1 Extension Domains

In this section we prove that if Ω is sufficiently regular, then a function $u \in W^{m,p}(\Omega)$ can be extended to $W^{m,p}(\mathbb{R}^N)$. The following results are due to Stein (see [2]).

Theorem 1 (Stein) *Let $f : \mathbb{R}^{N-1} \to \mathbb{R}$ be a Lipschitz function and let*

$$\Omega := \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \, x_N > f(x') \right\}. \tag{1}$$

Then for all $1 \leq p \leq \infty$ and $m \in \mathbb{N}$ there exists a continuous linear operator

$$\mathcal{E} : W^{m,p}(\Omega) \to W^{m,p}\left(\mathbb{R}^N\right)$$

such that for all $u \in W^{m,p}(\Omega)$,

$$\mathcal{E}(u)(x) = u(x) \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega \tag{2}$$

and

$$\|\mathcal{E}(u)\|_{L^p(\mathbb{R}^N)} \leq C(N,p) \|u\|_{L^p(\Omega)}, \tag{3}$$

$$\left\| \frac{\partial^\alpha \mathcal{E}(u)}{\partial x^\alpha} \right\|_{L^p(\mathbb{R}^N)} \leq C(N,p) \left(1 + (\operatorname{Lip} f)^{2|\alpha|-1}\right) \sum_{1 \leq |\beta| \leq |\alpha|} \left\| \frac{\partial^\beta u}{\partial x^\beta} \right\|_{L^p(\Omega)} \tag{4}$$

for every multi-index $\alpha \in \mathbb{N}_0^N$ with $1 \leq |\alpha| \leq m$.

Lemma 2 *Given $L \geq 0$, consider the cone*

$$C := \left\{ y \in \mathbb{R}^N : \, L|y'| < |y_N|, \, y_N < 0 \right\}.$$

Then for every $x_N < 0$,

$$\operatorname{dist}((0, x_N), \partial C) = \frac{-x_N}{\sqrt{L^2+1}}.$$

Proof. We give the proof only for $N = 2$. Consider the triangle of vertices $A = (0,0)$, $B = (0, x_2)$ and $C = \left(-\frac{x_2}{L}, x_2\right)$. The area of this triangle is given by

$$\text{area} = \frac{1}{2}(-x_2)\left(-\frac{x_2}{L}\right) = \frac{1}{2L}x_2^2.$$

On the other hand, by considering as base the segment of endpoints A and C, we have that the corresponding height h is $\operatorname{dist}((0, x_2), \partial C)$. Hence,

$$\text{area} = \frac{1}{2}h\sqrt{\left(-\frac{x_2}{L}\right)^2 + x_2^2} = \frac{1}{2}h(-x_2)\sqrt{\frac{1}{L^2}+1}.$$

By equating the two expressions for the area, we get

$$\operatorname{dist}((0, x_2), \partial C) = h = \frac{\frac{1}{2L}x_2^2}{\frac{1}{2}(-x_2)\sqrt{\frac{1}{L^2}+1}} = \frac{(-x_2)}{\sqrt{1+L^2}},$$

which concludes the proof. ■

We will also need a formula to calculate partial derivatives of a composition.

Theorem 3 *Let $\Omega \subset \mathbb{R}^N$ be open, let $w : \mathbb{R}^d \to \mathbb{R}$ be of class $C^m\left(\mathbb{R}^d\right)$ and let $g : \Omega \to \mathbb{R}^d$ be of class $C^m(\Omega)$. Then $v := w \circ g$ belongs to $C^m(\Omega)$ and for every multi-index $\alpha \in \mathbb{N}_0^N$, with $0 < |\alpha| \le m$,*

$$\frac{\partial^\alpha v}{\partial x^\alpha}(x) = \alpha! \sum_{|\beta|=1}^{|\alpha|} \frac{\partial^\beta w}{\partial z^\beta}(g(x)) \sum_{s=1}^{|\alpha|} \sum_{I_{s,\alpha,\beta}} \prod_{j=1}^{s} \frac{1}{\gamma_j!\,(\delta_j!)^{|\gamma_j|}} \left(\frac{\partial^{\delta_j} g}{\partial x^{\delta_j}}(x)\right)^{\gamma_j},$$

where $\beta \in \mathbb{N}_0^d$, $\gamma_j \in \mathbb{N}_0^d$, $\delta_j \in \mathbb{N}_0^N$, and

$$I_{s,\alpha,\beta} = \Bigg\{ (\gamma_1, \dots, \gamma_s, \delta_1, \dots, \delta_s) : |\gamma_j| > 0, \\ 0 \prec \delta_1 \prec \cdots \prec \delta_s, \sum_{j=1}^{s} \gamma_j = \beta, \sum_{j=1}^{s} |\gamma_j|\,\delta_j = \alpha \Bigg\}.$$

Here $\delta \prec \zeta$ means that either $|\delta| < |\zeta|$ or $|\delta| = |\zeta|$ and there is $i = 1, \dots, N$ such that $\delta_i < \zeta_i$ and, if $i \ge 2$, $\delta_j = \zeta_j$ for all $j = 1, \dots, i-1$. Also we define $0^0 := 1$.

We refer to [1] for a proof of the previous theorem.

We turn to the proof of Theorem 1.

Proof of Theorem 1. We divide the proof in several steps. Let $L := \operatorname{Lip} f$.

Step 1: Let $\psi : [1, \infty) \to \mathbb{R}$ be a continuous function such that

$$\int_1^\infty \psi(t)\, dt = 1 \tag{5}$$

and

$$\lim_{t \to \infty} \frac{\psi(t)}{t^n} = 0, \quad \int_1^\infty t^n \psi(t)\, dt = 0$$

for every integer $n \in \mathbb{N}$. Given a function $u \in C_c^\infty\left(\mathbb{R}^N\right)$, define the function $w : \mathbb{R}^{N+1} \to \mathbb{R}$ as follows

$$w(x, s) := \int_1^\infty u(x', x_N + st)\, \psi(t)\, dt.$$

Fix $n \ge 2$, $x \in \mathbb{R}^N$, and $s > 0$. By Theorem B. 53, we have that $w \in C^\infty\left(\mathbb{R}^{N+1}\right)$ and for every multi-index $\alpha \in \mathbb{N}_0^N$,

$$\frac{\partial^\alpha w}{\partial x^\alpha}(x, s) := \frac{\partial^{(\alpha,0)} w}{\partial x^\alpha \partial s^0}(x, s) = \int_1^\infty \frac{\partial^\alpha u}{\partial x^\alpha}(x', x_N + st)\, \psi(t)\, dt \tag{6}$$

for every $(x, s) \in \mathbb{R}^{N+1}$. Since $\frac{\partial^\alpha u}{\partial x^\alpha} \in C_c^\infty\left(\mathbb{R}^N\right)$, by (5) and the fundamental theorem of calculus,

$$\begin{aligned} \frac{\partial^\alpha w}{\partial x^\alpha}(x, s) - \frac{\partial^\alpha u}{\partial x^\alpha}(x) &= \int_1^\infty \left[\frac{\partial^\alpha u}{\partial x^\alpha}(x', x_N + st) - \frac{\partial^\alpha u}{\partial x^\alpha}(x)\right] \psi(t)\, dt \\ &= \int_1^\infty \int_0^{st} \frac{\partial^{\alpha + e_N} u}{\partial x^{\alpha + e_N}}(x', x_N + \tau)\, \psi(t)\, d\tau dt. \end{aligned}$$

