The following results are due to Hajłasz and Kałamajska [2]. In particular, Theorem 3 extends the Sobolev–Gagliardo–Nirenberg theorem to functions in $L^{1,p}\left(\mathbb{R}^N\right)$, $1 \le p < N$, that do not vanish at infinity. I thank Dejan Slepcev for bringing the paper to my attention.

Theorem 1 *Let $u \in L^{1,p}\left(\mathbb{R}^N\right)$, where $1 \le p < \infty$ and $N \in \mathbb{N}$. Then there exists a sequence of functions $\{u_n\} \subset C_c^\infty\left(\mathbb{R}^N\right)$ such that $\nabla u_n \to \nabla u$ in $L^p\left(\mathbb{R}^N;\mathbb{R}^N\right)$ if and only if either $N \ge 2$ or $p > 1$.*

Proof. Step 1: Assume first that $N = 1$ and $p = 1$. Let $\phi \in C_c^\infty(\mathbb{R})$ be such that $\int_{\mathbb{R}} \phi(t)\,dt \neq 0$ and define $u(x) := \int_0^x \phi(t)\,dt$. Then u belongs to $L^{1,1}(\mathbb{R})$. Assume, by contradiction, that there exists a sequence $\{u_n\} \subset C_c^\infty(\mathbb{R})$ such that $u_n' \to u'$ in $L^1(\mathbb{R})$. Then

$$0 = \int_{\mathbb{R}} u_n'(t)\,dt \to \int_{\mathbb{R}} \phi(t)\,dt \neq 0.$$

This is a contradiction.

Step 2: We consider the case $N = 1$ and $1 < p < \infty$. Fix $u \in L^{1,p}(\mathbb{R})$. Using standard mollifiers, we have that $C^\infty(\mathbb{R})$ is dense in $L^{1,p}(\mathbb{R})$. Thus, without loss of generality, we may assume that $u \in C^\infty(\mathbb{R}) \cap L^{1,p}(\mathbb{R})$. Consider a cut-off function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset \overline{B(0,2)}$, $\varphi = 1$ in $B(0,1)$ and $0 \le \varphi \le 1$. For $n \in \mathbb{N}$, define

$$u_n(x) := \varphi_n(x) \int_0^x u'(t)\,dt, \quad x \in \mathbb{R}.$$

where

$$\varphi_n(x) := \varphi\left(\frac{x}{n}\right), \quad x \in \mathbb{R}.$$

Then $u_n \in C_c^\infty(\mathbb{R})$. Thus, it remains to show that $u_n' \to u'$ in $L^p(\mathbb{R})$. We have

$$u_n'(x) = \varphi_n(x)\,u'(x) + \frac{1}{n}\varphi'\left(\frac{x}{n}\right)\int_0^x u'(t)\,dt, \quad x \in \mathbb{R}.$$

The fact that $\varphi_n u' \to u'$ in $L^p(\mathbb{R})$ follows from the Lebesgue dominated convergence theorem. On the other hand, by Hardy's inequality (see below)

$$\begin{aligned}\int_{\mathbb{R}} \left|\frac{1}{n}\varphi'\left(\frac{x}{n}\right)\int_0^x u'(t)\,dt\right|^p dx &\le \frac{c}{n^p}\int_{B(0,2n)\setminus B(0,n)} \left|\int_0^x u'(t)\,dt\right|^p dx \\ &\le c\int_{B(0,2n)\setminus B(0,n)} \left|\frac{1}{x}\int_0^x u'(t)\,dt\right|^p dx \to 0\end{aligned}$$

as $n \to \infty$.

