

A First Course in Sobolev Spaces, First edition
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For the original text I use the color **Red**, for corrections the color **Blue**, and for improvements and additions the color **Cyan**. Names in brackets refer to the persons who called the error to my attention (to the best of my recollection) or suggested improvements and additions.¹

CHAPTER 1:

- p. 5 In Exercise 1.5 one should assume that u is bounded from below, since otherwise the series could diverge to ∞ . Alternatively, one should choose a point $x_0 \in I^\circ$ at which u is continuous and then take

$$u_J(x) := \begin{cases} + \sum_{y \in I, x_0 \leq y < x} (u_+(y) - u_-(y)) + u(x) - u_-(x) & \text{if } x \geq x_0, \\ - \sum_{y \in I, x < y \leq x_0} (u_+(y) - u_-(y)) + u(x) - u_+(x), & \text{if } x \leq x_0. \end{cases} \quad (\boxplus)$$

- p. 32 In Exercise 1.44 **Consider the Banach space**

$$X := \{u : [0, 1] \rightarrow \mathbb{R} : u \text{ is continuous, } u(0) = 0, \text{ and } u(1) = 1\},$$

where we take the supremum norm

$$\|u\|_\infty := \max_{x \in [0, 1]} |u(x)|.$$

should be replaced by **Consider the complete metric space**

$$X := \{u : [0, 1] \rightarrow \mathbb{R} : u \text{ is continuous, } u(0) = 0, \text{ and } u(1) = 1\},$$

where we take the metric induced by supremum norm

$$\|u\|_\infty := \max_{x \in [0, 1]} |u(x)|.$$

[Rinat Kashaev]

CHAPTER 3:

- p. 94 In Exercise 3.43 $p \geq 1$ should be replaced by $p > 1$. [Pan Liu]
 p. 109 In Corollary 3.74 one should either assume that u is bounded from below or replace (3.33) with (\boxplus) (above in this file).

CHAPTER 4:

¹The style of this file is inspired by <http://www.hss.caltech.edu/~kcb/IDA-Errata.pdf>

- p. 116 In line 4 **By requiring parametric representations and parameter changes should be replaced by By requiring parametric representations, parameter changes and their inverses .**

CHAPTER 6:

- p. 188 In Proposition 6.1(ii) **If u vanishes at infinity, then**

$$\lim_{s \rightarrow \infty} \varrho_u(s) = 0$$

should be replaced either by **If u vanishes at infinity, then**

$$\lim_{s \rightarrow \infty} \varrho_u(s) = \mathcal{L}^1(\{x \in E : u(x) = \infty\})$$

or by **If u vanishes at infinity and is real-valued, then**

$$\lim_{s \rightarrow \infty} \varrho_u(s) = 0$$

[Allen]

- p. 189 In the proof of In Proposition 6.1(ii) **Then $E_n \supset E_{n+1}$ and**

$$\bigcap_{n=1}^{\infty} E_n = \emptyset.$$

Since $\mathcal{L}^1(E_n) < \infty$, it follows by Proposition B.9(ii) that

$$\lim_{n \rightarrow \infty} \varrho_u(s_n) = \lim_{n \rightarrow \infty} \mathcal{L}^1(E_n) = \mathcal{L}^1(\emptyset) = 0.$$

should be replaced either by **Then $E_n \supset E_{n+1}$ and**

$$\bigcap_{n=1}^{\infty} E_n = \{x \in E : u(x) = \infty\}.$$

Since $\mathcal{L}^1(E_n) < \infty$, it follows by Proposition B.9(ii) that

$$\lim_{n \rightarrow \infty} \varrho_u(s_n) = \lim_{n \rightarrow \infty} \mathcal{L}^1(E_n) = \mathcal{L}^1(\{x \in E : u(x) = \infty\}).$$

or one should assume in the statement that u is real-valued.

