ERRATA TO THE BOOK
"FROM STEIN TO WEINSTEIN AND BACK"

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We thank F. Laudenbach for pointing out a mistake in Lemma 9.29, and C. Wendl for detecting a large number errors throughout the book.

Chapter 2

Page 15, lines 7 and 9: "Hess(X, Y)" should be "Hess$_\phi(X, Y)$" (3 times)

Chapter 3

Page 31, line 9: "possible" should be "possibly"

Page 34, Remark 3.9: "$\max_\lambda \phi$" should be "$\max_\lambda \phi_\lambda$"

Chapter 4

Page 58: Corollary 4.3 should read:

For $a > 1$, $\gamma \in (0,1)$ and J as in Theorem 4.1 and any sufficiently small $\sigma > 0$ there exists an open subset $\Omega \subset \mathbb{C}^n$ containing $\{ar^2 - R^2 > -1-\sigma,\ r \geq \gamma\} \cup \{R \leq 1\}$ and a J-lc function $\Psi : \Omega \to (-1-\sigma, \infty)$ with the following properties (see Figure 4.1):

(i) *Ψ is of the form $\Psi(r, R)$ with $\frac{\partial \Psi}{\partial r} > 0$ and $\frac{\partial \Psi}{\partial R} \leq 0$;*
(ii) *$\Psi(r, R) = ar^2 - R^2$ on $\Omega \cap \{r \geq \gamma\}$;*
(iii) *there exists a diffeomorphism $f : (-1-\sigma, -1] \to (0, \delta]$ such that $f \circ \Psi(r, R) = r$ on the set $\{r \leq \delta,\ R \leq 1\}$;*
(iv) *Ψ is proper on $\Omega \cap C$ for any compact set $C \subset \mathbb{C}^n$.*

Chapter 5

Page 110: Add the following paragraph before the second paragraph from below.

In fact, J_ε is not even holomorphically fillable. One way to see this is by using Theorem 16.5 proven below in Chapter 16. Indeed, this theorem implies that any holomorphic filling of a J-convex S^3 can be blown down to a Stein filling (which is, moreover, diffeomorphic to a 4-ball).

Chapter 7

Page 137, Theorem 7.11: The displayed equation should be

$$\mathrm{Emb}_{\mathrm{isotr}}(\Lambda, M; A, h) \hookrightarrow \mathrm{Mon}^{\mathrm{emb}}_{\mathrm{isotr}}(T\Lambda, \xi; A, dh).$$

Page 141: The second half of the first paragraph beginning with the word "Indeed" in line 7 should be replaced with:

Indeed, suppose first that the overtwisted disc D_{ot} is *standard*, i.e., the contact structure in its neighborhood can be given by the normal form $\cos r\, dz + r \sin r\, d\phi$, where r, ϕ, z are cylindrical coordinates in $\mathbb{R}^3$ and the overtwisted disc is given in these coordinates as $\{z = 0, r \le \pi\}$. Then the vector field $\frac{\partial}{\partial z}$ is contact, and hence all parallel discs $\{z = c, r \le \pi\}$ are overtwisted. In the general case, take a slightly bigger disc $\widetilde{D} \supset D_{\mathrm{ot}}$ and note that the boundary $\partial D_{\mathrm{ot}} \subset \widetilde{D}$ is a stable limit cycle of the characteristic foliation. Hence it persists on all discs sufficiently C^2-close to $\widetilde{D}$, and the sub-discs bounded by these limit cycles are overtwisted.

Page 141: The following statement, analogous to Theorem 7.25(b), can be added to Theorem 7.19:

Let $f_t : S^1 \hookrightarrow M \setminus D$, $t \in [0, 1]$ be a smooth isotopy which begins with a Legendrian embedding f_0. Then there exists a Legendrian isotopy $\widetilde{f}_t : S^1 \hookrightarrow M \setminus D$ starting at f_0 and a C^0-small smooth isotopy $f'_t : S^1 \hookrightarrow M \setminus D$ connecting $\widetilde{f}_1$ to f_1 such that the concatenation of $\widetilde{f}_t$ and f'_t is homotopic to f_t through smooth isotopies with fixed endpoints.

