

Thus we see that the generalized Blaschke product  $B_Z^{z_0}(z)$  is *not* identically zero, has norm at most one in the multiplier space  $M_H$ , vanishes on  $Z$ , and is positive at  $z_0$ . In fact, using the left hand inequality in (C.3), this argument can be reversed and yields the following characterization of nontrivial generalized Blaschke products.

**Proposition 5.10** *Suppose  $H$  is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel  $k(x, y)$ , so that  $H = \mathcal{H}_k$ . Fix a sequence  $Z = \{z_j\}_{j=1}^\infty$  and  $z_0 \notin Z$ . Then  $B_Z^{z_0}(z)$  is not identically zero if and only if*

$$B_Z^{z_0}(z_0)^2 \equiv \prod_{n=1}^\infty d(z_0, z_n)^2 > 0,$$

if and only if  $\mu_Z = \sum_{j=1}^\infty \|k_{z_j}\|^{-2} \delta_{z_j}$  is a finite measure.

In particular, Proposition 5.10 gives a sufficient condition for  $Z$  to be a zero set for  $H$  (i.e. there is  $f \in H$  with  $f(z) = 0$  if and only if  $z \in Z$ ):  $\|\mu_Z\| < \infty$ . This recovers some results of Shapiro and Shields when  $H$  is the Dirichlet space  $\mathcal{D}(\mathbb{D})$  or a generalized Dirichlet space  $D_\varphi$  (see [44] where condition (12) there on  $\varphi$  shows that  $D_\varphi$  has a complete Nevanlinna-Pick kernel). This raises an interesting open question.

**Problem 5.11** Find a geometric characterization of the zero sets for  $H$ . When  $H = H^2(\mathbb{D})$  is the classical Hardy space on the disk,  $\|\mu_Z\| < \infty$  is necessary as well as sufficient. However, in general  $\|\mu_Z\| < \infty$  is *not* necessary for  $Z$  to be a zero set for  $H$ , in particular when  $H$  is the Dirichlet space  $\mathcal{D}(\mathbb{D})$ .

*consider*

We can also characterize separation and Carleson embedding for sequences  $Z$  using  $\int |k_{z_m}(z)|^2 d\mu_z(z) \leq \|\mu_z\|_{\text{Carleson}}$  for all  $m \geq 1$ .

**Proposition 5.12** *Suppose  $H$  is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel  $k(x, y)$ , so that  $H = \mathcal{H}_k$ . Then a sequence  $Z = \{z_j\}_{j=1}^\infty$  is separated  $\frac{|(k_{z_n}, k_{z_m})|}{\|k_{z_n}\| \|k_{z_m}\|} \leq 1 - \epsilon$ , and  $\mu_Z = \sum_{j=1}^\infty \|k_{z_j}\|^{-2} \delta_{z_j}$  is a Carleson measure for  $H$  if and only if*

$$\inf_{m \geq 1} B_{Z \setminus \{z_m\}}^{z_m}(z_m)^2 \equiv \inf_{m \geq 1} \prod_{n \neq 1} d(z_m, z_n)^2 > 0.$$

At this point we collect what can be said about interpolating sequences for a Hilbert space  $H$  with the NP property.

**Theorem 5.13** *Suppose  $H$  is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel  $k(x, y)$ , so that  $H = \mathcal{H}_k$ . Suppose  $Z = \{z_j\}_{j=1}^\infty$  is a sequence and let  $\mu_Z = \sum_{j=1}^\infty \|k_{z_j}\|^{-2} \delta_{z_j}$  be its associated positive measure. Denote by  $R$  the restriction map  $Rf = \{f(z_j)\}_{j=1}^\infty$ , and let  $B_{Z \setminus \{z_m\}}^{z_m}$  be the associated generalized Blaschke products. Then the relations*

$$1 \iff (2 + 3) \text{ and if } Z \text{ is separated } 3 \iff 4$$

hold among the following four conditions:

1.  $R$  maps  $M_H$  onto  $\ell^\infty(Z)$ ,
2.  $R$  maps  $H$  onto  $\ell^2(\mu_Z)$ ,
3.  $R$  maps  $H$  into  $\ell^2(\mu_Z)$ ,
4.  $\inf_{m \geq 1} B_{Z \setminus \{z_m\}}^{z_m}(z_m)^2 \equiv \delta_Z > 0$ .

