_0)$ is as on page 32.
It follows from (b) and (d) that $m_0 = \ell$ and therefore each sufficiently small nonzero complex number has precisely m_0 preimages under w' (again [1, Thm 11, p131] or [2, Satz 3.61]). Thus, for all $z, \zeta \in U_1$ sufficiently close to z_0, we have

$$z \sim \zeta \qquad \Longleftrightarrow \qquad w'(z) = w'(\zeta).$$

This shows that the map $U_0' := U_0/\sim \to \mathbb{C} : [z] \mapsto w'(z)$ is injective and hence is a holomorphic coordinate chart on $\Sigma/\sim$.

page 37: Exercise 2.6.6 is wrong. For example, every branched double cover of $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2$ with positive genus has positive self-intersection number and violates the adjunction inequality.

page 161, line 16: Replace π^E by ev^E.

page 162, line 20: The set Δ^E is always a submanifold of M^E.

page 165, line 13: Replace π^E by ev^E.

page 246: The proof of Lemma 7.5.5 contains a mistake. The element $\mathbf{w}_I \in \overline{\mathrm{M}}_{0,I}$ defined by (7.5.1) is not a regular value of the projection

$$\pi_{k,I} : \overline{\mathrm{M}}_{0,k} \to \overline{\mathrm{M}}_{0,I}, \tag{3}$$

and so $Y_{k,I} := \pi_{k,I}^{-1}(\mathbf{w}_I)$ is not a submanifold of $\overline{\mathrm{M}}_{0,k}$. Moreover, even if $\mathbf{w}_I$ is chosen as a regular value of the projection (3), and if $k \in I$ and $\#I \ge 4$, then, while $Y_{k,I}$ and $Y_{k-1,I\setminus\{k\}}$ have the same dimension $2(k - \#I)$, the projection

$$\pi_{0,k} : Y_{k,I} \to Y_{k-1,I\setminus\{k\}} \tag{4}$$

(which forgets the kth marked point) is not necessarily a holomorphic diffeomorphism; it may collapse certain submanifolds to points. Nevertheless, Lemma 7.5.5 is correct and the proof can be fixed as follows.

First, the class $\beta_{k,I}$ can be represented by any fibre of the projection (3), whether or not it is the preimage of a regular value. The fibres are all connected and a point $\mathbf{w}_I \in \overline{\mathrm{M}}_{0,I}$ is a regular value of (3) if and only if it belongs to the top stratum.

Second, if $\mathbf{w}_I$ is a regular value of the projection (3) and $k \in I$, then the point $\mathbf{w}_{I\setminus\{k\}} \in \overline{\mathrm{M}}_{0,I\setminus\{k\}}$ (obtained by deleting all the crossratios involving the index k) is a regular value of the projection $\pi_{k-1,I\setminus\{k\}} : \overline{\mathrm{M}}_{0,k-1} \to \overline{\mathrm{M}}_{0,I\setminus\{k\}}$ and the two preimages $Y_{k,I}$ and $Y_{k-1,I\setminus\{k\}}$ both have have dimension $2(k - \#I)$.

Third, the restricted projecton (4) will typically be a kind of blow-up map, collapsing some submanifolds. However, it has degree one and hence maps the fundamental class of $Y_{k,I}$ to that of $Y_{k-1,I\setminus\{k\}}$. So the forgetful map $\pi_{0,k} : \overline{\mathrm{M}}_{0,k} \to \overline{\mathrm{M}}_{0,k-1}$ sends the homology class $\beta_{k,I}$ represented by the fundamental class of $Y_{k,I}$ to the class $\beta_{k-1,I\setminus\{k\}}$ represented by the fundamental class of $Y_{k-1,I\setminus\{k\}}$. This proves part (ii) of Lemma 7.5.5 in the case $k \in I$.

page 342, line -8: To use Theorem 9.4.7 we must prove that $\widetilde{\mathrm{M}}$ is minimal.

page 343, line -12: To use Theorem 9.4.2 we must prove that $\widetilde{\mathrm{M}}$ is minimal.

page 368, line 15/16: Eliashberg–Mishachev.

page 534–546: The discussion of determinant bundles needs rewriting to correct signs [3].

page 584, line -9: In equation (C.1.8) replace $\Omega^{1,1}(\Sigma, E^*)$ by $\Omega^{1,1}(\Sigma)$.

page 637, line -12: Replace the first displayed equation in the second paragraph by the equation

$$w_{m,m+1,n,i} = \frac{w_{1,m,m+1,n} - 1}{w_{1,m,m+1,n} - w_{1,m,m+1,i}}.$$

This holds for $1 < i < m$ and for all w near w^0 by (D.4.3). To see this, one must verify that $(1, \infty, w_{1,m,m+1,n}, w_{1,m,m+1,i}) \notin \Delta_3$ at the relevant points. Indeed, $w_{1,m,m+1,n}(\mathbf{z}^0) = \infty$ by assumption and $w_{1,m,m+1,i}(\mathbf{z}^0) \neq \infty$, because the points $z_{\alpha 1}, z_{\alpha m}, z_{\alpha m+1}$ are pairwise distinct and $z_{\alpha i} \neq z_{\alpha m+1}$ when $i \leq m$.

page 644, line 5: In Exercise D.6.2 the set $\mathcal{M}_{0,n+1}$ is contained in but is not equal to the set of regular points of the projection $\pi : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ which forgets the $(n+1)$st marked point. The equivalence class of a tuple

$$\mathbf{z} = \left(\{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \leq i \leq n+1}\right) \in \mathcal{SC}_{0,n+1}$$

is a singular point of π if and only if $n_{\alpha_{n+1}} = 3$ and $\Lambda_{\alpha_{n+1}} = \{n+1\}$.
Exercise: Characterize this condition in terms of the corresponding tuple $\{w_{ijk\ell}\}_{1\le i,j,k,\ell\le n+1} := \mathbf{w}(\mathbf{z}) \in \overline{\mathrm{M}}_{0,n+1}$ of crossratios.

References

[1] Lars V. Ahlfors, *Complex Analysis*, Third Edition. McGraw-Hill Inc, 1979.

[2] Dietmar A. Salamon, *Funktionentheorie*. Birkhäuser, 2012.
`https://people.math.ethz.ch/~salamon/PREPRINTS/cxana.pdf`

[3] Dietmar A. Salamon, Notes on the universal determinant bundle. Preprint, 2013.
`https://people.math.ethz.ch/~salamon/PREPRINTS/det.pdf`