Page 146: In Theorem 7.25, the C^0-closeness assertions in parts (b) and (c) are false. The correct statement is

Theorem 7.25 (Murphy's h-principle for loose embeddings [**143**]). *Let (M, ξ) be a contact manifold of dimension $2n + 1 \ge 5$ and Λ an n-dimensional manifold.*

(a) Any formal Legendrian embedding $(f : \Lambda \hookrightarrow M, F^s : T\Lambda \to TM)$ can be C^0-approximated by a loose Legendrian embedding $\widetilde{f} : \Lambda \hookrightarrow M$ formally Legendrian isotopic to (f, F^s).

(b) Let $f_t : \Lambda \hookrightarrow M$, $t \in [0, 1]$ be a smooth isotopy which begins with a loose Legendrian embedding f_0. Then there exists a loose Legendrian isotopy $\widetilde{f}_t$ starting at f_0 and a C^0-small smooth isotopy f'_t connecting $\widetilde{f}_1$ to f_1 such that the concatenation of $\widetilde{f}_t$ and f'_t is homotopic to f_t through smooth isotopies with fixed endpoints.

(c) Let (f_t, F^s_t), $s, t \in [0, 1]$, be a formal Legendrian isotopy connecting two loose Legendrian knots f_0 and f_1. Then there exists a Legendrian isotopy $\widetilde{f}_t$ connecting

$\widetilde{f}_0 = f_0$ and $\widetilde{f}_1 = f_1$ which is homotopic to the formal isotopy (f_t, F_t^s) through formal isotopies with fixed endpoints.

Note that (b) is a direct consequence of (c) and a 1-parametric version of (a). Moreover, if in (b) we are given a compact subset $A \subset M$ with $f_1(\Lambda) \cap A = \varnothing$, then we can arrange that $\widetilde{f}_1$ is loose in $M \setminus A$ and $f_t'(\Lambda) \cap A = \varnothing$ for all $t \in [0,1]$.

Chapter 8

Page 161: The C^0-approximation assertion in Theorem 8.11(a) is false. The correct statement of the theorem (with (a) and (b) exchanged) is

Theorem 8.11 (Gromov [**82**],[**120**]).

(a) Let (W, J, ϕ) be a $2n$-dimensional almost complex Morse cobordism, where the function ϕ has no critical points of index $> n$. Suppose that J is integrable near $\partial_- W$. Let L be the skeleton of ϕ (with respect to some gradient-like vector field). Then J can be C^0-approximated by an almost complex structure which coincides with J on $\mathcal{O}p\,(\partial_- W)$ and outside a neighborhood of L and is integrable on $\mathcal{O}p\,(L \cup \partial_- W)$. In particular, J is homotopic to an integrable complex structure via a homotopy fixed on $\mathcal{O}p\,\partial_- W$.

(b) Let (W, J) be a $2n$-dimensional almost complex manifold which admits an exhausting Morse function ϕ without critical points of index $> n$. Then J is homotopic to an integrable complex structure.

Page 163: The proof of Theorem 8.11 needs to be modified as follows.