Since $u \in C_c^\infty\left(\mathbb{R}^N\right)$, it follows that

$$\lim_{s\to 0} \frac{\partial^\alpha w}{\partial x^\alpha}(x,s) = \frac{\partial^\alpha u}{\partial x^\alpha}(x) \tag{7}$$

uniformly for $x \in \mathbb{R}^N$.

Similarly, for $n \in \mathbb{N}$,

$$\frac{\partial^n w}{\partial s^n}(x,s) = \int_1^\infty \frac{\partial^n u}{\partial x_N^n}(x', x_N + st)\, t^n \psi(t)\, dt. \tag{8}$$

Fix x and $s > 0$ and consider the function $g(t) := \frac{\partial^n u}{\partial x_N^n}(x', x_N + st)$. Note that $\frac{d^i g}{dt^i}(t) = s^i \frac{\partial^{n+i} u}{\partial x_N^{n+i}}(x', x_N + st)$. Hence, by applying to the function g Taylor's formula of order $\ell \in \mathbb{N}_0$ with integral remainder at $t = 1$, we get

$$\begin{aligned}\frac{\partial^n u}{\partial x_N^n}(x', x_N + st) &= \sum_{i=0}^{\ell} \frac{1}{i!} \frac{\partial^{n+i} u}{\partial x_N^{n+i}}(x', x_N + s)\, s^i (t-1)^i \\ &\quad + \frac{1}{\ell!} \int_1^t (t-\tau)^\ell \frac{\partial^{n+\ell+1} u}{\partial x_N^{n+\ell+1}}(x', x_N + s\tau)\, s^{\ell+1}\, d\tau\end{aligned}$$

By the binomial theorem

$$\int_1^\infty (t-1)^i t^n \psi(t)\, dt = \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} \int_1^\infty t^{n+k} \psi(t)\, dt = 0,$$

and so

$$\begin{aligned}\frac{\partial^n w}{\partial s^n}(x,s) &= \frac{s^{\ell+1}}{\ell!} \int_1^\infty t^n \psi(t) \int_1^t (t-\tau)^\ell \frac{\partial^{n+\ell+1} u}{\partial x_N^{n+\ell+1}}(x', x_N + s\tau)\, d\tau dt \\ &= \frac{1}{\ell!} \int_1^\infty t^n \psi(t) \int_s^{st} (st - r)^\ell \frac{\partial^{n+\ell+1} u}{\partial x_N^{n+\ell+1}}(x', x_N + r)\, dr dt.\end{aligned}$$

Using the fact that $|\psi(t)| \le C t^{-a-n-\ell}$ for some $a > 2$ and Tonelli's theorem, we have

$$\begin{aligned}\left|\frac{\partial^n w}{\partial s^n}(x,s)\right| &\le C s^\ell \int_1^\infty \frac{1}{t^a} \int_s^{st} \left|\frac{\partial^{n+\ell+1} u}{\partial x_N^{n+\ell+1}}(x', x_N + r)\right| dr dt \\ &= C s^{\ell+a-1} \int_s^\infty \frac{1}{r^{a-1}} \left|\frac{\partial^{n+\ell+1} u}{\partial x_N^{n+\ell+1}}(x', x_N + r)\right| dr.\end{aligned} \tag{9}$$

Since the derivatives of u are bounded, it follows that

$$\left|\frac{\partial^n w}{\partial s^n}(x,s)\right| \le C M s^{\ell+1}$$

and so

$$\lim_{s\to 0^+}\frac{1}{s^\ell}\frac{\partial^n w}{\partial s^n}(x,s)=0$$

uniformly for $x\in\mathbb{R}^N$.

Similarly, for every multi-index $\delta\in\mathbb{N}_0^{N+1}$, with $\delta_{N+1}\geq 1$, setting $z:=(x,s)\in\mathbb{R}^N\times\mathbb{R}$, we have that

$$\frac{\partial^\delta w}{\partial z^\delta}(x,s)=\int_1^\infty \frac{\partial^{\left(\delta',\delta_N+\delta_{N+1}\right)}u}{\partial (x')^{\delta'}\,\partial x_N^{\delta_N+\delta_{N+1}}}(x',x_N+st)\,t^{\delta_{N+1}}\psi(t)\,dt. \tag{10}$$

Reasoning as before, we conclude that for every $\ell\in\mathbb{N}_0$,

$$\left|\frac{\partial^\delta w}{\partial z^\delta}(x,s)\right|\leq Cs^{\ell+a-1}\int_s^\infty\frac{1}{r^{a-1}}\left|\frac{\partial^{\left(\delta',\delta_N+\delta_{N+1}+\ell+1\right)}u}{\partial (x')^{\delta'}\,\partial x_N^{\delta_N+\delta_{N+1}+\ell+1}}(x',x_N+r)\right|\,dr. \tag{11}$$

As before, this implies that

$$\lim_{s\to 0^+}\frac{1}{s^\ell}\frac{\partial^\delta w}{\partial z^\delta}(x,s)=0 \tag{12}$$

uniformly for $x\in\mathbb{R}^N$.

Step 2: Given a point $x\in\mathbb{R}^N\setminus\overline{\Omega}$ consider the point $(x',f(x'))$ and let

$$C_x:=\left\{y\in\mathbb{R}^N:\,L\,|y'-x'|<|y_N-f(x')|\,,\,y_N<f(x')\right\}$$

be the cone with vertex $(x',f(x'))$. Note that $x\in C_x$. Moreover, since f is Lipschitz, C_x does not intersect the graph of f. Thus, $C_x\subset\mathbb{R}^N\setminus\overline{\Omega}$. By the previous lemma,

$$\operatorname{dist}(x,\partial C_x)=\frac{1}{\sqrt{L^2+1}}\left(f(x')-x_N\right).$$

It follows that

$$\operatorname{dist}\left(x,\overline{\Omega}\right)\geq\operatorname{dist}(x,\partial C_x)=\frac{1}{\sqrt{L^2+1}}\left(f(x')-x_N\right).$$

Let d_{reg} be the regularized distance from $\overline{\Omega}$ (see Exercise 12.5, where the role of Ω there is played here by $\mathbb{R}^N\setminus\overline{\Omega}$). Then

$$d_{\mathrm{reg}}(x)\geq c_1\operatorname{dist}\left(x,\overline{\Omega}\right)\geq\frac{c_1}{\sqrt{L^2+1}}\left(f(x')-x_N\right), \tag{13}$$

where $c_1=c_1(N)$. On the other hand, since $(x',f(x'))\in\overline{\Omega}$, we have that

$$\operatorname{dist}\left(x,\overline{\Omega}\right)\leq|x-(x',f(x'))|=\left(f(x')-x_N\right), \tag{14}$$

and so

$$d_{\mathrm{reg}}(x)\leq c_2\operatorname{dist}\left(x,\overline{\Omega}\right)\leq c_2\left(f(x')-x_N\right),$$

where $c_2 = c_2(N)$.