*Z*

*→*

*implies*

**Proof:** Conditions **2+3** are equivalent to condition **1** by Theorem 5.6, and condition **3** is ~~equivalent to~~ condition **4** by Proposition 5.12.

**Remark 5.14** Let  $\widetilde{k}_{z_n} = \frac{k_{z_n}}{\|k_{z_n}\|}$  be the normalized reproducing kernel with pole at  $z_n$ , and let

$$B^{(n)}(z) = B_{Z \setminus \{z_n\}}^{z_n}(z)$$

be the generalized Blaschke product associated to the set  $Z_n = Z \setminus \{z_n\}$  with pole at  $z_n$ . In order to prove that condition **4** implies condition **2**, it suffices to show that the Grammian  $[(g_m, g_n)]_{m,n=1}^\infty$  is bounded on  $\ell^2$  where

$$g_n = \frac{1}{B^{(n)}(z_n)} \mathcal{M}_{B^{(n)}} \widetilde{k}_{z_n} \in H, \quad n \geq 1.$$

Indeed,

$$\begin{aligned} \langle \widetilde{k}_{z_m}, g_n \rangle &= B^{(n)}(z_n)^{-1} \langle \widetilde{k}_{z_m}, \mathcal{M}_{B^{(n)}} \widetilde{k}_{z_n} \rangle \\ &= B^{(n)}(z_n)^{-1} \langle M_{B^{(n)}}^* \widetilde{k}_{z_m}, \widetilde{k}_{z_n} \rangle \\ &= \frac{B^{(n)}(z_m)}{B^{(n)}(z_n)} \langle \widetilde{k}_{z_m}, \widetilde{k}_{z_n} \rangle = \delta_n^m. \end{aligned}$$

If the Gram matrix  $[(g_m, g_n)]_{m,n=1}^\infty$  were bounded on  $\ell^2$ , we would then have for  $\xi = \{\xi_j\}_{j=1}^\infty \in \ell^2$ ,

$$\begin{aligned} \|\xi\|_{\ell^2} &= \sup_{\|\eta\|_{\ell^2}=1} \left| \sum_{n=1}^\infty \xi_n \overline{\eta_n} \right| = \sup_{\|\eta\|_{\ell^2}=1} \left| \sum_{m,n=1}^\infty \delta_n^m \xi_m \overline{\eta_n} \right| \\ &= \sup_{\|\eta\|_{\ell^2}=1} \left| \sum_{m,n=1}^\infty \langle \widetilde{k}_{z_m}, g_n \rangle \xi_m \overline{\eta_n} \right| \\ &= \sup_{\|\eta\|_{\ell^2}=1} \left| \left\langle \sum_{m=1}^\infty \xi_m \widetilde{k}_{z_m}, \sum_{n=1}^\infty \eta_n g_n \right\rangle \right| \\ &\leq C^* \left\| \sum_{m=1}^\infty \xi_m \widetilde{k}_{z_m} \right\|_H, \end{aligned}$$

where  $C^* \equiv \sup_{\|\eta\|_{\ell^2}=1} \left\| \sum_{n=1}^\infty \eta_n g_n \right\|_H$  coincides with the norm of the Grammian  $[(g_m, g_n)]_{m,n=1}^\infty$  on  $\ell^2$ . The inequality  $\|\xi\|_{\ell^2} \leq C^* \left\| \sum_{m=1}^\infty \xi_m \widetilde{k}_{z_m} \right\|_H$  implies condition **2** by standard functional analysis - Corollary A.39 (p. 164) in Appendix A.

Note that using  $\delta_Z \leq B^{(n)}(z_n) \leq 1$  and Proposition 5.12, we at least have that the functions  $g_n$  are uniformly bounded in  $H$ :

$$\|g_n\|_H = B^{(n)}(z_n)^{-1} \left\| M_{B^{(n)}} \widetilde{k}_{z_n} \right\|_H \leq B^{(n)}(z_n)^{-1} \|B^{(n)}\|_{M_H} \|\widetilde{k}_{z_n}\|_H \leq \frac{1}{\delta_Z},$$

since  $\|B^{(n)}\|_{M_H} \leq 1$ .