Step 3: Finally, we study the case $N \ge 2$ and $1 \le p < \infty$. Fix $u \in L^{1,p}\left(\mathbb{R}^N\right)$. As before we can assume that $u \in C^\infty\left(\mathbb{R}^N\right) \cap L^{1,p}\left(\mathbb{R}^N\right)$. Consider a cut-off function $\varphi \in C_c^\infty\left(\mathbb{R}^N\right)$ such that $\operatorname{supp}\varphi \subset \overline{B(0,2)}$, $\varphi = 1$ in $B(0,1)$ and $0 \le \varphi \le 1$. For $r > 0$, define the annulus $A_r := B(0,2r)\setminus B(0,r)$. By Poincaré's inequality applied to the annulus A_1, there exists a constant $C = C(p, N)$ such that for every $v \in W^{1,p}(A_1)$,

$$\int_{A_1} |v(x) - v_{A_1}|^p \, dx \le C \int_{A_1} |\nabla v|^p \, dx,$$

where for every measurable set with positive finite measure $E \subset \mathbb{R}^N$,

$$v_E := \frac{1}{\mathcal{L}^N(E)} \int_E v(x) \, dx.$$

A rescaling argument, shows that for every $v \in W^{1,p}(A_r)$, $r > 0$,

$$\int_{A_r} |v(x) - v_{A_r}|^p \, dx \le C r^p \int_{A_r} |\nabla v|^p \, dx. \tag{1}$$

For $n \in \mathbb{N}$, define

$$u_n(x) := \varphi_n(x) \left(u(x) - u_{A_n}\right), \quad x \in \mathbb{R}^N.$$

Then

$$\nabla u_n(x) = \varphi_n(x) \nabla u(x) + \frac{1}{n} \nabla \varphi\left(\frac{x}{n}\right) \left(u(x) - u_{A_n}\right), \quad x \in \mathbb{R}^N.$$

As in the previous step, $\varphi_n \nabla u \to \nabla u$ in $L^p\left(\mathbb{R}^N\right)$ as $n \to \infty$, while by (1),

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{1}{n} \nabla \varphi\left(\frac{x}{n}\right) \left(u(x) - u_{A_n}\right) \right|^p dx &\le \frac{c}{n^p} \int_{A_n} |u(x) - u_{A_n}|^p \, dx \\ &\le C \int_{A_n} |\nabla v|^p \, dx \to 0 \end{aligned}$$

as $n \to \infty$. This concludes the proof. ■

Remark 2 *If $N \ge 2$ and $u \in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$ is such that its distributional gradient Du belongs to $\mathcal{M}_{\text{b}}\left(\Omega; \mathbb{R}^N\right)$, then reasoning as in Step 3 of the previous proof, one can find a sequence $\{u_n\} \subset C_c^\infty\left(\mathbb{R}^N\right)$ such that*

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n| \, dx = |Du|\left(\mathbb{R}^N\right).$$

As a corollary of the previous theorem, we can extend the Sobolev–Gagliardo–Nirenberg theorem to functions in $L^{1,p}\left(\mathbb{R}^N\right)$ that do not vanish at infinity. More precisely, we have the following result.

Theorem 3 (Sobolev–Gagliardo–Nirenberg's embedding theorem) *Let $1 \le p < N$. Then for every function $u \in L^{1,p}\left(\mathbb{R}^N\right)$ there exists a unique constant $c \in \mathbb{R}$ (depending on u) such that*

$$\left(\int_{\mathbb{R}^N} |u(x) - c|^{p^*} \, dx\right)^{\frac{1}{p^*}} \le C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx\right)^{\frac{1}{p}}, \tag{2}$$

where $C = C(N, p) > 0$.

Proof. By the previous theorem, there exists a sequence $\{u_n\} \subset C_c^\infty(\mathbb{R}^N)$ such that $\nabla u_n \to \nabla u$ in $L^p(\mathbb{R}^N;\mathbb{R}^N)$. Since for every $n, m \in \mathbb{N}$, $u_n - u_m \in W^{1,p}(\mathbb{R}^N)$, by the Sobolev–Gagliardo–Nirenberg theorem,

$$\left(\int_{\mathbb{R}^N} |u_n(x) - u_m(x)|^{p^*}\, dx\right)^{\frac{1}{p^*}} \le C \left(\int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u_n(x)|^p\, dx\right)^{\frac{1}{p}}.$$