Step 3: Let $v : \mathbb{R}^N \setminus \partial\Omega \to \mathbb{R}$ be the function defined by

$$v(x) = \int_1^\infty u(x', x_N + \kappa_L d_{\text{reg}}(x) t) \psi(t)\, dt,$$

where

$$\kappa_L := \frac{2}{c_1}\sqrt{L^2+1}.$$

Note that if $x \in \mathbb{R}^N \setminus \overline{\Omega}$, then by (13), for $t > 1$,

$$x_N + \kappa_L d_{\text{reg}}(x) t > x_N + 2(f(x') - x_N) = f(x') + (f(x') - x_N) > f(x'),$$

and so

$$(x', x_N + \kappa_L d_{\text{reg}}(x) t) \in \Omega.$$

On the other hand, if $x \in \overline{\Omega}$, then $d_{\text{reg}}(x) = 0$, and so by (5), $v(x) = u(x)$. Thus, v is an extension of u outside $\mathbb{R}^N \setminus \overline{\Omega}$.

Next we study the regularity of v. Note that $v(x) = w(x, \kappa_L d_{\text{reg}}(x))$, $x \in \mathbb{R}^N \setminus \overline{\Omega}$. Since $d_{\text{reg}} \in C^\infty(\Omega)$ and $d_{\text{reg}} = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, by the chain rule, we have that $v \in C^\infty\left(\mathbb{R}^N \setminus \partial\Omega\right)$. It remains to study the behavior at the boundary. For all $i = 1, \ldots, N$, and $x \in \mathbb{R}^N \setminus \partial\Omega$, we have that

$$\frac{\partial v}{\partial x_i}(x) = \frac{\partial w}{\partial x_i}(x, \kappa_L d_{\text{reg}}(x)) + \kappa_L \frac{\partial w}{\partial s}(x, \kappa_L d_{\text{reg}}(x)) \frac{\partial d_{\text{reg}}}{\partial x_i}(x). \tag{15}$$

Since

$$\left|\frac{\partial d_{\text{reg}}}{\partial x_i}(x)\right| \le C(N),$$

and $d_{\text{reg}}(x) \to 0$ as $x \to x_0 \in \partial\Omega$, it follows from (7) and (12) that

$$\lim_{x \to x_0} \frac{\partial v}{\partial x_i}(x) = \frac{\partial u}{\partial x_i}(x).$$

The situation is more complicated for multi-indeces of length greater than one. Recall that $z := (x, s) \in \mathbb{R}^N \times \mathbb{R}$, and set $g_i(x) := x_i$, $i = 1, \ldots, N$, $g_{N+1}(x) := \kappa_L d_{\text{reg}}(x)$. Then by Theorem 3 for every multi-index $\alpha \in \mathbb{N}_0^N$,

$$\frac{\partial^\alpha v}{\partial x^\alpha}(x) = \alpha! \sum_{|\beta|=1}^{|\alpha|} \frac{\partial^\beta w}{\partial z^\beta}(g(x)) \sum_{s=1}^{|\alpha|} \sum_{I_{s,\alpha,\beta}} \prod_{j=1}^{s} \frac{1}{\gamma_j! \, (\delta_j!)^{|\gamma_j|}} \left(\frac{\partial^{\delta_j} g}{\partial x^{\delta_j}}(x)\right)^{\gamma_j}, \tag{16}$$

where $\beta, \gamma_j \in \mathbb{N}_0^{N+1}$, $\delta_j \in \mathbb{N}_0^N$, and

$$I_{s,\alpha,\beta} = \Bigg\{ (\gamma_1, \ldots, \gamma_s, \delta_1, \ldots, \delta_s) : |\gamma_j| > 0, \quad 0 \prec \delta_1 \prec \cdots \prec \delta_s, \sum_{j=1}^s \gamma_j = \beta, \sum_{j=1}^s |\gamma_j| \delta_j = \alpha \Bigg\}.$$

Since $g_i(x) := x_i$, $i = 1, \ldots, N$, if $\delta_j = e_k$ for some $k = 1, \ldots, N$, then $\frac{\partial^{\delta_j} g_i}{\partial x^{\delta_j}}(x) = 0$ for all $i = 1, \ldots, N$, $i \neq k$, and so (recalling that $0^0 = 1$), the only nonzero terms are those for which $\gamma_j = m e_k + n e_{N+1}$. For the same reason, if two of the first N components of δ_j are one, then the only nonzero terms are those for which $\gamma_j = n e_{N+1}$. Finally, if one of the first N components of δ_j is bigger than one, then again the only nonzero terms are those for which $\gamma_j = n e_{N+1}$.

In view of (16), we have that $\frac{\partial^\alpha v}{\partial x^\alpha}(x)$ is given by

$$\frac{\partial^\alpha w}{\partial x^\alpha}(x, \kappa_L d_{\text{reg}}(x))\, 1 \tag{17}$$

(this can be seen either by induction on $|\alpha|$ or by taking $\beta = (\alpha_1, \ldots, \alpha_N, 0)$ in (16)) and by a sum of products of the type

$$I_{\beta,s,j}(x) := \alpha! \frac{\partial^\beta w}{\partial z^\beta}(x, \kappa_L d_{\text{reg}}(x)) \prod_{j=1}^{s} \frac{\kappa_L^{|\delta_j|(\gamma_j)_{N+1}}}{\gamma_j! \,(\delta_j!)^{|\gamma_j|}} \left(\frac{\partial^{\delta_j} d_{\text{reg}}}{\partial x^{\delta_j}}(x)\right)^{(\gamma_j)_{N+1}}, \tag{18}$$

where if $|\delta_j| > 1$ for some $j = 1, \ldots, s$, then the corresponding γ_j can be assumed of the form $(\gamma_j)_{N+1} e_{N+1}$. By the properties of the regularized distance (see Exercise 12.5), we have that