Proof of Theorem 8.11. (a) After a C^∞-small perturbation of ϕ (keeping the gradient-like vector field and thus the skeleton fixed) we may assume that no two critical points have the same value. If $\partial_- W \neq \emptyset$, then we move down the value $\phi|_{\partial_- W}$ (without changing the skeleton) until it is the minimum and set $L_0 := \partial_- W$. If $\partial_- W = \emptyset$, then we set $L_0 := \{p_0\}$ for the point p_0 where ϕ attains its minimum. In both cases, we order the remaining critical points $p_1, p_2, \dots, p_N$ by increasing value and set $L_j := L_0 \cup \bigcup_{i=1}^{j} W_{p_i}^-$. Then each L_j is compact, $L_j \setminus L_{j-1} = W_{p_j}^-$ is the stable manifold of p_j, and $L = L_N$ is the skeleton. If $\partial_- W \neq \emptyset$, the J is integrable on $\mathcal{O}p\,L_0$ by hypothesis. If $\partial_- W = \emptyset$, then we deform J to make it integrable on $\mathcal{O}p\,L_0$. Proceeding inductively, suppose that for some $j \geq 1$ we already made J integrable on $\mathcal{O}p\,L_{j-1}$. Choose a compact domain Ω with smooth boundary such that $L_{j-1} \subset \operatorname{Int}\Omega$ and J is already integrable on Ω, and such that $L_j \setminus \operatorname{Int}\Omega$ is a disc transversely attached to Ω. (Such a domain can be obtained by picking a regular level c_j between $\phi(p_{j-1})$ and $\phi(p_j)$ and moving the set $\{\phi \leq c_j\}$ under the backward flow of the gradient-like vector field for sufficiently long time). Hence we can apply Corollary 8.14 to make J integrable on $\mathcal{O}p\,L_j$.

b) We pick a sequence of regular values $\min\phi < c_1 < c_2 \cdots$ converging to infinity and set $V_0 := \emptyset$ and $V_k := \{\phi \le c_k\}$, $k \in \mathbb{N}$. We will construct a continuous family J_t, $t \in [0,\infty)$, of almost complex structures on V with the following properties:

- $J_0 = J$;
- J_k is integrable on $\mathcal{O}p\,V_k$;
- $J_t = J_k$ on V_k for all $t \ge k$.

From such a family J_t we obtain an integrable complex structure $\widetilde{J}_1$ on V which equals J_k on V_k for all k, and a homotopy $(\widetilde{J}_t)_{t\in[0,1]}$ connecting J to $\widetilde{J}_1$ defined by $\widetilde{J}_t := J_{\tau(t)}$ for any homeomorphism $\tau : [0,1) \to [0,\infty)$.

To construct the family J_t, suppose inductively that we have already constructed it for $t \le k$ for some integer $k \ge 0$ (the case $k = 0$ is vacuous). By part (a), there exists a homotopy J_t, $t \in [k, k+\frac{1}{2}]$, fixed on $\mathcal{O}p\,V_k$ and outside V_{k+1}, such that $J_{k+\frac{1}{2}}$ is integrable on an open neighbourhood Ω of $V_k \cup L_k$, where L_k is the skeleton of $V_{k+1} \setminus \mathrm{Int}\,V_k$. Using a suitably cut off negative gradient flow of ϕ, we find a diffeotopy $f_s : V \to V$, fixed on $\mathcal{O}p\,V_k$ and outside V_{k+2}, such that $f_0 = \mathrm{id}$ and $f_1(V_{k+1}) \subset \Omega$. Now $J_{k+\frac{1}{2}+\frac{s}{2}} := f_s^* J_{k+\frac{1}{2}}$, $s \in [0,1]$, provides the desired extension of the family J_t up to $t \le k+1$. $\square$

Page 165: In the proof of Lemma 8.20, add after "...independently for all stable discs." the following sentence: "(More precisely, we inductively modify each disc by applying Theorem 7.34 in the complement of the previous discs.)"

Chapter 9

Page 187: The sentence after Lemma 9.1 should read: "More precisely, this means that for a function ϕ on a neighborhood of $0 \in \mathbb{R}^m$ with a nondegenerate critical point of index k at 0 there exists a diffeomorphism g between neighborhoods of 0 such that $g^*\phi$ has the form (9.1)."

Page 197, line 4: "imaginary" can be replaced by "nonreal".