This shows that $\{u_n\}$ is a Cauchy sequence in $L^{p^*}(\mathbb{R}^N)$, and thus there exists $v \in L^{p^*}(\mathbb{R}^N)$ such that $u_n \to v$ in $L^{p^*}(\mathbb{R}^N)$. Since $\nabla u_n \to \nabla u$ in $L^p(\mathbb{R}^N;\mathbb{R}^N)$, it follows that $v \in L^{1,p}(\mathbb{R}^N) \cap L^{p^*}(\mathbb{R}^N)$, with $\nabla v = \nabla u$ $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$. Hence, there exists a constant $c \in \mathbb{R}$ such that $u(x) = v(x) + c$ for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$. Again by the Sobolev–Gagliardo–Nirenberg theorem, this time applied to $u_n \in W^{1,p}(\mathbb{R}^N)$, $n \in \mathbb{N}$, we have

$$\left(\int_{\mathbb{R}^N} |u_n(x)|^{p^*}\, dx\right)^{\frac{1}{p^*}} \le C \left(\int_{\mathbb{R}^N} |\nabla u_n(x)|^p\, dx\right)^{\frac{1}{p}}.$$

Letting $n \to \infty$ gives

$$\left(\int_{\mathbb{R}^N} |u(x) - c|^{p^*}\, dx\right)^{\frac{1}{p^*}} \le C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx\right)^{\frac{1}{p}}.$$

To prove the uniqueness of the constant c, let $c_1 \in \mathbb{R}$ be another constant for which such that (2) holds. Then

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |c_1 - c|^{p^*}\, dx\right)^{\frac{1}{p^*}} &= \left(\int_{\mathbb{R}^N} |c_1 \pm u(x) - c|^{p^*}\, dx\right)^{\frac{1}{p^*}} \\ &\le \left(\int_{\mathbb{R}^N} |u(x) - c_1|^{p^*}\, dx\right)^{\frac{1}{p^*}} + \left(\int_{\mathbb{R}^N} |u(x) - c|^{p^*}\, dx\right)^{\frac{1}{p^*}} \\ &\le 2C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx\right)^{\frac{1}{p}} < \infty,\end{aligned}$$

which gives that $c_1 = c$. This completes the proof. ■

Remark 4 *If $N \ge 2$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is such that its distributional gradient Du belongs to $\mathcal{M}_{\mathrm{b}}(\Omega;\mathbb{R}^N)$, then reasoning as in the previous proof, one can show that there exists a unique constant $c \in \mathbb{R}$ (depending on u) such that*

$$\left(\int_{\mathbb{R}^N} |u(x) - c|^{1^*}\, dx\right)^{\frac{1}{1^*}} \le C\, |Du|\,(\mathbb{R}^N),$$

where $C = C(N) > 0$. See Theorem 3.47 in [1] for a different proof of this result.

A similar result fails if $p \ge N$.

Exercise 5 *Assume that* $p > N$ *and consider a function* $u \in C^\infty\left(\mathbb{R}^N\right)$ *such that* $u(x) = |x|^\varepsilon$ *for all* $x \in \mathbb{R}^N \setminus B(0,1)$, *where* $0 < \varepsilon < 1$. *Prove that if* $0 < \varepsilon < 1$ *is chosen appropriately, then* $u \in L^{1,p}\left(\mathbb{R}^N\right)$, *but there is no constant* $c \in \mathbb{R}$ *such that* $u - c$ *belongs to* $L^q\left(\mathbb{R}^N\right)$ *for some* $1 \le q \le \infty$.

Exercise 6 *Assume that* $p = N > 1$ *and consider a function* $u \in C^\infty\left(\mathbb{R}^N\right)$ *such that* $u(x) = \log\log|x|$ *for all* $x \in \mathbb{R}^N \setminus B(0,e)$. *Prove that if* $0 < \varepsilon < 1$ *is chosen appropriately, then* $u \in L^{1,N}\left(\mathbb{R}^N\right)$, *but there is no constant* $c \in \mathbb{R}$ *such that* $u - c$ *belongs to* $L^q\left(\mathbb{R}^N\right)$ *for some* $1 \le q \le \infty$.