$$\left|\frac{\partial^{\delta_j} d_{\text{reg}}}{\partial x^{\delta_j}}(x)\right| \leq C_{\delta_j} \left(\operatorname{dist}\left(x, \overline{\Omega}\right)\right)^{1-|\delta_j|}, \tag{19}$$

and so

$$|I_{\beta,s,j}(x)| \leq C \kappa_L^{\sum_{j=1}^{s} |\delta_j|(\gamma_j)_{N+1}} \left|\frac{\partial^\beta w}{\partial z^\beta}(x, \kappa_L d_{\text{reg}}(x))\right| (d_{\text{reg}}(x))^{\sum_{|\delta_j|>1}(1-|\delta_j|)|\gamma_j|}, \tag{20}$$

where we have used the fact that if $|\delta_j| > 1$, then $(\gamma_j)_{N+1} = |\gamma_j|$. Since $d_{\text{reg}}(x) \to 0$ as $x \to x_0 \in \partial\Omega$, it follows from (7) and (12) that

$$\lim_{x \to x_0} \frac{\partial^\alpha v}{\partial x^\alpha}(x) = \frac{\partial^\alpha u}{\partial x^\alpha}(x).$$

Note that this implies that v can be extended to $\mathbb{R}^N$ in such a way that $v \in C^\infty\left(\mathbb{R}^N\right)$ (why?).

Step 4: We prove that

$$\|v\|_{L^p\left(\mathbb{R}^N \setminus \overline{\Omega}\right)} \leq C \|u\|_{L^p(\Omega)}. \tag{21}$$

Using the fact that $|\psi(t)| \leq C t^{-b}$, for some $b \geq 1$, and the change of variables $r = x_N + st$, we get

$$|w(x,s)| \leq C \int_1^\infty \frac{1}{t^b} |u(x', x_N + st)|\, dt = C s^{b-1} \int_{x_N+s}^\infty \frac{1}{(r - x_N)^b} |u(x', r)|\, dr, \tag{22}$$

Using (13), and (14), we have

$$2\left(f\left(x'\right)-x_N\right)\leq\kappa_L d_{\text{reg}}\left(x\right)\leq\kappa_L c_2\left(f\left(x'\right)-x_N\right)\tag{23}$$

for all $x\in\mathbb{R}^N\setminus\overline{\Omega}$, and so by (22) and using the fact that $r-x_N>r-f\left(x'\right)$, we obtain

$$\begin{aligned}\left|v\left(x\right)\right|&=\left|w\left(x,\kappa_L d_{\text{reg}}\left(x\right)\right)\right| &(24)\\ &\leq\kappa_L^{b-1}C\left(f\left(x'\right)-x_N\right)^{b-1}\int_{2f(x')-x_N}^{\infty}\frac{1}{\left(r-f\left(x'\right)\right)^b}\left|u\left(x',r\right)\right|\,dr.\end{aligned}$$

It follows by Hardy's inequality (see the file densityL1p.pdf) that

$$\begin{aligned}&\left(\int_{-\infty}^{f(x')}\left|v\left(x',x_N\right)\right|^p\,dx_N\right)^{\frac{1}{p}}\\ &\leq\kappa_L^{b-1}C\left(\int_{-\infty}^{f(x')}\left|\left(f\left(x'\right)-x_N\right)^{b-1}\int_{2f(x')-x_N}^{\infty}\frac{1}{\left(r-f\left(x'\right)\right)^b}\left|u\left(x',r\right)\right|\,dr\right|^p\,dx_N\right)^{\frac{1}{p}}\\ &=\kappa_L^{b-1}C\left(\int_0^{\infty}\left|t^{b-1}\int_t^{\infty}\frac{1}{s^b}\left|u\left(x',s+f\left(x'\right)\right)\right|\,ds\right|^p\,dt\right)^{\frac{1}{p}}\\ &\leq\frac{\kappa_L^{b-1}C}{\frac{1}{p}+b-1}\left(\int_0^{\infty}\left|u\left(x',t+f\left(x'\right)\right)\right|^p\,dt\right)^{\frac{1}{p}}=\frac{\kappa_L^{b-1}C}{\frac{1}{p}+b-1}\left(\int_{f(x')}^{\infty}\left|u\left(x',x_N\right)\right|^p\,dx_N\right)^{\frac{1}{p}}.\end{aligned}$$

Taking $b=1$, raising both sides to the power p, and integrating with respect to $x'\in\mathbb{R}^{N-1}$ gives (21).

Step 5: We show that

$$\left\|\frac{\partial^\alpha v}{\partial x^\alpha}\right\|_{L^p\left(\mathbb{R}^N\setminus\overline{\Omega}\right)}\leq C\left\|\frac{\partial^\alpha u}{\partial x^\alpha}\right\|_{L^p(\Omega)}$$

for every multi-index $\alpha\in\mathbb{N}_0^N$ with $0<|\alpha|\leq m$.

By (15) and (19), for all $i=1,\ldots,N$, and $x\in\mathbb{R}^N\setminus\partial\Omega$, we have that

$$\left|\frac{\partial v}{\partial x_i}\left(x\right)\right|\leq\left|\frac{\partial w}{\partial x_i}\left(x,\kappa_L d_{\text{reg}}\left(x\right)\right)\right|+C\kappa_L\left|\frac{\partial w}{\partial s}\left(x,\kappa_L d_{\text{reg}}\left(x\right)\right)\right|.$$

In view of (6) and (8), we may proceed as in the previous step, to obtain

$$\left\|\frac{\partial v}{\partial x_i}\right\|_{L^p\left(\mathbb{R}^N\setminus\overline{\Omega}\right)}\leq C\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}+C\kappa_L\left\|\frac{\partial u}{\partial x_N}\right\|_{L^p(\Omega)}.$$

On the other hand, for higher order multi-indeces $\alpha\in\mathbb{N}_0^N$, the function (17) can be estimated exactly as in the previous step leading to

$$\left\|\frac{\partial^\alpha w}{\partial x^\alpha}\left(\cdot,\kappa_L d_{\text{reg}}\left(\cdot\right)\right)\right\|_{L^p\left(\mathbb{R}^N\setminus\overline{\Omega}\right)}\leq C\left\|\frac{\partial^\alpha u}{\partial x^\alpha}\right\|_{L^p(\Omega)}.$$

Similarly, for (18), if $|\delta_j| = 1$ for all $j = 1, \ldots, s$, then

$$|\beta| = \sum_{j=1}^{s} |\gamma_j| = \sum_{j=1}^{s} |\gamma_j| \, |\delta_j| = |\alpha| \, ,$$

and so by (10) and (20),

$$\begin{aligned}\|I_{\beta,s,j}\|_{L^p(\mathbb{R}^N \setminus \overline{\Omega})} &\le C\kappa_L^{\sum_{j=1}^{s} |\delta_j| (\gamma_j)_{N+1}} \left\| \frac{\partial^\beta w}{\partial z^\beta} \left(\cdot, \kappa_L d_{\mathrm{reg}}(\cdot)\right) \right\|_{L^p(\mathbb{R}^N \setminus \overline{\Omega})} \\ &\le C\kappa_L^{|\alpha|} \sum_{1 \le |\delta| \le |\alpha|} \left\| \frac{\partial^\delta u}{\partial x^\delta} \right\|_{L^p(\Omega)} ,\end{aligned}$$

where the second inequality has been obtained as in the previous step.