Page 203: Lemma 9.29 is false as stated. In the following corrected statement the hypothesis $\phi'(p_i) = \phi(p_i)$ for all $i = 1,\dots,k$ is replaced by to $\phi' = \phi$ on Δ. This stronger hypothesis is satisfied in all the applications of Lemma 9.29 in the book, namely in the proofs of Lemma 8.20, Theorem 13.6, and Lemma 14.19. In the proofs of Theorem 13.6 and Lemma 14.19 this is stated explicitly; in the proof of Lemma

8.20 it is not stated explicitly, but it holds because the applications of Corollary 8.9 and Theorem 8.4 in the proof both preserve the function on Δ.

Lemma 9.29. *Let (W, X, ϕ) be an elementary Smale cobordism with critical points $p_1, \dots, p_k$ and skeleton (= union of the stable discs) $\Delta = \bigcup_{i=1}^k D_{p_i}^-$. Let (W', X', ϕ') be another Smale cobordism with the following properties:*

- *$W' \subset W$ and $\partial_- W = \partial_- W'$;*
- *(X', ϕ') has the same critical points and stable discs as (X, ϕ);*
- *$\phi' = \phi$ on $\Delta \cup \mathcal{O}p\,(\partial_- W)$, and $\phi'(\partial_+ W') = \phi(\partial_+ W)$.*

Then there exists an isotopy $h_t : W \hookrightarrow W$, $t \in [0,1]$, with $h_0 = \mathrm{id}$, $h_t(\Delta) = \Delta$ and $h_t = \mathrm{id}$ on $\mathcal{O}p\,(\partial_- W)$, such that $h_1(W) = W'$ and $\phi = \phi' \circ h_1$. Moreover, the construction can be done smoothly in families.

Proof. **Step 1.** Applying the Morse Lemma 9.1 and Remark 9.2 near each critical point and extending the diffeomorphisms to all of W, we find a diffeotopy $h_t : W \to W$, fixed on $\Delta \cup \partial_- W$, such that $\phi' \circ h_1 = \phi$ on $\bigcup_{i=1}^k \mathcal{O}p\, p_i$. After renaming $\phi' \circ h_1$ back to ϕ' and modifying the gradient-like vector field X', we may hence assume that $(X', \phi') = (X, \phi)$ on $\Delta \cup \bigcup_{i=1}^k \mathcal{O}p\, p_i$.

Step 2. The identity on $\bigcup_{i=1}^k \mathcal{O}p\, p_i$ extends to a unique diffeomorphism $h_1 : \mathcal{O}p\,\Delta \to \mathcal{O}p\,\Delta$ mapping trajectories of X to trajectories of X' and such that $\phi' \circ h_1 = \phi$ on $\mathcal{O}p\,\Delta$. Since $(X', \phi') = (X, \phi)$ on Δ, this diffeomorphism restricts to the identity on Δ. Hence we can adjust h_1 near $\partial_- W$ and extend it to a diffeotopy $h_t : W \to W$, fixed on $\Delta \cup \bigcup_{i=1}^k \mathcal{O}p\, p_i \cup \mathcal{O}p\,(\partial_- W)$, such that $\phi' \circ h_1 = \phi$ on $\mathcal{O}p\,\Delta$. After renaming $\phi' \circ h_1$ back to ϕ' and modifying the gradient-like vector field X', we may hence assume that $(X', \phi') = (X, \phi)$ on a neighborhood U of $\partial_- W \cup \Delta$.

Step 3 remains unchanged. □

Page 205, line 27: Should be “...along which W_q^+ and W_p^- intersect transversely.”

Page 214, last line: “$\pi_1 \mathcal{P}(M)$” should be “$\pi_0 \mathcal{P}(M)$”.

Chapter 11

Page 244, Definition 11.14: “triple” should be “pair”.

Page 244, line 16: “coarser that” should be “coarser than”.

Chapter 12

Page 255, lines 17 and 22: “Liouville structure” should be “Weinstein structure”.

Page 268, line 3: “gradient trajectory” should be “X-trajectory”.

