A revision for *Exploring Continued Fractions* by A. Simoson

Revised Proposition IX.28: A mean-value rule for O

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Theorem 28: A mean-value rule for O. Let ω be a positive irrational number. At step $k \geq 0$ with respect to Algorithm O, let $C_{k-1} = \frac{a}{b}$ and $C_{k-2} = \frac{c}{d}$. With $\omega = s_k C_{k-1} \oplus C_{k-2}$, then $s_k = \frac{c - \omega d}{\omega b - a}$ and $\epsilon_k = \mathrm{sgn}(s_k)$. If $|s_k| - \lfloor |s_k| \rfloor < \delta_k$ where

$$\delta_k = \frac{d + \epsilon_k b(\lceil |s_k| \rceil + 1)}{2d + \epsilon_k b(2\lceil |s_k| \rceil + 1)} \tag{11}$$

then $n_k = \lceil |s_k| \rceil$, otherwise $n_k = \lfloor |s_k| \rfloor$. Equivalently, $n_k = \lceil |s_k| - \delta_k \rceil$.

PROOF: When $k = 0$, then $s_0 = \omega$ and $\delta_0 = \frac{1}{2}$, so $n_0 = [\omega]$. That is, $C_0 = \frac{n_0}{1}$ is a really-good approximation for ω because $|\omega - [\omega]| < \frac{1}{2} = \frac{1}{2 \cdot 1^2}$.

Now let $k \geq 1$. For simplicity, let $s = s_k$ and $\epsilon = \epsilon_k$. By Exercise IV.6c, we know that at least one of $\epsilon\lfloor |s| \rfloor A \oplus B$ and $\epsilon\lceil |s| \rceil A \oplus B$ is a really-good approximation to $\omega = sA \oplus B$. When

$$\left|\omega - \frac{a\epsilon\lceil |s| \rceil + c}{b\epsilon\lceil |s| \rceil + d}\right| < \frac{1}{2(b\epsilon\lceil |s| \rceil + d)^2} \tag{12}$$

then $\epsilon\lceil |s| \rceil A \oplus B$ is a good approximation to ω, otherwise $\epsilon\lfloor |s| \rfloor A \oplus B$ is a good approximation to ω. Since the expressions $b\epsilon\lceil |s| \rceil + d$ and $bs + d$ have the same sign and since $|ad - bc| = 1$, then (12) is equivalent to

$$\lceil |s| \rceil - |s| = \left|(as + c)(b\epsilon\lceil |s| \rceil + d) - (a\epsilon\lceil |s| \rceil + c)(bs + d)\right| < \frac{bs + d}{2(b\epsilon\lceil |s| \rceil + d)} = \frac{b|s| + \epsilon d}{2(b\lceil |s| \rceil + \epsilon d)}. \tag{13}$$

Multiplying (13) by the positive expression $2(b\lceil |s| \rceil + \epsilon d)$ and simplifying gives

$$\lceil |s| \rceil \left(2(b\lceil |s| \rceil + \epsilon d)\right) < \epsilon d + |s| \left(b(2\lceil |s| \rceil + 1) + 2\epsilon d\right). \tag{14}$$

Rewriting (14) using $\lceil |s| \rceil = \lfloor |s| \rfloor + 1$ and simplifying gives

$$\lfloor |s| \rfloor \left(2(b\lceil |s| \rceil + \epsilon d)\right) + b(\lfloor |s| \rfloor + 1) + b\lceil |s| \rceil + \epsilon d < |s| \left(b(2\lceil |s| \rceil + 1) + 2\epsilon d\right), \tag{15}$$

which is

$$\lfloor |s| \rfloor \left(b(2\lceil |s| \rceil + 1) + 2\epsilon d\right) + b(\lceil |s| \rceil + 1) + \epsilon d < |s| \left(b(2\lceil |s| \rceil + 1) + 2\epsilon d\right). \tag{16}$$

Dividing (16) through by $b(2\lceil |s| \rceil + 1) + 2\epsilon d$ and simplifying gives

$$\delta_k = \frac{d + \epsilon b(\lceil |s| \rceil + 1)}{2d + \epsilon b(2\lceil |s| \rceil + 1)} < |s| - \lfloor |s| \rfloor. \tag{17}$$

Whenever (17) is true, then $n_k = \lceil |s_k| \rceil$, and is $\lfloor |s_k| \rfloor$ otherwise. $\square$

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Example 29: Euler's constant via O. Let $\omega = \gamma \approx 0.577216$. By Proposition 28,

Step 0: $s_0 = \gamma$, $\delta_0 = \frac{1}{2}$. Since $\gamma - \lfloor \gamma \rfloor \approx -.5777 > \delta_0$, then $n_0 = 1$.

Step 1: $s_1 \approx -2.365$, $\delta = \frac{4}{7} \approx 0.571$, $\epsilon_1 = -1$, $|s_1| - \lfloor |s_1| \rfloor \approx 0.365 < \delta_1$. So $\epsilon_1 n_1 = -2$.

Step 2: $s_2 \approx 2.738$, $\delta_2 = \frac{9}{16} \approx 0.562$, $\epsilon_2 = 1$, $|s_2| - \lfloor |s_2| \rfloor \approx 0.738 > \delta_2$. So $\epsilon_2 n_2 = 3$.

Step 3: $s_3 \approx -3.81$, $\delta_3 = \frac{37}{67} \approx 0.552$, $\epsilon_3 = -1$, $|s_3| - \lfloor |s_3| \rfloor \approx 0.81 > \delta_3$. So $\epsilon_3 n_3 = -4$.

Step 4: $s_4 \approx -5.35$, $\delta_4 = \frac{189}{352} \approx 0.537$, $\epsilon_4 = -1$, $|s_4| - \lfloor |s_4| \rfloor \approx 0.325 < \delta_3$. So $\epsilon_4 n_4 = -5$.

Thus, $\gamma = [1; -2, 3, -4, -5, 3, 13, \ldots]_O$ which is the same expansion as given by the nearest integer continued fraction algorithm. However the reader may check that O's seventh convergent is $O_7 = \frac{15403}{26685}$ and Z's seventh convergent is $Z_7 = \frac{18438}{31943}$, while their common sixth convergent is $\frac{3035}{5258}$. The reader may check that O_7 is a really-good approximation for γ whereas Z_7 fails to be one.