On the other hand, if $|\delta_j| > 1$ for some $j = 1, \ldots, s$, then the corresponding γ_j can be assumed of the form $(\gamma_j)_{N+1} e_{N+1}$ and

$$\sum_{j=1}^{s} |\gamma_j| = |\beta| < \sum_{j=1}^{s} |\gamma_j| \, |\delta_j| = |\alpha| \, ,$$

and so we take

$$\begin{aligned}\ell := |\alpha| - |\beta| - 1 &= \sum_{j=1}^{s} |\gamma_j| \left(|\delta_j| - 1\right) - 1 \\ &= \sum_{|\delta_j| > 1} |\gamma_j| \left(|\delta_j| - 1\right) - 1\end{aligned}$$

and use (11) and the change of variables $t = x_N + r$ to obtain

$$\left| \frac{\partial^\beta w}{\partial z^\beta} (x, s) \right| \le C s^{\ell + a - 1} \int_{x_N + s}^{\infty} \frac{1}{(t - x_N)^{a-1}} \left| \frac{\partial^{(\beta', \beta_N + \ell + 1)} u}{\partial (x')^{\beta'} \partial x_N^{\beta_N + \ell + 1}} (x', t) \right| dt.$$

Note that $|(\beta', \beta_N + \ell + 1)| = |\alpha|$. It follows from (20) that

$$|I_{\beta,s,j}(x)| \le C\kappa_L^{2|\alpha| - |\beta| + a - 2} \left(d_{\mathrm{reg}}(x)\right)^{a-2} \int_{x_N + \kappa_L d_{\mathrm{reg}}(x)}^{\infty} \frac{1}{(t - x_N)^{a-1}} \left| \frac{\partial^{(\beta', \beta_N + \ell + 1)} u}{\partial (x')^{\beta'} \partial x_N^{\beta_N + \ell + 1}} (x', t) \right| dt.$$

Taking $a = 2$, using (23), we can now continue as in the previous step (see (24)) to conclude that

$$\|I_{\beta,s,j}\|_{L^p(\mathbb{R}^N \setminus \overline{\Omega})} \le \frac{C\kappa_L^{2|\alpha| - |\beta|}}{\frac{1}{p}} \sum_{1 \le |\delta| \le |\alpha|} \left\| \frac{\partial^\delta u}{\partial x^\delta} \right\|_{L^p(\Omega)} .$$

Step 6: We have proved that the linear operator

$$\mathcal{E} : C_c^\infty \left(\mathbb{R}^N\right) \to W^{m,p} \left(\mathbb{R}^N\right)$$

defined by

$$\mathcal{E}(u)(x) = v(x) \quad \text{for } x \in \mathbb{R}^N$$

satisfies (2), (3), and (4). Since functions in $C_c^\infty(\mathbb{R}^N)$ are dense in $W^{m,p}(\mathbb{R}^N)$, we can extend $\mathcal{E}$ to a linear operator defined on $W^{m,p}(\mathbb{R}^N)$ and with the same properties. ■

Remark 4 *Note that the operator $\mathcal{E}$ does not depend on m and p.*

Next we consider uniformly Lipschitz domains. We recall the definition.

Definition 5 *The boundary $\partial\Omega$ of an open set $\Omega \subset \mathbb{R}^N$ is* uniformly Lipschitz *if there exist $\varepsilon, L > 0$, $M \in \mathbb{N}$, and a locally finite countable open cover $\{\Omega_n\}$ of $\partial\Omega$ such that*

(i) if $x \in \partial\Omega$, then $B(x,\varepsilon) \subset \Omega_n$ for some $n \in \mathbb{N}$,

(ii) no point of $\mathbb{R}^N$ is contained in more than M of the Ω_n's,

(iii) for each n there exist local coordinates $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and a Lipschitz function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ (both depending on n), with Lip $f \le L$, *such that*

$$\Omega_n \cap \Omega = \Omega_n \cap \left\{ (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : y_N > f(y') \right\}.$$

Exercise 6 *Prove that for every $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and for every $n \in \mathbb{N}$,*

$$(x_1 + \cdots + x_n)^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!} x^\alpha.$$

Exercise 7 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u, v \in C^m(\Omega)$. Prove that for every multi-index $\alpha \in \mathbb{N}_0^N$, with $0 < |\alpha| \le m$,*

$$\frac{\partial^\alpha (uv)}{\partial x^\alpha}(x) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \frac{\partial^\beta u}{\partial x^\beta}(x) \frac{\partial^\gamma v}{\partial x^\gamma}(x)$$

for $x \in \Omega$. Hint: Use induction on N.

Theorem 8 (Stein) *Let $\Omega \subset \mathbb{R}^N$ be an open set with uniformly Lipschitz boundary. Then for all $1 \le p \le \infty$ and $m \in \mathbb{N}$ there exists a continuous linear operator*

$$\mathcal{E} : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^N)$$

such that for all $u \in W^{m,p}(\Omega)$,

$$\mathcal{E}(u)(x) = u(x) \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega$$

and

$$\|\mathcal{E}(u)\|_{L^p(\mathbb{R}^N)} \leq C(1+M)\|u\|_{L^p(\Omega)},$$

$$\left\|\frac{\partial^\alpha \mathcal{E}(u)}{\partial x^\alpha}\right\|_{L^p(\mathbb{R}^N)} \leq C\left(1+M\left(1+L^{2|\alpha|-1}\right)\right)\sum_{0\leq|\beta|\leq|\alpha|}\left(\frac{M}{\varepsilon}\right)^{|\alpha|-|\beta|}\left\|\frac{\partial^\beta u}{\partial x^\beta}\right\|_{L^p(\Omega)}.$$

for every multi-index $\alpha \in \mathbb{N}_0^N$ *with* $1 \leq |\alpha| \leq m$*.*

Proof. We only prove the case $1 \leq p < \infty$ and leave the easier case $p = \infty$ as an exercise. For every set $E \subset \mathbb{R}^N$ and every $r > 0$, we define

$$E^r := \left\{x \in \mathbb{R}^N : B(x,r) \subset E\right\}.$$

We observe that $E^r \subset E$ and that condition (i) in Definition 5 reads

$$\partial\Omega \subset \bigcup_n \Omega_n^\varepsilon.$$

Define the regularized functions

$$\phi_n := \varphi_{\frac{\varepsilon}{4}} * \chi_{\Omega_n^{3\varepsilon/4}}, \tag{25}$$

where $\varphi_{\frac{\varepsilon}{4}}$ is a standard mollifier. Then

$$\operatorname{supp}\phi_n \subset \Omega_n, \quad \phi_n = 1 \text{ in } \Omega_n^{\varepsilon/2}. \tag{26}$$