Chapter 14

Page 279, Theorem 14.1: The second sentence should be "Let $\psi : W \to \mathbb{R}$ be a Morse function without critical points of index $> n$ and having $\partial_\pm W$ as regular level sets."

Page 280, line -9: "consequence the" should be "consequence of the".

Pages 281-284: The statements of Lemmas 14.10 and 14.11 should include the condition that the Weinstein cobordisms $\mathfrak{W}_t$, $t \in [0, 1/2]$, are flexible. Their proofs need to be adjusted to the corrected version of Theorem 7.25 as follows.

Proof of Lemma 14.10. **Step 1.** "$\Sigma_j^- = \partial_+ V_j$" should be "$\Sigma_j^- = \partial_+ V_{j-1}$".
The sentence "Moreover, after modifying Y near ∂W we may assume that $Y = X$ on $\mathcal{O}p\,\partial W$." is superfluous and can be dropped.

Step 2. In the application of Theorem 7.25 the C^0-smallness assertions need to be dropped. Here is the corrected third paragraph:

Since $\Gamma_{Y_0} = \Gamma_X$ is a contactomorphism, this implies that the embedding g_0 is loose isotropic. Hence, by the h-principles in Chapter 7 (Theorem 7.11 for the subcritical case, Theorem 7.19 for the Legendrian overtwisted case in dimension 4, and Theorem 7.25 in the Legendrian loose case in dimension $2n > 4$), there exists a (loose) isotropic isotopy $\widetilde{g}_t$ starting at g_0 and a C^0-small smooth isotopy g'_t connecting $\widetilde{g}_1$ to g_1 such that the concatenation of $\widetilde{g}_t$ and g'_t is homotopic to g_t through smooth isotopies with fixed endpoints. More precisely, by the isotopy extension theorem and the discussion following Theorem 7.25, we find diffeotopies $\delta_t, \widehat{\delta}_t : \Sigma_j^+ \to \Sigma_j^+$ with the following properties:

- $\delta_0 = \widehat{\delta}_0 = \mathrm{Id}$ and $\delta_1 = \widehat{\delta}_1$;
- the diffeotopies δ_t and $\widehat{\delta}_t$ are homotopic with fixed endpoints;
- $\widehat{\delta}_t$ is C^0-small and $\widehat{\delta}_t \circ g_1(\mathbf{S}_{j+1}^-) \cap \mathbf{S}_j^+ = \varnothing$ for all $t \in [0, 1]$;
- $\delta_t \circ g_t$ is loose isotropic in Σ_j^+ with respect to the contact structure ξ_j^+;
- $\delta_1 \circ g_1$ is loose isotropic in $\Sigma_j^+ \setminus \mathbf{S}_j^+$.

The path Γ_{Y_t}, $t \in [0, 1]$, in $\mathrm{Diff}(\Sigma_{j+1}^-, \Sigma_j^+)$ is homotopic with fixed endpoints to the concatenation of the paths $\delta_t \circ \Gamma_{Y_t}$ (from Γ_{Y_0} to $\delta_1 \circ \Gamma_{Y_1}$) and $\widehat{\delta}_t^{-1} \circ \delta_1 \circ \Gamma_{Y_1}$ (from $\delta_1 \circ \Gamma_{Y_1}$ to Γ_{Y_1}). Hence by Lemma 9.41 we find paths Y'_t and Y''_t, $t \in [0, 1]$, in $\mathcal{X}(V_j, \phi)$ such that

- $Y'_0 = X$, $Y'_1 = Y''_0$ and $Y''_1 = Y$;
- $\Gamma_{Y'_t} = \delta_t \circ \Gamma_{Y_t}$ and $\Gamma_{Y''_t} = \widehat{\delta}_t^{-1} \circ \delta_1 \circ \Gamma_{Y_1}$, $t \in [0, 1]$.