- p. 190 One line after formula (6.6): one should either replace **Note that if u vanishes at infinity, then $\varrho_u(s) \rightarrow 0$ as $s \rightarrow \infty$ by the previous proposition, and so for $t > 0$ the set**

$$\{s \in [0, \infty) : \varrho_u(s) \leq t\}$$

is nonempty. Thus, $u^*(t) < \infty$ for all $t > 0$, while $u^*(0)$ could be infinite. with **Note that if u vanishes at infinity, then $\varrho_u(s) \rightarrow T$ as $s \rightarrow \infty$, where**

$T := \mathcal{L}^1(\{x \in E : u(x) = \infty\})$, by the previous proposition, and so for $t > T$ the set

$$\{s \in [0, \infty) : \varrho_u(s) \leq t\}$$

is nonempty. Thus, $u^*(t) < \infty$ for all $t > T$. or replace **Note that if u vanishes at infinity** with **Note that if u vanishes at infinity and is real-valued**

CHAPTER 7:

- p. 217 An alternative proof of Step 2 in the proof of Lemma 7.3:

Step 2: In the general case, let (a_n, b_n) be an increasing sequence of open intervals such that $[a_n, b_n] \subset (a_{n+1}, b_{n+1})$ and

$$\bigcup_{n=1}^{\infty} (a_n, b_n).$$

Then for every n ,

$$\int_{(a_n, b_n)} u \varphi' dx = 0$$

for all $\varphi \in C_c^1((a_n, b_n))$. Hence, by Step 1, there exists $c_n \in \mathbb{R}$ such that $u(x) = c_n$ for \mathcal{L}^1 a.e. $x \in (a_n, b_n)$. But since, $[a_n, b_n] \subset (a_{n+1}, b_{n+1})$, it follows that $c_n = c_{n+1}$ for all n .

CHAPTER 10:

- p. 282 In Exercise 10.11(4), $1 \leq p < \infty$ should be replaced by $1 < p < \infty$ [Alberto Venni]
- p. 287 In formula (10.11) and in the formula on the last line of the page $(y', y_N) \in \Omega \cap A$ should be replaced by $(y', y_N) \in A$.
- p. 290 In formula (10.12) $(y', y_N) \in \Omega \cap A_{x_0}$ should be replaced by $(y', y_N) \in A_{x_0}$.
- p. 305 In Exercise 10.51, p should be ∞ . Also the fact that $W^{1,\infty}(\Omega)$ is a dual space is not obvious. See the file “Lipschitz functions as a dual space”. [M.G. Mora]

CHAPTER 11:

- p. 305 In line 8, by (1.15) (with u replaced by u_n) should be replaced by reasoning somewhat as in (1.15). [Maria Giovanna Mora]

p. 316 In Step 4

$$v_n(x) := \begin{cases} |u(x)| - \frac{1}{n} & \text{if } \frac{1}{n} \leq |u(x)| \leq n, \\ 0 & \text{if } |u(x)| < \frac{1}{n}, \\ n - \frac{1}{n} & \text{if } |u(x)| > \frac{1}{n}. \end{cases}$$

should be replaced by

$$v_n(x) := \begin{cases} |u(x)| - \frac{1}{n} & \text{if } \frac{1}{n} \leq |u(x)| \leq n, \\ 0 & \text{if } |u(x)| < \frac{1}{n}, \\ n - \frac{1}{n} & \text{if } |u(x)| > n. \end{cases}$$

[Robert Simione]

p.322 On line 6 of the proof of Theorem 11.10 we may find a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ should be replaced by we may find a subsequence of $\{u_n\}$, not relabeled, and then in the remaining of the proof u_{n_k} should be replaced by u_n .

p. 326 In the formula on line 4 of the proof of Theorem 11.21 $(y', y_N) \in \Omega \cap A_{x_0}$ should be replaced by $(y', y_N) \in A_{x_0}$.

p. 326 In the formula on line 19 $(y', y_N) \in \Omega \cap A$ should be replaced by $(y', y_N) \in A$.

p. 330 In Exercise 11.25 $\left\| \frac{\partial^N u}{\partial x_1 \cdots \partial x_N} \right\|_{L^N(\mathbb{R}^N)}$ should be replaced by $\left\| \frac{\partial^N u}{\partial x_1 \cdots \partial x_N} \right\|_{L^1(\mathbb{R}^N)}$.
 . [Pan Liu]