By Theorem C.20, for every multi-index $\alpha \in \mathbb{N}_0^N$, we have that

$$\frac{\partial^\alpha \phi_n}{\partial x_\alpha} = \frac{\partial^\alpha \varphi_{\frac{\varepsilon}{4}}}{\partial x_\alpha} * \chi_{\Omega_n^{3\varepsilon/4}},$$

and so for all $x \in \Omega_n$ we have that

$$\begin{aligned}\left|\frac{\partial^\alpha \phi_n}{\partial x_\alpha}(x)\right| &\leq \int_{\mathbb{R}^N}\left|\frac{\partial^\alpha \varphi_{\frac{\varepsilon}{4}}}{\partial x_\alpha}(y-x)\right|\left|\chi_{\Omega_n^{3\varepsilon/4}}(y)\right| dy \\ &\leq \left\|\frac{\partial^\alpha \varphi_{\frac{\varepsilon}{4}}}{\partial x_\alpha}\right\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} \leq \frac{C}{\varepsilon^{|\alpha|}}.\end{aligned} \tag{27}$$

Next consider the three open sets

$$\begin{aligned}\Omega_0 &:= \left\{x \in \mathbb{R}^N : \operatorname{dist}(x,\Omega) < \frac{\varepsilon}{4}\right\}, \\ \Omega_+ &:= \left\{x \in \mathbb{R}^N : \operatorname{dist}(x,\partial\Omega) < \frac{3\varepsilon}{4}\right\}, \\ \Omega_- &:= \left\{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \frac{\varepsilon}{4}\right\},\end{aligned} \tag{28}$$

and define the regularized functions

$$\phi_0 := \varphi_{\frac{\varepsilon}{4}} * \chi_{\Omega_0}, \quad \phi_\pm := \varphi_{\frac{\varepsilon}{4}} * \chi_{\Omega_\pm}. \tag{29}$$

Then $\phi_0 = 1$ in $\overline{\Omega}$, $\phi_+(x) = 1$ if $x \in \mathbb{R}^N$ and $\operatorname{dist}(x, \partial\Omega) \le \frac{\varepsilon}{2}$, and $\phi_-(x) = 1$ if $x \in \Omega$ and $\operatorname{dist}(x, \partial\Omega) \ge \frac{\varepsilon}{2}$. Moreover, the supports of ϕ_0, ϕ_+, and ϕ_- are contained, respectively, in an $\frac{\varepsilon}{2}$ neighborhood of Ω, in an ε neighborhood of $\partial\Omega$, and in Ω. Finally, reasoning as in (27), for every multi-index $\alpha \in \mathbb{N}_0^N$, we have that

$$\left\| \frac{\partial^\alpha \phi_0}{\partial x_\alpha} \right\|_\infty, \left\| \frac{\partial^\alpha \phi_\pm}{\partial x_\alpha} \right\|_\infty \le \frac{C}{\varepsilon^{|\alpha|}}. \tag{30}$$

Note that

$$\operatorname{supp} \phi_0 \subset \left\{ x \in \mathbb{R}^N : \phi_+(x) + \phi_-(x) \ge 1 \right\}. \tag{31}$$

Thus, with a slight abuse of notation, we may define

$$\psi_+ := \phi_0 \frac{\phi_+}{\phi_+ + \phi_-}, \quad \psi_- := \phi_0 \frac{\phi_-}{\phi_+ + \phi_-}, \tag{32}$$

where we interpret the right-hand sides to be zero whenever $\phi_0 = 0$. Again by (31) we have that all the derivatives of $\psi_\pm$ are bounded by $\frac{C}{\varepsilon}$. Also, $\psi_+ + \psi_- = 1$ in $\overline{\Omega}$ and $\psi_+ = \psi_- = 0$ outside an $\frac{\varepsilon}{2}$ neighborhood of Ω. By Exercise 7,

$$\frac{\partial^\alpha \psi_\pm}{\partial x^\alpha} = \frac{\partial^\alpha}{\partial x^\alpha} \left(\phi_0 \phi_\pm (\phi_+ + \phi_-)^{-1} \right) = \sum_{\beta+\delta=\alpha} \frac{\alpha!}{\beta!\delta!} \frac{\partial^\beta (\phi_0 \phi_\pm)}{\partial x^\beta} \frac{\partial^\delta}{\partial x^\delta} \left((\phi_+ + \phi_-)^{-1} \right)$$

$$= \sum_{\beta+\delta=\alpha} \frac{\alpha!}{\beta!\delta!} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \frac{\partial^{\beta_1} \phi_0}{\partial x^{\beta_1}} \frac{\partial^{\beta_2} \phi_\pm}{\partial x^{\beta_2}} \frac{\partial^\delta}{\partial x^\delta} \left((\phi_+ + \phi_-)^{-1} \right),$$

while by Theorem 3 with $w(t) = t^{-1}$ and $g(x) = \sum_k \phi_k^2(x)$,

$$\frac{\partial^\delta}{\partial x^\delta} (\phi_+ + \phi_-)^{-1}$$

$$= \delta! \sum_{n=1}^{|\delta|} (-1)^n n! (\phi_+ + \phi_-)^{-n-1} \sum_{s=1}^{|\delta|} \sum_{I_{s,\delta,n}} \prod_{j=1}^{s} \frac{1}{\gamma_j! (\delta_j!)^{|\gamma_j|}} \left(\frac{\partial^{\delta_j}}{\partial x^{\delta_j}} (\phi_+ + \phi_-) \right)^{\gamma_j}$$

$$= \delta! \sum_{n=1}^{|\delta|} (-1)^n n! (\phi_+ + \phi_-)^{-n-1} \sum_{s=1}^{|\delta|} \sum_{I_{s,\delta,n}} \prod_{j=1}^{s} \frac{1}{\gamma_j! (\delta_j!)^{|\gamma_j|}} \left(\sum_{\delta'+\delta''=\delta_j} \frac{\delta_j!}{\delta'!\delta''!} \frac{\partial^{\delta'} \phi_+}{\partial x^{\delta'}} \frac{\partial^{\delta''} \phi_-}{\partial x^{\delta''}} \right)^{\gamma_j}$$

and

$$I_{s,\delta,n} = \left\{ (\gamma_1, \ldots, \gamma_s, \delta_1, \ldots, \delta_s) : |\gamma_j| > 0, \right.$$
$$\left. 0 \prec \delta_1 \prec \cdots \prec \delta_s, \sum_{j=1}^{s} \gamma_j = n, \sum_{j=1}^{s} |\gamma_j| \delta_j = \delta \right\}.$$