Note that $\Gamma_{Y'_t}|_{\mathbf{S}_{j+1}^-}$ is loose isotropic. Moreover, $\Gamma_{Y''_t}(\mathbf{S}_{j+1}^-) \cap \mathbf{S}_j^+ = \varnothing$ in Σ_j^+ and $\Gamma_{Y'_1}(\mathbf{S}_{j+1}^-)$ is loose in $\Sigma_j^+ \setminus \mathbf{S}_j^+$.

The rest of the proof remains unchanged. □

Proof of Lemma 14.11. In the application of Theorem 7.25 the C^0-smallness assertions need to be dropped. Here is the corrected sentence:

Hence by Theorems 7.11, 7.19 and 7.25 find an isotropic isotopy $\widetilde{S}_t$ with the following properties:

- $\widetilde{S}_0 = S'_0 = S_0$;
- $\widetilde{S}_1$ is connected to S'_1 by a C^0-small smooth isotopy that coincides with S'_1 near q and has q as its unique transverse intersection point with S^+.

The rest of the proof is unchanged. □

Page 291, line 2: "to a a" should be "to a".

Page 293: In diagram (14.1), surjectivity of the map $\pi_0\mathcal{P}(M,\xi) \longrightarrow \pi_0\mathrm{Diff}_{\mathcal{P}}(M,\xi)$ may not hold. To make the diagram true as stated, the definition of $\mathrm{Diff}_{\mathcal{P}}(M,\xi)$ on page 292 should be changed to be the set of all F_+ associated to $F \in \mathcal{P}(M,\xi)$. It would be interesting to better understand the relation between these two definitions.

Page 298, lines 11-12: The sentence is correct, but would more appropriately refer to more recently stated results and read: "According to Corollary A.6 in Appendix A.1, Conjecture 15.6 combined with Corollary 13.7 would imply".

Chapter 16

Page 315: The following remark may be added after the proof of Theorem 16.5:

Note that although *a posteriori* the assumption of holomorphic minimality in Theorem 16.5 implies symplectic minimality, i.e., the absence of embedded symplectic spheres of self-intersection index -1, a priori this is not the case. If one replaces the minimality hypothesis in Theorem 16.5 by symplectic minimality, then Step 1 of the proof simplifies and it is not necessary to use the result of Bogomolov and Oliverira from [20].

Page 315, Theorem 16.7: The last sentence should read "... admits a J-convex Morse function (having the boundary components as regular level sets) with exactly..."

Page 318, line 20: "$h : (\widehat{W}_0, \widehat{\omega}_0) \to (\widehat{W}_0, \widehat{\omega}_0)$" should be "$h : (\widehat{W}_0, \widehat{\omega}_0) \to (\widehat{W}_1, \widehat{\omega}_1)$"

Page 320, proof of Lemma 16.13, second line: Should be "$B \times I \subset B' \times I' \subset B \times J \subset B' \times J'$"

Page 321, Theorem 16.15: The second sentence should be "If finite type Stein structures on the interiors of W_1, W_2 are unique up to deformation equivalence, then so are finite type Stein structures on the interior of the ..."

Chapter 17

Page 323, line -4: "critical points on $\mathcal{A}_H$" should be "critical points of $\mathcal{A}_H$"

Page 325, line 14: "$SH(W)$" should be "$SH_*(W)$"

Page 325, line 15: The sentence should be "Since $SH_*(B^{2n}) = 0$, this implies that every subcritical Weinstein manifold has vanishing symplectic homology.

Page 325, line -15: Reference [185] should be replaced by "C. Viterbo, *Functors and computations in Floer cohomology with applications, Part II* (unpublished)".

Appendix A

Page 332, third line after the end of proof of Lemma A.7: "$p : G_{n,k} \to V_{n,k}$" should be "$p : V_{n,k} \to G_{n,k}$"

Appendix B

Page 341: Bennequin's inequality is lacking a minus sign. The correct inequality is

$$tb(L) + |r(L)| \leq -\chi(\Sigma).$$

Appendix C

Page 347, last line: "habititation" should be "habilitation"

Page 348, line 12: "second half of 20th century" should be "second half of the 20th century"