CHAPTER 12:

p. 353 In Exercises 12.6 and 12.7, d_{reg} be its regularized distance should be replaced by d_{reg} be the regularized distance corresponding to $\mathbb{R}^N \setminus \bar{\Omega}$.

p. 353 In Exercises 12.7, $\lim_{t \rightarrow \infty} \frac{\psi(t)}{t^m} = 0$ should be replaced by $\lim_{t \rightarrow \infty} t^m \psi(t) = 0$ [Adrian Hagerty]

p. 354 In Line 2, $td_{\text{reg}}(x)$ should be replaced by $Ctd_{\text{reg}}(x)$, where C is the constant given in Exercises 12.6. See also the file “Extension domains for higher order Sobolev spaces” for a complete proof.

p. 354 In line 5 of Definition 12.9 $(y', y_N) \in \Omega \cap A$ should be replaced by $(y', y_N) \in A$.

p. 356 In formula (12.7), $\frac{1}{\varepsilon}$ should be replaced by $\frac{M}{\varepsilon}$.

p. 359 In lines 2-6 and in Remark 12.16, $\frac{1}{\varepsilon}$ should be replaced by $\frac{M}{\varepsilon}$.

p.362 Exercise 12.25 is wrong. Playing with the parameters in Exercise 11.7 should provide a counter-example. See however Exercise 11.17.

p. 370 In Remark 12.35, $\frac{1}{4}$ should be replaced by $\frac{1}{2}$. [Marco Barchiesi]

CHAPTER 13:

p. 402 Thanks to Paolo Piovano for pointing out that the proof of Theorem 13.30 is too fast. It should be divided in two steps. Assuming first that $u \in L^{1^*}(\mathbb{R}^N)$, by mollification it follows that

$$\left(\int_{\mathbb{R}^N} |u(x)|^{1^*} dx \right)^{\frac{1}{1^*}} \leq C |Du|(\mathbb{R}^N).$$

To remove the additional hypothesis that $u \in L^{1^*}(\mathbb{R}^N)$, one should truncate u as in Step 4 of the proof of Theorem 11.2. The problem here is the fact the mollification of a function vanishing at infinity needs not vanish at infinity. See also the file “An extension of the Sobolev–Gagliardo–Nirenberg theorem”.

CHAPTER 14:

p. 422 In Exercise 14.13 **Prove that if g is decreasing, then there exists**

$$\lim_{s \rightarrow 0^+} \|g\|_{s,\infty} = \lim_{h \rightarrow \infty} g(h).$$

What happens if we remove the hypothesis that g is decreasing? should be replaced by **Prove that if g is increasing, then there exists**

$$\lim_{s \rightarrow 0^+} \|g\|_{s,\infty} = \lim_{h \rightarrow \infty} g(h).$$

What happens if we remove the hypothesis that g is increasing? [Xiang Xu]

p. 422 In Exercise 14.13 **Prove that if $\frac{g(h)}{h}$ is increasing, then there exists**

$$\lim_{s \rightarrow 1^-} \|g\|_{s,\infty} = \lim_{h \rightarrow 0^+} \frac{g(h)}{h}.$$

What happens if we remove the hypothesis that $\frac{g(h)}{h}$ is increasing? should be replaced by **Prove that if $\frac{g(h)}{h}$ is decreasing, then there exists**

$$\lim_{s \rightarrow 1^-} \|g\|_{s,\infty} = \lim_{h \rightarrow 0^+} \frac{g(h)}{h}.$$

What happens if we remove the hypothesis that $\frac{g(h)}{h}$ is decreasing? [Xiang Xu]

CHAPTER 15:

- p. 461-462 In formula (15.16) and in several formulas afterwards $\int_{\partial\Omega\cap\partial A_{x_n}}$ should be $\int_{\partial\Omega\cap\partial A_n}$. [Joshua Ciappara]
- p. 461 Two lines above formula (15.16) $\Omega_n \subset A_n$ should be Ω_n . [Joshua Ciappara]

More to come for sure... :(