Using the facts that $\beta+\delta=\alpha$, $\beta_1+\beta_2=\beta$, $\delta'+\delta''=\delta_j$, $\sum_{j=1}^{s}|\gamma_j|\,\delta_j=\delta$, and (30), we have that

$$\left\|\frac{\partial^\alpha\psi_\pm}{\partial x_\alpha}\right\|_\infty\le\frac{C}{\varepsilon^{|\alpha|}}. \tag{33}$$

We are finally ready to construct the linear extension operator. Given a function $u\in W^{m,p}(\Omega)$, $1\le p<\infty$, since $\operatorname{supp}(\phi_n u)\subset\Omega_n$ for each n by (26), by condition (i) in Definition 5 and Theorem 1 we can extend $\phi_n u$ to a function $v_n\in W^{m,p}\left(\mathbb{R}^N\right)$ in such a way that

$$\begin{aligned}\|v_n\|_{L^p(\mathbb{R}^N)}&=C(N,p)\,\|\phi_n u\|_{L^p(\Omega\cap\Omega_n)}\,, \qquad (34)\\ \left\|\frac{\partial^\alpha v_n}{\partial x^\alpha}\right\|_{L^p(\mathbb{R}^N)}&\le C(N,p)\left(1+L^{2|\alpha|-1}\right)\sum_{1\le|\delta|\le|\alpha|}\left\|\frac{\partial^\delta(\phi_n u)}{\partial x^\delta}\right\|_{L^p(\Omega\cap\Omega_n)}.\end{aligned}$$

Again with a slight abuse of notation we define

$$\mathcal{E}(u)(x):=\psi_+(x)\,\frac{\sum\limits_n\phi_n(x)\,v_n(x)}{\sum\limits_k\phi_k^2(x)}+\psi_-(x)\,u(x)\,,\quad x\in\mathbb{R}^N. \tag{35}$$

Note that if $x\in\mathbb{R}^N$ is such that $\operatorname{dist}(x,\partial\Omega)\le\frac{\varepsilon}{2}$, then there exists an n such that $y\in\Omega_n^{\varepsilon/2}$, and so $\phi_n(x)=1$ by (26). In particular, since all the functions ϕ_n are nonnegative, it follows that

$$\text{if } x\in\operatorname{supp}\psi_+,\ \text{then }\sum_n\phi_n(x)\ge1, \tag{36}$$

and thus the first term on the right-hand side of (35) is well-defined, provided we interpret it to be zero whenever $\psi_+=0$. Similarly, since $\operatorname{supp}\psi_-\subset\operatorname{supp}\phi_-\subset\Omega$, the term $\psi_-(x)\,u(x)$ is well-defined, provided we set it to be zero outside Ω.

It remains to show that the linear operator $\mathcal{E}(u)$ is an extension operator and that it is bounded. For the former, it suffices to observe that if $x\in\Omega$, then $v_n(x)=\phi_n(x)\,u(x)$, and so

$$\mathcal{E}(u)(x)=\psi_+(x)\,u(x)+\psi_-(x)\,u(x)=u(x)$$

by (32). To obtain the latter, we observe that by (35), condition (ii) in Definition 5, Exercise 12.14, (36), the fact that $0\le\psi_\pm,\phi_n\le1$, and (34), in this order, we have

$$\begin{aligned}\|\mathcal{E}(u)\|_{L^p(\mathbb{R}^N)}&\le M^{\frac{1}{p'}}\left(\sum_n\int_{\Omega_n}|v_n|^p\,dx\right)^{\frac1p}+\|u\|_{L^p(\Omega)}\\&\le 2M^{\frac{1}{p'}}\left(\int_\Omega|u|^p\sum_n|\phi_n|^p\,dx\right)^{\frac1p}+\|u\|_{L^p(\Omega)}\\&\le(1+2M)\,\|u\|_{L^p(\Omega)}\,,\end{aligned}$$

where in the last inequality we have used the fact that $\sum_n |\phi_n|^p \leq M$.

To estimate $\left\| \frac{\partial^\alpha \mathcal{E}(u)}{\partial x^\alpha} \right\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}$, we use the fact that since $\{\Omega_n\}$ is locally finite, any bounded neighborhood of every point $x \in \mathbb{R}^N$ intersects only finitely many Ω_n's. Hence, by Exercise 7,

$$\begin{aligned}\frac{\partial^\alpha \mathcal{E}(u)}{\partial x^\alpha} &= \sum_n \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \frac{\partial^\beta v_n}{\partial x^\beta} \frac{\partial^\gamma}{\partial x^\gamma} \left(\psi_+ \phi_n \left(\sum_k \phi_k^2 \right)^{-1} \right) \\ &\quad + \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \frac{\partial^\beta u}{\partial x^\beta} \frac{\partial^\gamma \psi_-}{\partial x^\gamma}.\end{aligned}$$

Again by Exercise 7,

$$\begin{aligned}\frac{\partial^\gamma}{\partial x^\gamma} \left(\psi_+ \phi_n \left(\sum_k \phi_k^2 \right)^{-1} \right) &= \sum_{\beta+\delta=\gamma} \frac{\gamma!}{\beta!\delta!} \frac{\partial^\beta (\psi_+ \phi_n)}{\partial x^\beta} \frac{\partial^\delta}{\partial x^\delta} \left(\left(\sum_k \phi_k^2 \right)^{-1} \right) \\ = \sum_{\beta+\delta=\gamma} \frac{\gamma!}{\beta!\delta!} &\sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \frac{\partial^{\beta_1}\psi_+}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}\phi_n}{\partial x^{\beta_2}} \frac{\partial^\delta}{\partial x^\delta} \left(\left(\sum_k \phi_k^2 \right)^{-1} \right),\end{aligned}$$

while by Theorem 3 with $w(t) = t^{-1}$ and $g(x) = \sum_k \phi_k^2(x)$,

$$\begin{aligned}&\frac{\partial^\delta}{\partial x^\delta} \left(\left(\sum_k \phi_k^2 \right)^{-1} \right) \\ &= \delta! \sum_{m=1}^{|\delta|} (-1)^m m! \left(\sum_k \phi_k^2 \right)^{-m-1} \sum_{s=1}^{|\delta|} \sum_{I_{s,\delta,m}} \prod_{j=1}^{s} \frac{1}{\gamma_j! (\delta_j!)^{|\gamma_j|}} \left(\frac{\partial^{\delta_j}}{\partial x^{\delta_j}} \left(\sum_k \phi_k^2 \right) \right)^{\gamma_j} \\ &= \delta! \sum_{m=1}^{|\delta|} (-1)^m m! \left(\sum_k \phi_k^2 \right)^{-m-1} \sum_{s=1}^{|\delta|} \sum_{I_{s,\delta,m}} \prod_{j=1}^{s} \frac{1}{\gamma_j! (\delta_j!)^{|\gamma_j|}} \left(\sum_k \sum_{\delta'+\delta''=\delta_j} \frac{\delta_j!}{\delta'!\delta''!} \frac{\partial^{\delta'}\phi_k}{\partial x^{\delta'}} \frac{\partial^{\delta''}\phi_k}{\partial x^{\delta''}} \right)^{\gamma_j}\end{aligned}$$

