1 A Simpler Proof of Theorem 10.29

In this section we present a simpler proof of Theorem 10.29.

Theorem 1 *Let $\Omega \subset \mathbb{R}^N$ be an open set whose boundary is of class C. Then the restriction to Ω of functions in $C_c^\infty\left(\mathbb{R}^N\right)$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$.*

Proof. Fix $u \in W^{1,p}(\Omega)$. By the Meyers–Serrin theorem, without loss of generality, we may assume that $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$.
Step 1: We first approximate u with a function with compact support. Consider a cut-off function $\varphi \in C_c^\infty\left(\mathbb{R}^N\right)$ such that $\operatorname{supp}\varphi \subset \overline{B(0,2)}$, $\varphi = 1$ in $B(0,1)$ and $0 \leq \varphi \leq 1$. For $n \in \mathbb{N}$, define

$$u_n(x) := \varphi_n(x)\, u(x), \quad x \in \Omega.$$

By the Lebesgue dominated convergence theorem, we have that $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$, while

$$\nabla u_n(x) = \varphi_n(x)\, \nabla u(x) + \frac{1}{n} \nabla\varphi\left(\frac{x}{n}\right) u(x), \quad x \in \Omega.$$

Again by the Lebesgue dominated convergence theorem, $\varphi_n \nabla u \to \nabla u$ in $L^p\left(\Omega; \mathbb{R}^N\right)$ as $n \to \infty$, while

$$\int_\Omega \left|\frac{1}{n}\nabla\varphi\left(\frac{x}{n}\right) u(x)\right|^p dx \leq \frac{c}{n^p}\int_\Omega |u(x)|^p\, dx \to 0$$

as $n \to \infty$. This concludes the proof.
Step 2: By Step 1, we can assume that $u = 0$ outside a compact set. For every $x_0 \in \partial\Omega$ there exist a neighborhood A_{x_0} of x_0, local coordinates $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, with $y = 0$ at $x = x_0$, and a function $f \in C\left(\overline{Q_{N-1}(0,r)}\right)$, $r > 0$, such that

$$\partial\Omega \cap A_{x_0} = \{(y', f(y')) : \, y' \in Q_{N-1}(0,r)\},$$

and (see (10.11))

$$\Omega \cap A_{x_0} = \{(y', y_N) \in \Omega \cap A_{x_0} : \, y' \in Q_{N-1}(0,r),\, y_N > f(y')\}. \tag{1}$$

If the set $\Omega \setminus \bigcup_{x \in \partial\Omega} A_x$ is nonempty, for every $x_0 \in \Omega \setminus \bigcup_{x \in \partial\Omega} A_x$ let A_{x_0} be any open ball centered at x_0 and contained in Ω. The family $\{A_x\}_{x \in \overline{\Omega}}$ is an open cover of $\overline{\Omega}$. Since $u = 0$ outside a compact set K, we have that $\overline{\Omega} \cap K$ is compact. Hence, there is a finite cover $A_{x_1}, \ldots, A_{x_\ell}$ that covers $\overline{\Omega} \cap K$. Let $\{\psi_n\}_{n=1}^{\ell}$ be a smooth partition of unity subordinated to $A_{x_1}, \ldots, A_{x_\ell}$.

Fix $n \in \{1, \ldots, \ell\}$ and define $u_n := u\psi_n \in W^{1,p}(\Omega)$ (see Exercise 10.18), where we extend u_n to be zero outside $\operatorname{supp}\psi_n$. There are two cases.

If $\operatorname{supp}\psi_n$ is contained in Ω, then we set $v_n := \phi_n u \in C_c^\infty\left(\mathbb{R}^N\right)$. If $\operatorname{supp}\psi_n$ is not contained in Ω, let $x_n \in \partial\Omega$ be such that $\operatorname{supp}\psi_n \subset A_{x_n}$. For $t > 0$, using local coordinates in A_{x_n}, define the function $u_{n,t} : A_{x_n}^t \to \mathbb{R}$ by

$$u_{n,t}(y', y_N) := u_n(y', y_N + t),$$

where

$$A_{x_n}^t := \left\{ (y', y_N) \in \mathbb{R}^N : y' \in Q_{N-1}(0, r), \, y_N > f(y') - t \right\}.$$

Note that $u_{n,t}$ is well-defined for $t > 0$ sufficiently small and that $A_{x_n}^t \supset \overline{\Omega} \cap A_{x_n}$.

Fix $\eta > 0$. By the previous lemma there exists $t_n > 0$ so small that $\operatorname{supp} \psi_n + B(0, t_n) \subset\subset A_{x_n}$ and

$$\|u_{n,t_n} - u_n\|_{W^{1,p}(\Omega)} \leq \frac{\eta}{2^n}. \tag{2}$$

Construct a function $\phi_n \in C^\infty\left(\mathbb{R}^N\right)$ such that $\phi_n(y', y_N) = 1$ if $y' \in Q_{N-1}(0, r)$ and $y_N > f(y') - \frac{t_n}{2}$ and $\phi_n(y', y_N) = 0$ if $y' \in Q_{N-1}(0, r)$ and $y_N \leq f(y') - \frac{3t_n}{4}$ and define

$$v_n := \phi_n u_{n,t_n}.$$

Then $v_n \in C_c^\infty\left(\mathbb{R}^N\right)$ (provided we define u_{n,t_n} to be zero, whenever ϕ_n is zero) and $\operatorname{supp} v_n \subset A_{x_n}^{t_n}$. Note that $v_n = u_{n,t_n}$ in Ω, and so (2) may be rewritten as

$$\|v_n - u\psi_n\|_{W^{1,p}(\Omega)} \leq \frac{\eta}{2^n}. \tag{3}$$

Define the function

$$v := \sum_{n=1}^{\ell} v_n.$$

Then $v \in C_c^\infty\left(\mathbb{R}^N\right)$ and so, $v \in W^{1,p}(\Omega)$. Moreover,

$$\|u - v\|_{W^{1,p}(\Omega)} \leq \sum_{i=1}^{\ell} \|\psi_n u - v_n\|_{W^{1,p}(\Omega)} \leq \sum_{i=1}^{\ell} \frac{\eta}{2^i} \leq \eta.$$

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