Exercise 7 *Prove that if* $u \in L^{1,1}(\mathbb{R})$ *then there is a constant* $c \in \mathbb{R}$ *such that*

$$\|u - c\|_{L^\infty(\mathbb{R})} \le 2 \int_{\mathbb{R}} |u'(x)|\, dx.$$

1 Appendix

Theorem 8 (Hardy's inequality) *Let* $u : (a,b) \to \mathbb{R}$ *be a measurable function, where* $0 \le a < b \le \infty$, *and let* $1 \le p < \infty$ *and* $s \in \mathbb{R}$. *If* $s > \frac{1}{p}$, *then*

$$\left(\int_a^b \left|\frac{1}{x^s}\int_a^x u(t)\, dt\right|^p dx\right)^{\frac{1}{p}} \le \frac{1}{s - \frac{1}{p}} \left(\int_a^b \left|\frac{1}{x^{s-1}} u(x)\right|^p dx\right)^{\frac{1}{p}}, \tag{3}$$

while if $s < \frac{1}{p}$, *then*

$$\left(\int_a^b \left|\frac{1}{x^s}\int_x^b u(t)\, dt\right|^p dx\right)^{\frac{1}{p}} \le \frac{1}{\frac{1}{p} - s} \left(\int_a^b \left|\frac{1}{x^{s-1}} u(x)\right|^p dx\right)^{\frac{1}{p}}. \tag{4}$$

Proof. Extend u by zero in $(0,\infty) \setminus (a,b)$. Assume that $s > \frac{1}{p}$ and define the function $v : (0,\infty) \times (0,\infty) \to \mathbb{R}$ by

$$v(x,y) := \frac{1}{x^{s-1}} u(xy), \quad x, y > 0.$$

Note that by the change of variables $t = xy$,

$$\frac{1}{x^s}\int_0^x u(t)\, dt = \int_0^1 v(x,y)\, dy$$

Then by Corollary B.83 and the change of variables $t = xy$,

$$\left(\int_0^\infty \left|\frac{1}{x^s}\int_0^x u(t)\, dt\right|^p dx\right)^{\frac{1}{p}} = \left(\int_0^\infty \left|\int_0^1 v(x,y)\, dy\right|^p dx\right)^{\frac{1}{p}}$$

$$\le \int_0^1 \left(\int_0^\infty |v(x,y)|^p\, dx\right)^{\frac{1}{p}} dy = \int_0^1 y^{s-1-\frac{1}{p}} \left(\int_0^\infty \left|\frac{1}{t^{s-1}} u(t)\right|^p dt\right)^{\frac{1}{p}} dy$$

$$= \frac{1}{s - \frac{1}{p}} \left(\int_0^\infty \left|\frac{1}{x^{s-1}} u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

Recalling that $u=0$ outside (a,b), we obtain

$$\begin{aligned}\left(\int_a^b \left|\frac{1}{x^s}\int_a^x u(t)\ dt\right|^p dx\right)^{\frac{1}{p}} &\le \left(\int_0^\infty \left|\frac{1}{x^s}\int_0^x u(t)\ dt\right|^p dx\right)^{\frac{1}{p}}\\ &\le \frac{1}{s-\frac{1}{p}}\left(\int_0^\infty \left|\frac{1}{x^{s-1}}u(x)\right|^p dx\right)^{\frac{1}{p}}\\ &= \frac{1}{s-\frac{1}{p}}\left(\int_a^b \left|\frac{1}{x^{s-1}}u(x)\right|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

The case $s<\frac{1}{p}$ is similar. ■

References

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[2] P. Hajłasz and A. Kałamajska, *Polynomial asympototics and approximation of Sobolev functions,* Studia. Math. **113** (1995), no. 1, 55–64.