and

$$\begin{aligned}I_{s,\delta,m} = \Bigg\{ &(\gamma_1, \ldots, \gamma_s, \delta_1, \ldots, \delta_s) : |\gamma_j| > 0, \\ &0 \prec \delta_1 \prec \cdots \prec \delta_s, \sum_{j=1}^{s} \gamma_j = m, \sum_{j=1}^{s} |\gamma_j| \delta_j = \delta \Bigg\}.\end{aligned}$$

Using (10) and (36), condition (ii) in Definition 5, the previous exercise, and the facts that $0 \leq \psi_\pm, \phi_n \leq 1$, $|\nabla \psi_\pm|, |\nabla \phi_n| \leq \frac{C}{\varepsilon}$,

$$\left\| \frac{\partial^\delta}{\partial x^\delta} \left(\left(\sum_k \phi_k^2 \right)^{-1} \right) \right\|_\infty \leq C \frac{M^{|\delta|}}{\varepsilon^{|\delta|}},$$

and so by (10) and (33),

$$\left\| \frac{\partial^\gamma}{\partial x^\gamma} \left(\psi_+ \phi_n \left(\sum_k \phi_k^2 \right)^{-1} \right) \right\|_\infty \le C \frac{M^{|\gamma|}}{\varepsilon^{|\gamma|}}.$$

Hence,

$$\left| \frac{\partial^\alpha \mathcal{E}(u)}{\partial x^\alpha} \right| \le C \sum_n \sum_{\beta+\gamma=\alpha} \frac{M^{|\gamma|}}{\varepsilon^{|\gamma|}} \left| \frac{\partial^\beta v_n}{\partial x^\beta} \right| + C \sum_{\beta+\gamma=\alpha} \frac{1}{\varepsilon^{|\gamma|}} \left| \frac{\partial^\beta u}{\partial x^\beta} \right|.$$

By Exercise 12.14 and the fact that for each $x \in \mathbb{R}^N$ at most M terms in the sum are different from zero, we have

$$\begin{aligned} \left\| \frac{\partial^\alpha \mathcal{E}(u)}{\partial x^\alpha} \right\|_{L^p(\mathbb{R}^N)} \le{}& CM^{\frac{1}{p'}} \left(\sum_n \sum_{\beta+\gamma=\alpha} \frac{M^{|\gamma|p}}{\varepsilon^{|\gamma|p}} \int_{\Omega_n} \left| \frac{\partial^\beta v_n}{\partial x^\beta} \right|^p dx \right)^{\frac{1}{p}} \\ &+ C \sum_{\beta+\gamma=\alpha} \frac{1}{\varepsilon^{|\gamma|}} \left\| \frac{\partial^\beta u}{\partial x^\beta} \right\|_{L^p(\Omega)}. \end{aligned}$$

Finally, by (34), if $\beta = 0$,

$$\begin{aligned} \sum_n \int_{\Omega_n} |v_n|^p \, dx &\le C \sum_n \int_{\Omega \cap \Omega_n} |\phi_n u|^p \, dx \le C \int_\Omega \sum_n |\phi_n u|^p \, dx \\ &\le CM \int_\Omega |u|^p \, dx \end{aligned}$$

where we have used the facts that $\operatorname{supp} \phi_n \subset \Omega_n$ and that $\sum_n |\phi_n|^p \le M$. Similarly, if $|\beta| \ge 1$,

$$\begin{aligned} \sum_n \int_{\Omega_n} \left| \frac{\partial^\beta v_n}{\partial x^\beta} \right|^p dx &\le C \left(1 + L^{2|\beta|-1} \right)^p \sum_{1 \le |\delta| \le |\beta|} \sum_n \int_{\Omega \cap \Omega_n} \left| \frac{\partial^\delta (\phi_n u)}{\partial x^\delta} \right|^p dx \\ &\le C \left(1 + L^{2|\beta|-1} \right)^p \sum_{1 \le |\delta| \le |\beta|} \int_\Omega \sum_n \left| \frac{\partial^\delta (\phi_n u)}{\partial x^\delta} \right|^p dx \\ &\le CM \left(1 + L^{2|\beta|-1} \right)^p \sum_{1 \le |\delta| \le |\beta|} \frac{1}{\varepsilon^{(|\beta|-|\delta|)p}} \int_\Omega \left| \frac{\partial^\delta u}{\partial x^\delta} \right|^p dx, \end{aligned}$$

where we have used Exercise 7 to estimate

$$\sum_n \left| \frac{\partial^\delta (\phi_n u)}{\partial x^\delta} \right|^p \le C \sum_n \sum_{\gamma'+\gamma''=\delta} \left| \frac{\partial^{\gamma'} \phi_n}{\partial x^{\gamma'}} \frac{\partial^{\gamma''} u}{\partial x^{\gamma''}} \right|^p \le CM \sum_{\gamma'+\gamma''=\delta} \frac{1}{\varepsilon^{|\gamma'|p}} \left| \frac{\partial^{\gamma''} u}{\partial x^{\gamma''}} \right|^p.$$

In turn,

$$\left\| \frac{\partial^\alpha \mathcal{E}\left(u\right)}{\partial x^\alpha} \right\|_{L^p(\mathbb{R}^N)} \le CM\left(1 + L^{2|\alpha|-1}\right) \sum_{0\le|\delta|\le|\alpha|} \frac{M^{|\alpha|-|\delta|}}{\varepsilon^{|\alpha|-|\delta|}} \left\| \frac{\partial^\delta u}{\partial x^\delta} \right\|_{L^p(\Omega)} + C \sum_{0\le|\delta|\le|\alpha|} \frac{1}{\varepsilon^{|\alpha|-|\delta|}} \left\| \frac{\partial^\delta u}{\partial x^\delta} \right\|_{L^p(\Omega)}.$$

To conclude the proof, it suffices to observe that since $\{\Omega_n\}$ is locally finite, in a neighborhood of every point the infinite sum in (35) is finite. Hence, we can now invoke Theorem 10.35 and the previous estimates to conclude that $\mathcal{E}\left(u\right) \in W^{m,p}\left(\mathbb{R}^N\right)$. ■

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